Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **17**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

$\Psi-$ BOUNDED SOLUTIONS FOR A LYAPUNOV MATRIX DIFFERENTIAL EQUATION

Aurel Diamandescu
University of Craiova

Abstract

It is proved a necessary and sufficient condition for the existence of at least one $\Psi-$ bounded solution of a linear nonhomogeneous Lyapunov matrix differential equation.

Key Words: $\Psi-$ bounded solution, Lyapunov matrix differential equation.
2000 Mathematics Subject Classification: 34C11, 34D05.

1. Introduction.

This work is concerned with linear nonhomogeneous Lyapunov matrix differential equation

$$\frac{dZ}{dt} = A(t)Z + ZB(t) + F(t) \tag{1}$$

where A, B and F are continuous $n\times n$ matrix-valued function on $\mathbb{R}_+ = [0,\infty)$.

The basic problem under consideration is the determination of necessary and sufficient conditions for the existence of a solution with some specified boundedness condition.

The problem of $\Psi-$ boundedness of the solutions for systems of ordinary differential equations was studied in many papers, as e.q. [1], [3], [5], [6]. In these papers, the function Ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\Psi(t) \geq 1$ on $\mathbb{R}_+$ in [3]).

Recently, in [7] was studied the existence of $\Psi-$ bounded solutions for the corresponding Kronecker product system (6) associated with (1) (i.e. a linear nonhomogeneous differential system of the form $\frac{dx}{dt} = G(t)x + f(t)$) in the hypothesis that the free term f of the system is a Lebesgue $\Psi-$ integrable function defined on $\mathbb{R}_+$. But the obtained results in [7] are particular cases of our general results stated in [4]. Indeed, if in Theorems 2.1 and 2.2 ([4]), the fundamental matrix Y is replaced with the fundamental matrix $Z\otimes Y$ of the linear system (3) of [7], the Theorems 1 and 2 ([7]) follow. In addition, in Theorems 1 and 2 ([7]) there are a few mistakes in connection with the matrix Ψ.

The purpose of present paper is to give a necessary and sufficient conditions so that the linear nonhomogeneous Lyapunov matrix differential equation (1) have at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrable matrix-valued function F on $\mathbb{R}_+$.

Here, as in [4], Ψ will be a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

The results of this paper include our results [4] as a particular case, when $B = O_n$.

2. Preliminaries.

In this section we present some basic definitions, notations, hypotheses and results which are useful later on.

Let $\mathbb{R}^n$ be the Euclidean n - space. For $x = (x_1, x_2, x_3, ..., x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, ..., |x_n|\}$ be the norm of x (^{T} denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all m×n real valued matrices.

For a n×n real matrix $A = (a_{ij})$, we define the norm $|A|$ by $|A| = \sup\limits_{\|x\|\leq 1} \|Ax\|$.

It is well-known that $|A| = \max\limits_{1\leq i\leq n}\{\sum\limits_{j=1}^{n} |a_{ij}|\}$.

Definition 1. ([2]) Let $A = (a_{ij}) \in \mathbb{M}_{m\times n}$ and $B = (b_{ij}) \in \mathbb{M}_{p\times q}$. The Kronecker product of A and B written $A\otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Obviously, $A\otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 1. The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:

1). $A\otimes(B\otimes C) = (A\otimes B)\otimes C$;
2). $(A\otimes B)^T = A^T\otimes B^T$;
3). $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$;
4). $(A\otimes B)\cdot(C\otimes D) = AC\otimes BD$;
5). $A\otimes(B + C) = A\otimes B + A\otimes C$;
6). $(A + B)\otimes C = A\otimes C + B\otimes C$;
7). $I_p\otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8). $\frac{d}{dt}(A(t)\otimes B(t)) = \frac{dA(t)}{dt}\otimes B(t) + A(t)\otimes\frac{dB(t)}{dt}$.
(here, $\frac{dX}{dt}$ denotes the derivative of X with respect to t).

Proof. See in [2].

Definition 2. The application $\mathcal{V}\mathrm{ec} : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by

$$\mathcal{V}\mathrm{ec}(A) = (a_{11},a_{21},\cdots,a_{m1},a_{12},a_{22},\cdots,a_{m2},\cdots,a_{1n},a_{2n},\cdots,a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2. The vectorization operator $\mathcal{V}\mathrm{ec} : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, $\mathcal{V}\mathrm{ec}$ and $\mathcal{V}\mathrm{ec}^{-1}$ are continuous operators.

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for $A = (a_{ij}) \in \mathbb{M}_{n\times n}$, we have

$$\| \mathcal{V}\mathrm{ec}(A) \| = \max_{1\le i,j\le n} \{| a_{ij}|\} \le \max_{1\le i\le n}\{\sum_{j=1}^{n} | a_{ij}|\} = | A | .$$

Thus, the vectorization operator is continuous and $\| \mathcal{V}\mathrm{ec}\| \le 1$.

In addition, for $A = I_n$ (identity $n\times n$ matrix) we have $\| \mathcal{V}\mathrm{ec}(I_n)\| = 1 = |I_n|$ and then, $\| \mathcal{V}\mathrm{ec}\| = 1$.

Obviously, the inverse of the vectorization operator, $\mathcal{V}\mathrm{ec}^{-1} : \mathbb{R}^{n^2} \longrightarrow \mathbb{M}_{n\times n}$, is defined by

$$\mathcal{V}\mathrm{ec}^{-1}(u) = \begin{pmatrix} u_1 & u_{n+1} & \cdots & u_{n^2-n+1} \\ u_2 & u_{n+2} & \cdots & u_{n^2-n+2} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & u_{2n} & \cdots & u_{n^2} \end{pmatrix},$$

where $u = (u_1,u_2,\cdots,u_{n^2})^T \in \mathbb{R}^{n^2}$.

We have

$$| \mathcal{V}\mathrm{ec}^{-1}u | = \max_{1\le i\le n}\{\sum_{j=0}^{n-1} | u_{nj+i} |\} \le n\cdot \max_{1\le i\le n^2} \{| u_i |\} = n\cdot \| u \| .$$

Thus, $\mathcal{V}\mathrm{ec}^{-1}$ is a continuous operator.

Remark. Obviously, if F is a continuous matrix function on $\mathbb{R}_+$, then $f = \mathcal{V}\mathrm{ec}(F)$ is a continuous vector function on $\mathbb{R}_+$ and reciprocally.

The next Lemma gives some basic properties of vectorization operator $\mathcal{V}\mathrm{ec}$.

Lemma 3. If A, B, $M \in \mathbb{M}_{n\times n}$, then
1). $\mathcal{V}\mathrm{ec}(AMB) = (B^T\otimes A)\cdot\mathcal{V}\mathrm{ec}(M)$;
2). $\mathcal{V}\mathrm{ec}(MB) = (B^T\otimes I_n)\cdot\mathcal{V}\mathrm{ec}(M)$;
3). $\mathcal{V}\mathrm{ec}(AM) = (I_n\otimes A)\cdot\mathcal{V}\mathrm{ec}(M)$;
4). $\mathcal{V}\mathrm{ec}(AM) = (M^T\otimes A)\cdot\mathcal{V}\mathrm{ec}(I_n)$.
Proof. It is a simple exercise.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0,\infty)$, $i = 1,2,...,n$, be continuous functions and

$$\Psi = \mathrm{diag}\,[\Psi_1, \Psi_2, \cdots \Psi_n].$$

Definition 3. ([4]). A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ is said to be $\Psi-$ bounded on $\mathbb{R}_+$ if Ψf is bounded on $\mathbb{R}_+$ (i.e. $\sup_{t\geq 0} \| \Psi(t)f(t)\| < +\infty$).

Extend this definition for matrix functions.

Definition 4. A matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is said to be $\Psi-$ bounded on $\mathbb{R}_+$ if the matrix function ΨM is bounded on $\mathbb{R}_+$ (i.e. $\sup_{t\geq 0} | \Psi(t)M(t)| < +\infty$).

Definition 5. ([4]). A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ is said to be Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$ if f is measurable and Ψf is Lebesgue integrable on $\mathbb{R}_+$ (i.e. $\int_0^\infty \| \Psi(t)f(t)\| dt < \infty$).

Extend this definition for matrix functions.

Definition 6. A matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is said to be Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$ if M is measurable and ΨM is Lebesgue integrable on $\mathbb{R}_+$ (i.e. $\int_0^\infty | \Psi(t)M(t)| dt < \infty$).

Now, we shall assume that A and B are continuous $n\times n$ - matrices on $\mathbb{R}_+$ and F is a Lebesgue $\Psi-$ integrable matrix function on $\mathbb{R}_+$.

By a solution of (1), we mean an absolutely continuous matrix function $Z(t)$ satisfying the equation (1) for almost all $t \geq 0$.

The following lemmas play a vital role in the proof of main result.

Lemma 4. The matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$ if and only if the vector function $\mathcal{V}ecM(t)$ is Lebesgue $I_n \otimes \Psi-$ integrable on $\mathbb{R}_+$.

Proof. From the proof of Lemma 2, it results that

$$\frac{1}{n} | A | \quad \leq \quad \| \mathcal{V}ecA \|_{\mathbb{R}^{n^2}} \quad \leq \quad | A |,$$

for every $A \in \mathbb{M}_{n\times n}$.

Setting $A = \Psi(t)M(t)$, $t \geq 0$ and using Lemma 3, we have the inequality

$$\frac{1}{n} | \Psi(t)M(t) | \quad \leq \quad \| (I_n \otimes \Psi(t)) \cdot \mathcal{V}ecM(t) \|_{\mathbb{R}^{n^2}} \quad \leq \quad | \Psi(t)M(t) | \,, \ t \ \geq \ 0 \qquad (2)$$

for all matrix function $M(t)$.

Now, the Lemma follows immediately.

Lemma 5. The matrix function $M(t)$ is $\Psi-$ bounded on $\mathbb{R}_+$ if and only if the vector function $\mathcal{V}ecM(t)$ is $I_n \otimes \Psi$ $-$ bounded on $\mathbb{R}_+$.

Proof. It follows from the above inequality (2).

The next Lemma is Lemma 1 of [7]. Because the proof is incomplete, we present it with a complete proof.

Lemma 6. Let $X(t)$ and $Y(t)$ be the fundamental matrices for the equations

$$\frac{dX}{dt} = A(t)X \tag{3}$$

and

$$\frac{dY}{dt} = YB(t) \tag{4}$$

respectively.

Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the equation

$$\frac{dZ}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)Z. \tag{5}$$

If, in addition, $X(0) = I_n$ and $Y(0) = I_n$, then $Z(0) = I_{n^2}$.

Proof. Using the above properties of the Kronecker product, one obtains

$$\frac{dZ(t)}{dt} = \frac{d}{dt}(Y^T(t) \otimes X(t)) =$$

$$= \frac{dY^T(t)}{dt} \otimes X(t) + Y^T(t) \otimes \frac{dX(t)}{dt} =$$

$$= (\frac{dY(t)}{dt})^T \otimes X(t) + Y^T(t) \otimes \frac{dX(t)}{dt} =$$

$$= (Y(t)B(t))^T \otimes X(t) + Y^T(t) \otimes A(t)X(t) =$$

$$= B^T(t)Y^T(t) \otimes I_n X(t) + I_n Y^T(t) \otimes A(t)X(t) =$$

$$= (B^T(t) \otimes I_n) \cdot (Y^T(t) \otimes X(t)) + (I_n \otimes A(t)) \cdot (Y^T(t) \otimes X(t)) =$$

$$= (I_n \otimes A(t) + B^T(t) \otimes I_n)Z(t),$$

for all $t \in \mathbb{R}_+$.

On the other hand, the matrix $Z(t)$ is a nonsingular matrix for all $t \geq 0$ (because $X(t)$ and $Y(t)$ are nonsingular matrices for all $t \geq 0$).

Thus, the matrix Z is a fundamental matrix on $\mathbb{R}_+$ for the equation (5).

Obviously, $Z(0) = Y(0) \otimes X(0) = I_n \otimes I_n = I_{n^2}$.

The proof is now complete.

Lemma 7. The matrix function $Z(t)$ is a solution of (1) if and only if the vector valued function $z(t) = \mathcal{V}ecZ(t)$ is a solution of the differential system

$$\frac{dz}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z + f(t), \tag{6}$$

where $f(t) = \mathcal{V}ecF(t)$.

Proof. Using Kronecker product notation, the vectorization operator $\mathcal{V}\mathrm{ec}$ and the above properties, we can rewrite the equality

$$\frac{dZ(t)}{dt} = A(t)Z(t) + Z(t)B(t) + F(t), \text{ for almost all } t \geq 0,$$

in the equivalent form

$$\mathcal{V}\mathrm{ec}\frac{dZ(t)}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)\mathcal{V}\mathrm{ec}Z(t) + \mathcal{V}\mathrm{ec}F(t), \text{ for almost all } t \geq 0.$$

If we denote $\mathcal{V}\mathrm{ec}Z(t) = z(t)$, $\mathcal{V}\mathrm{ec}F(t) = f(t)$, we have $\frac{dz(t)}{dt} = \mathcal{V}\mathrm{ec}\frac{dZ(t)}{dt}$ and then, the above equality becomes

$$\frac{dz(t)}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z(t) + f(t), \text{ for almost all } t \geq 0.$$

The proof is now complete.

Definition 7. The above system (6) is called "corresponding Kronecker product system associated with (1)"

Now, let $Z(t)$ the above fundamental matrix for the system (5) with $Z(0) = I_{n^2}$.

Let $\widetilde{X}_1$ denote the subspace of $\mathbb{R}^{n^2}$ consisting of all vectors which are values of $I_n \otimes \Psi-$ bounded solutions of (5) on $\mathbb{R}_+$ for $t = 0$ and let $\widetilde{X}_2$ an arbitrary fixed subspace of $\mathbb{R}^{n^2}$, supplementary to $\widetilde{X}_1$. Let $\widetilde{P}_1$, $\widetilde{P}_2$ denote the corresponding projections of $\mathbb{R}^{n^2}$ onto $\widetilde{X}_1$, $\widetilde{X}_2$ respectively.

Finally, we remaind of two theorems which be useful in the proofs of our main results.

Theorem (2.1). ([4]). Let A be a continuous $n\times n$ real matrix function on $\mathbb{R}_+$ and let Y the fundamental matrix of the homogeneous linear system $\frac{dx}{dt} = A(t)x$ for which $Y(0) = I_n$.

Then, the system $\frac{dx}{dt} = A(t)x + f(t)$ has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrable function f on $\mathbb{R}_+$ if and only if there is a positive constant K such that

$$\begin{cases} \mid \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \mid & \leq K, \quad \text{for } 0 \leq s \leq t \\ \mid \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \mid & \leq K, \quad \text{for } 0 \leq t \leq s \end{cases} \tag{7}$$

Theorem (2.2). ([4]). Suppose that:

1°. The fundamental matrix $Y(t)$ of the system $\frac{dx}{dt} = A(t)x$ satisfies:

a). the above condition (7), where K is a positive constant;

b). the condition $\lim_{t\to\infty} \Psi(t)Y(t)P_1 = 0$;

2°. The function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ is Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$.

Then, every $\Psi-$ bounded solution x of the system $\frac{dx}{dt} = A(t)x + f(t)$ is such that

$$\lim_{t\to+\infty} \| \Psi(t)x(t) \| = 0.$$

3. The main result.

The main result of this paper is the following:

Theorem 1. Let A and B be continuous n×n real matrix functions on $\mathbb{R}_+$ and let X and Y be the fundamental matrices of the homogeneous linear equations (3) and (4) respectively for which $X(0) = Y(0) = I_n$.

Then, the equation (1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrable matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ if and only if there is a positive constant K such that

$$\begin{array}{lll} |\,(Y^T(t) \otimes (\Psi(t)X(t)))\widetilde{P}_1((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s)))\,| & \leq K, & 0 \leq s \leq t \\ |\,(Y^T(t) \otimes (\Psi(t)X(t)))\widetilde{P}_2((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s)))\,| & \leq K, & 0 \leq t \leq s \end{array} \tag{8}$$

Proof. First, we prove the "only if" part.

Suppose that the equation (1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrable matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$.

Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ a Lebesgue $I_n \otimes \Psi-$ integrable function on $\mathbb{R}_+$. From Lemma 4, it follows that the matrix function $F(t) = \mathcal{V}ec^{-1}f(t)$ is Lebesgue $\Psi-$ integrable matrix function on $\mathbb{R}_+$. From the hypothesis, the equation

$$\frac{dZ}{dt} = A(t)Z + ZB(t) + \mathcal{V}ec^{-1}f(t)$$

has at least one $\Psi-$ bounded solution Z(t) on $\mathbb{R}_+$.

From Lemma 7 and Lemma 5, it follows that the vector valued function $z(t) = \mathcal{V}ecZ(t)$ is a $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of the differential system

$$\frac{dz}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z + f(t).$$

Thus, this system has at least one $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $I_n \otimes \Psi-$ integrable function f on $\mathbb{R}_+$.

From the Theorem 2.1 ([4]), there is a positive constant K such that the fundamental matrix Z(t) of the equation (5) satisfies the condition

$$\begin{array}{lll} |\,(I_n \otimes \Psi(t))Z(t)\widetilde{P}_1Z^{-1}(s)(I_n \otimes \Psi(s))^{-1}\,| & \leq K, & 0 \leq s \leq t \\ |\,(I_n \otimes \Psi(t))Z(t)\widetilde{P}_2Z^{-1}(s)(I_n \otimes \Psi(s))^{-1}\,| & \leq K, & 0 \leq t \leq s \end{array}.$$

We replace $Z(t) = Y^T(t)\otimes X(t)$. After computation, it follows that (8) holds.

Now, we prove the "if" part.

Suppose that (8) holds for some $K > 0$.

Let $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ a Lebesgue $\Psi-$ integrable matrix function on $\mathbb{R}_+$. From Lemma 4, it follows that the vector valued function $f(t) = \mathcal{V}ecF(t)$ is a Lebesgue $I_n \otimes \Psi-$ integrable function on $\mathbb{R}_+$.

From Theorem 2.1 ([4]), it follows that the differential system

$$\frac{dz}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z + f(t)$$

has at least one $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$. Let $z(t)$ be this solution.

From Lemma 5 and Lemma 7, it follows that the matrix function $Z(t) = \mathcal{V}ec^{-1}z(t)$ is a $\Psi-$ bounded solution of the equation (1) on $\mathbb{R}_+$ (because $F(t) = \mathcal{V}ec^{-1}f(t)$).

Thus, the differential equation (1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrable matrix function F on $\mathbb{R}_+$.

The proof is now complete.

Remark. 1. The Theorem generalizes the Theorem 2.1, [4].

Indeed, if X and Y are fundamental matrices for the equations (3) and (4) respectively, then, the unique solution Z of the nonhomogeneous equation (1) which takes the value Z_0 for $t = t_0$ is given by

$$Z(t) = X(t)X^{-1}(t_0)Z_0Y^{-1}(t_0)Y(t) + \int_{t_0}^{t} X(t)X^{-1}(s)F(s)Y^{-1}(s)Y(t)ds,\ t \in \mathbb{R}_+.$$

(The proof is similar to the proof of well-known Variation of constants formula for the linear differential system $\frac{dx}{dt} = A(t)x + f(t)$).

In particular case $B(t) = O_n$, we have $Y = I_n$ and then, the solution Z of the nonhomogeneous equation $\frac{dZ}{dt} = A(t)Z + F(t)$ which takes the value Z_0 for $t = t_0$ is given by

$$Z(t) = X(t)X^{-1}(t_0)Z_0 + \int_{t_0}^{t} X(t)X^{-1}(s)F(s)ds,\ t \in \mathbb{R}_+.$$

If, in addition

$$F(t) = \begin{pmatrix} f_1(t) & f_1(t) & \cdots & f_1(t) \\ f_2(t) & f_2(t) & \cdots & f_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(t) & f_n(t) & \cdots & f_n(t) \end{pmatrix} \text{ and } Z_0 = \begin{pmatrix} z_1^0 & z_1^0 & \cdots & z_1^0 \\ z_2^0 & z_2^0 & \cdots & z_2^0 \\ \vdots & \vdots & \vdots & \vdots \\ z_n^0 & z_n^0 & \cdots & z_n^0 \end{pmatrix},$$

it is easy to see that the solution of the equation $\frac{dZ}{dt} = A(t)Z + F(t)$ is

$$Z(t) = \begin{pmatrix} x_1(t) & x_1(t) & \cdots & x_1(t) \\ x_2(t) & x_2(t) & \cdots & x_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n(t) & x_n(t) & \cdots & x_n(t) \end{pmatrix},$$

where $x = (x_1(t),x_2(t),\cdots,x_n(t))^T$ is the solution of the problem

$$\frac{dx}{dt} = A(t)x + f(t),\ x(t_0) = (z_1^0,z_2^0,\cdots,z_n^0)^T,$$

with $f(t) = (f_1(t),f_2(t),\cdots,f_n(t))^T$.

In this case, the condition (8) becomes the condition (7), because the solution $x(t)$ is $\Psi-$ bounded on $\mathbb{R}_+$ if and only if the solution $Z(t)$ is $\Psi-$ bounded on $\mathbb{R}_+$.

Thus, the Theorem generalizes the result from [4].

2. The particular case $A = O_n$ come to the above case.

3). Other particular cases: $A = I_n$ or $B = I_n$ or $A = B$ or $A = B = O_n$, etc.

In the end, we prove a theorem which shows that the asymptotic behavior of solutions of (1) is determined completely by the asymptotic behavior of the fundamental matrices X and Y of (3) and (4) respectively as $t \longrightarrow \infty$.

Theorem 2. Suppose that:

1). The fundamental matrices $X(t)$ and $Y(t)$ of (3) and (4) respectively ($X(0) = Y(0) = I_n$) satisfy:

a). the condition (8) for some $K > 0$;

b). the condition

$$\lim_{t\to\infty} (Y^T(t) \otimes (\Psi(t)X(t)))\widetilde{P}_1 = 0;$$

2). The matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$.

Then, every $\Psi-$ bounded solution $Z(t)$ of (1) satisfies the condition

$$\lim_{t\to\infty} \mid \Psi(t)Z(t) \mid\ = 0.$$

Proof. Let $Z(t)$ be a $\Psi-$ bounded solution of (1). From Lemma 7 and Lemma 5, it follows that the function $z(t) = \mathcal{V}ecZ(t)$ is a $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of the differential system

$$\frac{dz}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z + f(t),$$

where $f(t) = \mathcal{V}ecF(t)$.

Also, from Lemma 4, the function $f(t)$ is Lebesgue $I_n \otimes \Psi-$ integrable on $\mathbb{R}_+$.

From the Theorem 2.2 ([4]), it follows that

$$\lim_{t\to\infty} \parallel (I_n \otimes \Psi(t)) \cdot z(t) \parallel_{\mathbb{R}^{n^2}}\ = 0.$$

Now, from the inequality (2) we have

$$\mid \Psi(t)Z(t) \mid\ \leq\ n \parallel (I_n \otimes \Psi(t)) \cdot z(t) \parallel_{\mathbb{R}^{n^2}},\ t \geq 0$$

and then

$$\lim_{t\to\infty} \mid \Psi(t)Z(t) \mid\ = 0.$$

The proof is now complete.

Remark. The Theorem generalizes the Theorem 2.2, [4].

Note that Theorem 2 is no longer true if we require that the matrix function F is $\Psi-$ bounded on $\mathbb{R}_+$, instead of the condition 2) of the Theorem. Even if the matrix function F is such that $\lim_{t\to\infty} |\,\Psi(t)F(t)| = 0$.

This is shown by the next

Example. Consider the equation (1) with

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix},\ B(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix},$$

where $a \in \mathbb{R}$, $a < 0$.

The fundamental matrices for the equations (3) and (4) are

$$X(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{at} \end{pmatrix},\ Y(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

respectively.

Consider

$$\Psi(t) = \begin{pmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{e^{-at}}{t+1} \end{pmatrix},\ t \geq 0.$$

It is easy to see that the conditions of the Theorem 2 are satisfied with

$$\widetilde{P}_1 = I_4,\ \widetilde{P}_2 = O_4 \text{ and } K = 2.$$

In addition, we have that the function matrix F is $\Psi-$ bounded on $\mathbb{R}_+$, but it is not Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$.

On the other hand, the solutions of the equation (1) are

$$Z(t) = \begin{pmatrix} (c_1 - 1)\cos t - c_2 \sin t + 1 & (c_1 - 1)\sin t + c_2 \cos t + t \\ e^{at}(c_3 \cos t - c_4 \sin t) & e^{at}(c_3 \sin t + c_4 \cos t) \end{pmatrix},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

It is easy to see that each solution Z is $\Psi-$ bounded on $\mathbb{R}_+$, but

$$\lim_{t\to\infty} |\,\Psi(t)Z(t)\,| = 1.$$

Note that the asymptotic properties of the components of the solutions are not the same. On the other hand, we see that the asymptotic properties of the components of the solutions are the same, via matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

Acknowledgement. The author would like to thank the anonymous referee for careful reading, valuable comments and suggestions.

References

[1] Akinyele, O. $-$ *On partial stability and boundedness of degree k,* Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), 65(1978), 259 - 264.

[2] Bellman, R. $-$ *Introduction to Matrix Analysis,* McGraw-Hill Book Company, Inc. New York 1960 (translated in Romanian).

[3] Constantin, A. – *Asymptotic Properties of Solutions of Differential Equations,* Analele Universităţii din Timişoara, Vol. XXX, fasc. 2-3, 1992, Seria Ştiinţe Matematice, 183 - 225.

[4] Diamandescu, A. – *Existence of* Ψ *- bounded solutions for a system of differential equations,* Electronic Journal of Differential Equations, Vol. 2004(2004), No. 63, pp. 1-6. URL: http://ejde.math.txstate.edu

[5] Hallam, T. G. – *On asymptotic equivalence of the bounded solutions of two systems of differential equations,* Mich. math. Journal, Vol. 16(1969), 353-363.

[6] Morchalo, J. $-$ *On* $(\Psi - L_p) -$ *stability of nonlinear systems of differential equations,* Analele Universităţii "Al. I. Cuza", Iaşi, XXXVI, I, Matematică, (1990), 4, 353-360.

[7] Murty M.S.N. and Suresh Kumar G. $-$ *On* $\Psi -$ *boundedness and* $\Psi -$ *stability of matrix Lyapunov systems,* J. Appl. Math. Comput(2008) 26: 67 - 84.

(Received November 12, 2008)

Aurel Diamandescu
University of Craiova,
Department of Applied Mathematics,
13, "Al. I. Cuza" st., 200585 Craiova, Romania.
E - mail address: adiamandescu@central.ucv.ro