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# Solutions to a class of nonlinear differential equations of fractional order

Nickolai Kosmatov

Department of Mathematics and Statistics

University of Arkansas at Little Rock

Little Rock, AR 72204-1099, USA

nxkosmatov@ualr.edu

**Abstract**

In this paper we investigate the formulation of a class of boundary value problems of fractional order with the Riemann-Liouville fractional derivative and integral-type boundary conditions. The existence of solutions is established by applying a fixed point theorem of Krasnosel'skiĭ and Zabreiko for asymptotically linear mappings.

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# 1 Preliminaries

In this section we introduce basic facts of fractional calculus involving the Riemann-Liouville fractional differential operator and list several recent and classical results dealing with initial and boundary value problems of fractional order. In the present section we also formulate the boundary value problems and establish the equivalence of those to certain integral equations. The existence of solutions of the integral equations is shown in Section 2 by applying a fixed point theorem of Krasnosel'skiĭ and Zabreiko [10].

The study of primarily initial value problems for differential equations of fractional order with various types of integro-differential operators such as Riemann-Liouville is extensive. It includes several well-known monographs [8, 13, 14, 15] and papers [2, 3, 4, 5, 6, 9, 11, 12]. Boundary value problems for fractional order have received less attention than initial value problems [1]. In this work we obtain several existence results for a Riemann-Liouville differential equation fractional order $1 < \alpha < 2$. We point out that if the solutions are sought in the class of continuous functions, one of the conditions loses its meaning as a boundary condition and should instead be interpreted as a well-posedness condition. In this regard the paper clears certain misconceptions about formulation of boundary value problems for differential equations of fractional order.

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^p(0, 1)$,

$1 \le p < \infty$, is the integral

$$\mathcal{I}_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n = [\alpha] + 1$, is defined by

$$\mathcal{D}_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

It is well-known [9, 13] that if the fractional derivative $\mathcal{D}_{0+}^{\alpha} u$, $n-1 < \alpha < n$, of a function $u$ is integrable, then

$$\mathcal{I}_{0+}^{\alpha} \mathcal{D}_{0+}^{\alpha} u(t) = u(t) - \sum_{k=1}^{n} \frac{\mathcal{D}_{0+}^{\alpha-k} u(0)}{\Gamma(\alpha-k+1)} t^{\alpha-k}. \tag{1}$$

For an integrable function $g$, it holds that

$$\mathcal{D}_{0+}^{\alpha} \mathcal{I}_{0+}^{\alpha} g(t) = g(t). \tag{2}$$

For $\beta < 0$, it is convenient to introduce the notation $\mathcal{I}_{0+}^{\beta} = \mathcal{D}_{0+}^{-\beta}$. If $\alpha > 0$ and $\beta + \alpha > 0$, then a composition rule (also called a semigroup property)

$$\mathcal{I}_{0+}^{\beta} \mathcal{I}_{0+}^{\alpha} g(t) = \mathcal{I}_{0+}^{\alpha+\beta} g(t)$$

holds (see, e.g., [9, 13]).

If $u \in C[0,1]$, then (1) becomes

$$u(t) = \mathcal{D}_{0+}^{\alpha-1} u(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \mathcal{I}_{0+}^{\alpha} \mathcal{D}_{0+}^{\alpha} u(t) \tag{3}$$

since $\mathcal{D}_{0+}^{\alpha-2} u(0) = \mathcal{I}_{0+}^{2-\alpha} u(0) = 0$.

We study the Riemann-Liouville integro-differential equation

$$\mathcal{D}_{0+}^{\alpha} u(t) = f(t, u(t)), \quad t \in (0,1), \tag{4}$$

of fractional order $1 < \alpha < 2$. We assume throughout the note that

$(H_1)$ $f \in C([0,1] \times \mathbf{R}, \mathbf{R})$.

We seek solutions of (4) in the class of continuous functions satisfying the boundary condition

$$\mathcal{I}_{0+}^{\beta} u(1) = 0, \tag{5}$$

where $\beta > -\alpha$. In addition, we assume that $\gamma > 1 - \alpha$ and impose the condition

$$\mathcal{I}_{0+}^{\gamma} u(0) = 0, \tag{6}$$

to which we refer as the well-posedness condition. The condition (6) plays the role of the second boundary condition.

By a solution of the boundary value problem (4)-(6) we understand a function $u \in C[0,1]$ with $\mathcal{D}_{0+}^{\alpha} u \in C[0,1]$ satisfying the equation (4) and the conditions (5) and (6).

Since $\alpha + \gamma > 1$, setting $g = \mathcal{D}_{0+}^{\alpha} u \in C[0,1]$, we have by the semigroup property (3) that

$$\mathcal{I}_{0+}^{\gamma} u(t) = \mathcal{D}_{0+}^{\alpha-1} u(0) \frac{t^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} + \mathcal{I}_{0+}^{\alpha+\gamma} g(t).$$

In particular, the right side of the above equation vanishes at $t = 0$ thus justifying the use of (6) and the applicability of the term "well-posedness" condition to the scope of this work.

Similarly,

$$\mathcal{I}_{0+}^{\beta} u(t) = \mathcal{D}_{0+}^{\alpha-1} u(0) \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \mathcal{I}_{0+}^{\alpha+\beta} g(t).$$

It follows from (5) that

$$\mathcal{D}_{0+}^{\alpha-1}u(0) = -\Gamma(\alpha+\beta)\mathcal{I}_{0+}^{\alpha+\beta}g(1)$$

and

$$u(t) = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\mathcal{I}_{0+}^{\alpha+\beta}g(1)t^{\alpha-1} + \mathcal{I}_{0+}^{\alpha}g(t).$$

Replacing $g$ with the inhomogeneous term of (4), we obtain that if $u \in C[0,1]$ is a solution of the problem (4)-(6), then $u \in C[0,1]$ is the solution of the integral equation

$$u(t) = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\mathcal{I}_{0+}^{\alpha+\beta}f(\cdot,u(\cdot))(1)t^{\alpha-1} + \mathcal{I}_{0+}^{\alpha}f(\cdot,u(\cdot))(t). \tag{7}$$

The converse is also true in view of (2).

Since the solvability of (4)-(6) is equivalent the existence of a solution of the integral equation (7), we will seek a fixed point of the integral mapping, for $t \in [0,1]$,

$$Tu(t) = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 (1-s)^{\alpha+\beta-1}f(s,u(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))\,ds. \tag{8}$$

Note that the parameter $\gamma$ is absent from the integral equation (7) simply because the shape of the mapping above is dictated by the class of admissible functions rather than the condition (6) at $t = 0$ (see, e. g., [1] with $u(0) = 0$ which is the case $\gamma = 0$ in this note).

Let $X = C[0,1]$ be endowed with the sup-norm denoted by $\|\cdot\|$. It is clear that $T : X \to X$ is a completely continuous mapping. The existence of a fixed point of $T$ will follow from the Krasnosel'skiĭ-Zabreiko fixed point theorem [10]:

**Theorem 1.1** *Let $X$ be a Banach space. Assume that $T : X \to X$ is completely continuous mapping and $L : X \to X$ is a bounded linear mapping such that 1 is not an eigenvalue of $L$ and*

$$\lim_{\|u\|\to\infty} \frac{\|Tu - Lu\|}{\|u\|} = 0. \tag{9}$$

*Then $T$ has a fixed point in $X$.*

For applications of Theorem 1.1 we refer the reader to [7] and the references therein. Although the inhomogeneous term in the above mentioned work was considered in the form $\phi(t)g(u(t))$, we observe that there is no need in separating the variables so that a slightly more general scenario $f(t, u(t))$ is considered in the present note.

## 2 The existence result

The first existence result is obtained for the problem (4)-(6).

In addition to the hypothesis $(H_1)$ we impose the following conditions:

$(H_2)$ $\displaystyle\lim_{|u|\to\infty} \frac{f(t,u)}{u} = \phi(t)$ uniformly in $[0, 1]$;

$(H_3)$ $f(t, 0)$ does not vanish identically in $[0, 1]$.

**Theorem 2.1** *Let the assumptions $(H_1)$-$(H_3)$ be satisfied. Assume that*

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha+\beta-1} |\phi(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |\phi(s)|\, ds < 1. \tag{10}$$

*Then the boundary value problem (4)-(6) has a nontrivial solution.*

*Proof.* Define a bounded linear mapping, $L : X \to X$, by

$$Lu(t) = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 (1-s)^{\alpha+\beta-1}\phi(s)u(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)u(s)\,ds,$$

for $t \in [0,1]$. Suppose that $\lambda = 1$ is an eigenvalue of $L$. Since (10) holds,

$$\begin{aligned}\|Lu\| &\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+\beta-1}|\phi(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|\phi(s)|\,ds\right)\|u\| \\ &< \|u\|,\end{aligned}$$

together with $Lu = u$ with $u \neq 0$ lead to a contradiction. Thus $\lambda = 1$ is not an eigenvalue of $L$.

Let $\epsilon > 0$. There exists an $A > 0$ such that

$$\left|\frac{f(t,z)}{z} - \phi(s)\right| < \epsilon \tag{11}$$

for all $t \in [0,1]$ provided $|z| > A$. Set

$$B = \sup\{|f(t,z)| : t \in [0,1], |z| \in [0,A|\}.$$

If $|u(t)| \leq A$, then $|f(t,u(t)) - \phi(t)u(t)| \leq B + \|\phi\|A$. Choose $M > A$ so that $B + \|\phi\|A \leq M\epsilon$. Then, for $u \in X$ with $\|u\| > M$, either $|u(s)| \leq A$, in which case $|f(t,u(t)) - \phi(t)u(t)| < \epsilon\|u\|$, or $|u(s)| > A$, in which case, by (11), we also have $|f(t,u(t)) - \phi(t)u(t)| < \epsilon\|u\|$.

Now, if $\|u\| > M$, then

$$\begin{aligned}\|Tu - Lu\| &= \sup_{t\in[0,1]} |Tu(t) - Lu(t)| \\ &= \sup_{t\in[0,1]}\left| -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 (1-s)^{\alpha+\beta-1}(f(s,u(s)) - \phi(s)u(s))\,ds\right.\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s,u(s)) - \phi(s)u(s))\, ds \Bigg| \\
&\leq \quad \left( \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha+\beta-1}\, ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\, ds \right) \epsilon \|u\| \\
&\leq \quad \left( \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right) \epsilon \|u\|.
\end{aligned}
$$

We see that the condition (9) is verified. Thus, by Theorem 1.1, $T$ has a fixed point in $X$ which is a solution of the boundary value problem (4)-(6). It follows from $(H_3)$ that the boundary value problem does not possess the trivial solution. □

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