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On Oscillation Theorems for Differential Polynomials

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Abstract. In this paper, we investigate the relationship between small functions and differential polynomials $g_f(z) = d_2 f'' + d_1 f' + d_0 f$, where $d_0(z), d_1(z), d_2(z)$ are meromorphic functions that are not all equal to zero with finite order generated by solutions of the second order linear differential equation

$$f'' + Af' + Bf = F,$$

where A, B, $F \not\equiv 0$ are finite order meromorphic functions having only finitely many poles.

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1 Introduction and Statement of Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [7], [10]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, where $T(r, f)$ is the Nevanlinna characteristic function of f.

To give the precise estimate of fixed points, we define:

Definition 1.1 ([9], [11], [12]) Let f be a meromorphic function and let $z_1, z_2, ...$ ($|z_j| = r_j$, $0 < r_1 \leq r_2 \leq ...$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.1}$$

where $\overline{N}\left(r, \frac{1}{f-z}\right)$ is the counting function of distinct fixed points of $f(z)$ in $\{|z| < r\}$.

Recently the complex oscillation theory of the complex differential equations has been investigated actively [1, 2, 3, 4, 5, 6, 8, 9, 11, 12]. In the study of the differential equation

$$f'' + Af' + Bf = F, \tag{1.2}$$

where $A, B, F \not\equiv 0$ are finite order meromorphic functions having only finitely many poles, Chen [4] has investigated the complex oscillation of (1.2) and has obtained the following results:

Theorem A [4] *Suppose that $A, B, F \not\equiv 0$ are finite order meromorphic functions having only finitely many poles and $F \not\equiv CB$ for any constant C. Let $\alpha > 0$, $\beta > 0$ be real constants and we have $\rho(B) < \beta$, $\rho(F) < \beta$. Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\phi_m\}$ and $\{\theta_m\}$ that satisfy*

$$\phi_1 < \theta_1 < \phi_2 < \theta_2 < ... < \phi_n < \theta_n < \phi_{n+1} = \phi_1 + 2\pi \tag{1.3}$$

and

$$\sum_{m=1}^{n}\left(\phi_{m+1}-\theta_{m}\right)<\varepsilon \tag{1.4}$$

such that

$$\left|A\left(z\right)\right|\geq\exp\left\{\left(1+o\left(1\right)\right)\alpha\left|z\right|^{\beta}\right\} \tag{1.5}$$

as $z\rightarrow\infty$ *in* $\phi_{m}\leq\arg z\leq\theta_{m}\ \left(m=1,...,n\right).$

If the second order non-homogeneous linear differential equation (1.2) *has a meromorphic solution* $f\left(z\right)$, *then*

$$\lambda\left(f\right)=\overline{\lambda}\left(f\right)=\rho\left(f\right)=\infty. \tag{1.6}$$

Theorem B [4] *Suppose that* $A,B,F\not\equiv 0$ *are finite order meromorphic functions having only finitely many poles. Let* $\alpha>0,\ \beta>0$ *be real constants and we have* $\rho\left(B\right)<\beta\leq\rho\left(F\right)$. *Suppose that for any given* $\varepsilon>0$, *there exist two finite collections of real numbers* $\{\phi_{m}\}$ *and* $\{\theta_{m}\}$ *that satisfy* (1.3) *and* (1.4) *such that* (1.5) *holds as* $z\rightarrow\infty$ *in* $\phi_{m}\leq\arg z\leq\theta_{m}\ \left(m=1,...,n\right).$ *If equation* (1.2) *has a meromorphic solution* $f\left(z\right)$, *then*
(*a*) *If* $B\not\equiv 0$, *then* (1.2) *has at most one finite order meromorphic solution* f_{0} *and all other meromorphic solutions of* (1.2) *satisfy* (1.6). *If* $B\equiv 0$, *then any two finite order solutions* $f_{0},\ f_{1}$ *of* (1.2) *satisfy* $f_{1}=f_{0}+C$ *for some constant* C. *If all the solutions of* (1.2) *are meromorphic, then* (1.2) *has a solution which satisfies* (1.6).
(*b*) *If there exists a finite order meromorphic solution* f_{0} *in case* (*a*), *then* f_{0} *satisfies*

$$\rho\left(f_{0}\right)\leq\max\left\{\rho\left(F\right),\rho\left(A\right),\overline{\lambda}\left(f_{0}\right)\right\}. \tag{1.7}$$

If $\overline{\lambda}\left(f_{0}\right)<\rho\left(f_{0}\right),\ \rho\left(F\right)\neq\rho\left(A\right)$, *then* $\rho\left(f_{0}\right)=\max\left\{\rho\left(F\right),\rho\left(A\right)\right\}.$

Recently, in [6] Chen Zongxuan and Shon Kwang Ho have studied the growth of solutions of the differential equation

$$f^{''}+A_{1}\left(z\right)e^{az}f^{'}+A_{0}\left(z\right)e^{bz}f=0 \tag{1.8}$$

and have obtained the following result:

Theorem C [6] *let* $A_j(z)$ $(\not\equiv 0)$ $(j=0,1)$ *be meromorphic functions with* $\rho(A_j) < 1$ $(j=0,1)$, a, b *be complex numbers such that* $ab \neq 0$ *and* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of equation* (1.8) *has infinite order.*

In the same paper, Z. X. Chen and K. H. Shon have investigated the fixed points of solutions, their 1st and 2nd derivatives and the differential polynomials and have obtained :

Theorem D [6] *Let* $A_j(z)$ $(j=0,1)$, a, b, c *satisfy the additional hypotheses of Theorem C. Let* d_0, d_1, d_2 *be complex constants that are not all equal to zero. If* $f(z) \not\equiv 0$ *is any meromorphic solution of equation* (1.8), *then:*
(i) f, f', f'' *all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = \infty, \tag{1.9}$$

(ii) *the differential polynomial*

$$g(z) = d_2 f'' + d_1 f' + d_0 f \tag{1.10}$$

has infinitely many fixed points and satisfies $\overline{\tau}(g) = \infty$.

The main purpose of this paper is to study the growth, the oscillation and the relation between small functions and differential polynomials generated by solutions of second order linear differential equation (1.2).

Before we state our results, we denote by

$$\alpha_1 = d_1 - d_2 A, \;\; \beta_0 = d_2 AB - (d_2 B)' - d_1 B + d_0', \tag{1.11}$$

$$\alpha_0 = d_0 - d_2 B, \;\; \beta_1 = d_2 A^2 - (d_2 A)' - d_1 A - d_2 B + d_0 + d_1', \tag{1.12}$$

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 \tag{1.13}$$

and

$$\psi(z) = \frac{\alpha_1 \left(\varphi' - (d_2 F)' - \alpha_1 F\right) - \beta_1 (\varphi - d_2 F)}{h}, \tag{1.14}$$

where $A, B,\ F \not\equiv 0$ are meromorphic functions having only finitely many poles and $d_j\ (j = 0, 1, 2)$, φ are meromorphic functions with finite order.

Theorem 1.1 *Suppose that $A, B, F,\ \alpha, \beta, \varepsilon, \{\phi_m\}$ and $\{\theta_m\}$ satisfy the hypotheses of Theorem A. Let $d_0(z), d_1(z), d_2(z)$ be meromorphic functions that are not all equal to zero with $\rho(d_j) < \infty\ (j = 0, 1, 2)$ such that $h \not\equiv 0$, and let $\varphi(z)$ be a meromorphic function with finite order. If $f(z)$ is a meromorphic solution of (1.2), then the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies*

$$\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty. \tag{1.15}$$

Theorem 1.2 *Suppose that $A, B, F,\ \alpha, \beta, \varepsilon, \{\phi_m\}$ and $\{\theta_m\}$ satisfy the hypotheses of Theorem 1.1, and let $\varphi(z)$ be a meromorphic function with finite order. If $f(z)$ is a meromorphic solution of (1.2), then we have*

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}\left(f' - \varphi\right) = \overline{\lambda}\left(f'' - \varphi\right) = \infty. \tag{1.16}$$

Setting $\varphi(z) = z$ in Theorem 1.2, we obtain the following corollary:

Corollary 1.1 *Suppose that $A, B, F,\ \alpha, \beta, \varepsilon, \{\phi_m\}$ and $\{\theta_m\}$ satisfy the hypotheses of Theorem 1.1. If $f(z)$ is a meromorphic solution of (1.2), then f, f', f'' all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = \infty. \tag{1.17}$$

Theorem 1.3 *Suppose that $A, B, F,\ \alpha, \beta, \varepsilon, \{\phi_m\}$ and $\{\theta_m\}$ satisfy the hypotheses of Theorem B. Let $d_0(z), d_1(z), d_2(z)$ be meromorphic functions that are not all equal to zero with $\rho(d_j) < \infty\ (j = 0, 1, 2)$ such that $h \not\equiv 0$, and let $\varphi(z)$ be a meromorphic function with finite order such that $\psi(z)$ is not a solution of (1.2). If $f(z)$ is an infinite order meromorphic solution of (1.2), then the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies*

$$\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty. \tag{1.18}$$

Remark 1.1 In Theorem 1.1, Theorem 1.3, if we don't have the condition $h \not\equiv 0$, then the differential polynomial can be of finite order. For example if $d_2(z) \not\equiv 0$ is of finite order meromorphic function and $d_0(z) = Bd_2(z)$, $d_1(z) = Ad_2(z)$, then $g_f(z) = d_2(z) F$ is of finite order.

In the next, we investigate the relation between the solutions of a pair non-homogeneous linear differential equations and we obtain the following result :

Theorem 1.4 *Suppose that* $A, B, d_0(z), d_1(z), d_2(z), \alpha, \beta, \varepsilon, \{\phi_m\}$ *and* $\{\theta_m\}$ *satisfy the hypotheses of Theorem 1.1. Let* $F_1 \not\equiv 0$ *and* $F_2 \not\equiv 0$ *be meromorphic functions having only finitely many poles such that* $\max\{\rho(F_1), \rho(F_2)\} < \beta$, $F_1 - CF_2 \not\equiv C_1 B$ *for any constants* C, C_1, *and let* $\varphi(z)$ *be a meromorphic function with finite order. If* f_1 *is a meromorphic solution of equation*

$$f'' + Af' + Bf = F_1, \tag{1.19}$$

and f_2 *is a meromorphic solution of equation*

$$f'' + Af' + Bf = F_2, \tag{1.20}$$

then the differential polynomial $g_{f_1 - Cf_2}(z) = d_2\left(f_1'' - Cf_2''\right) + d_1\left(f_1' - Cf_2'\right) + d_0(f_1 - Cf_2)$ *satisfies*

$$\overline{\lambda}(g_{f_1-Cf_2} - \varphi) = \rho(g_{f_1-Cf_2}) = \infty \tag{1.21}$$

for any constant C.

2 Auxiliary Lemmas

Lemma 2.1 [5] *Let* $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions. If* f *is a meromorphic solution with* $\rho(f) = +\infty$ *of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F, \tag{2.1}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.2 *Suppose that* $A, B, F, \alpha, \beta, \varepsilon, \{\phi_m\}$ *and* $\{\theta_m\}$ *satisfy the hypotheses of Theorem A or Theorem B. Let* $d_0(z), d_1(z), d_2(z)$ *be meromorphic functions that are not all equal to zero with* $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ *such that* $h \not\equiv 0$, *where* h *is defined in* (1.13). *If* $f(z)$ *is an infinite order meromorphic solution of* (1.2), *then the differential polynomial* $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ *satisfies*

$$\rho(g_f) = \rho(f) = \infty. \tag{2.2}$$

Proof. Suppose that f is a meromorphic solution of equation (1.2) with $\rho(f) = \infty$. First we suppose that $d_2 \not\equiv 0$. Substituting $f'' = F - Af' - Bf$ into g_f, we get

$$g_f - d_2 F = (d_1 - d_2 A) f' + (d_0 - d_2 B) f. \tag{2.3}$$

Differentiating both sides of equation (2.3) and replacing f'' with $f'' = F - Af' - Bf$, we obtain

$$g_f' - (d_2 F)' - (d_1 - d_2 A) F = \left[d_2 A^2 - (d_2 A)' - d_1 A - d_2 B + d_0 + d_1'\right] f'$$

$$+ \left[d_2 AB - (d_2 B)' - d_1 B + d_0'\right] f. \tag{2.4}$$

Set

$$\alpha_1 = d_1 - d_2 A, \quad \alpha_0 = d_0 - d_2 B, \tag{2.5}$$

$$\beta_1 = d_2 A^2 - (d_2 A)' - d_1 A - d_2 B + d_0 + d_1', \tag{2.6}$$

$$\beta_0 = d_2 AB - (d_2 B)' - d_1 B + d_0'. \tag{2.7}$$

Then we have

$$\alpha_1 f' + \alpha_0 f = g_f - d_2 F, \tag{2.8}$$

$$\beta_1 f' + \beta_0 f = g_f' - (d_2 F)' - \alpha_1 F. \tag{2.9}$$

Set

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 = (d_1 - d_2 A)\left(d_2 AB - (d_2 B)' - d_1 B + d_0'\right)$$

$$- (d_0 - d_2 B)\left(d_2 A^2 - (d_2 A)' - d_1 A - d_2 B + d_0 + d_1'\right). \tag{2.10}$$

By $h \not\equiv 0$ and $(2.8) - (2.10)$, we obtain

$$f = \frac{\alpha_1 \left(g_f^{'} - (d_2F)^{'} - \alpha_1 F \right) - \beta_1 (g_f - d_2F)}{h} \tag{2.11}$$

If $\rho(g_f) < \infty$, then by (2.11) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\rho(g_f) = \infty$.

3 Proof of Theorem 1.1

Suppose that f is a meromorphic solution of equation (1.2). Then by Theorem A, we have $\rho(f) = \infty$. It follows by Lemma 2.2, $\rho(g_f) = \rho(f) = \infty$. Set $w(z) = d_2 f^{''} + d_1 f^{'} + d_0 f - \varphi$. Since $\rho(\varphi) < \infty$, then $\rho(w) = \rho(g_f) = \rho(f) = \infty$. In order to the prove $\overline{\lambda}(g_f - \varphi) = \infty$, we need to prove only $\overline{\lambda}(w) = \infty$. Using $g_f = w + \varphi$, we get from (2.11)

$$f = \frac{\alpha_1 w^{'} - \beta_1 w}{h} + \psi, \tag{3.1}$$

where ψ is defined in (1.14) with $\rho(\psi) < \infty$. Substituting (3.1) into equation (1.2), we obtain

$$\begin{aligned} &\frac{\alpha_1}{h} w^{'''} + \phi_2 w^{''} + \phi_1 w^{'} + \phi_0 w \\ &= F - \left(\psi^{''} + A(z) \psi^{'} + B(z) \psi \right) = W, \end{aligned} \tag{3.2}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\psi(z)$ is of a meromorphic function of finite order, then by Theorem A, it follows that $W \not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \infty$.

4 Proof of Theorem 1.2

Suppose that f is a meromorphic solution of equation (1.2). Then by Theorem A, we have $\rho(f) = \rho\left(f^{'}\right) = \rho\left(f^{''}\right) = \infty$. Since $\rho(\varphi) < \infty$, then

$\rho\left(f-\varphi\right)=\rho\left(f^{'}-\varphi\right)=\rho\left(f^{''}-\varphi\right)=\infty$. By using similar reasoning to that in the proof of Theorem 1.1, the proof of Theorem 1.2 can be completed.

5 Proof of Theorem 1.3

By hypothesis of Theorem 1.3, $\psi\left(z\right)$ is not a solution of equation $\left(1.2\right)$. Then

$$F-\left(\psi^{''}+A\left(z\right)\psi^{'}+B\left(z\right)\psi\right)\not\equiv 0.$$

By using a similar reasoning to that in the proof of Theorem 1.1, we can prove Theorem 1.3.

6 Proof of Theorem 1.4

Suppose that f_1 is a meromorphic solution of equation (1.19) and f_2 is a meromorphic solution of equation (1.20). Set $w=f_1-Cf_2$. Then w is a solution of equation $w^{''}+Aw^{'}+Bw=F_1-CF_0$. By $F_1-CF_2\not\equiv C_1B$, $\rho\left(F_1-CF_0\right)<\beta$ and Theorem A, we have $\rho\left(w\right)=\infty$. Thus, by Lemma 2.2, we have $\rho\left(g_{f_1-Cf_2}\right)=\rho\left(f_1-Cf_2\right)=\infty$. Let φ be a finite order meromorphic function. Then by Theorem 1.1, we get

$$\overline{\lambda}\left(g_{f_1-Cf_2}-\varphi\right)=\rho\left(g_{f_1-Cf_2}\right)=\infty.$$

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