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ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF SECOND ORDER NEUTRAL QUANTUM EQUATIONS WITH MAXIMA

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Abstract. In this study, the behavior of solutions to certain second order quantum (q-difference) equations with maxima are considered. In particular, the asymptotic behavior of non-oscillatory solutions is described, and sufficient conditions for oscillation of all solutions are obtained.

1. introduction

Quantum calculus has been utilized since at least the time of Pierre de Fermat [8, Chapter B.5] to augment mathematical understanding gained from the more traditional continuous calculus and other branches of the discipline; see Kac and Cheung [4], for example. In this study we will analyze a second order neutral quantum (q-difference) equation

$$D_q^2\Big(x(t)+p(t)x(q^{-k}t)\Big)+r(t)\max_{s\in\{0,\cdots,\ell\}}x(q^{-s}t)=0, \tag{1.1}$$

where the real scalar $q>1$ and the q-derivatives are given, respectively, by the difference quotient

$$D_qy(t)=\frac{y(qt)-y(t)}{qt-t}, \quad \text{and} \quad D_q^2y(t)=D_q\left(D_qy(t)\right).$$

Equation (1.1) is a quantum version of

$$\Delta^2\Big(x_n+p_nx_{n-k}\Big)+q_n\max_{\{n-\ell,\cdots,\ell\}}x_s=0, \tag{1.2}$$

studied by Luo and Bainov [5]; there the usual forward difference operator $\Delta y_n:=y_{n+1}-y_n$ was used. For more results on differential and difference equations related to (1.1) and (1.2), please see the work by Bainov, Petrov, and Proytcheva [1, 2, 3], Luo and Bainov [5], Luo and Petrov [6], and Petrov [7]. The particular appeal of (1.1) is that it is still a discrete problem, but with non-constant step size between domain points.

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2. PRELIMINARY RESULTS

For $q > 1$, define the quantum half line by

$$(0,\infty)_q := \{\cdots, q^{-2}, q^{-1}, 1, q, q^2, \cdots\}.$$

Let k, ℓ be non-negative integers, $r : (0,\infty)_q \to [0,\infty)$, $p : (0,\infty)_q \to \mathbb{R}$, and consider the second order neutral quantum (q-difference) equation

$$D_q^2\Big(x(t) + p(t)x(q^{-k}t)\Big) + r(t) \max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}t\right) = 0, \tag{2.1}$$

where we assume

$$\sum_{\eta\in[t_0,\infty)_q} \eta r(\eta) = \infty, \quad t_0 \in (0,\infty)_q. \tag{2.2}$$

Definition 2.1. *A function $f : (0,\infty)_q \to \mathbb{R}$ eventually enjoys property $\mathcal{P}$ if and only if there exists $t^* \in (0,\infty)_q$ such that for $t \in [t^*,\infty)_q$ the function f enjoys property $\mathcal{P}$. A solution x of* (2.1) *is non-oscillatory if and only if $x(t) < 0$ or $x(t) > 0$ eventually; otherwise x is oscillatory.*

Define the function $z : (0,\infty)_q \to \mathbb{R}$ via

$$z(t) := x(t) + p(t)x(q^{-k}t). \tag{2.3}$$

Then from (2.1) we have that

$$D_q^2 z(t) = -r(t) \max_{s\in\{0,\cdots,\ell\}} x(q^{-s}t), \tag{2.4}$$

and

$$D_q z(t) = D_q z(t_0) - (q-1) \sum_{\eta\in[t_0,t)_q} \eta r(\eta) \max_{s\in\{0,\cdots,\ell\}} x(q^{-s}\eta). \tag{2.5}$$

We will use these expressions involving z in the following lemmas.

Lemma 2.2. *Assume x is a solution of* (2.1)*, r satisfies* (2.2)*, z is given by* (2.3)*, and*

$$p \le p(t) \le P < -1 \quad \textit{for all} \quad t \in [t_0,\infty)_q. \tag{2.6}$$

(a) *If $x(t) > 0$ eventually, then either*

$$z(t) < 0, \quad D_q z(t) < 0, \quad \textit{and} \quad D_q^2 z(t) \le 0 \quad \textit{eventually and} \tag{2.7}$$

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} D_q z(t) = -\infty, \tag{2.8}$$

or

$$z(t) < 0, \quad D_q z(t) > 0, \quad \textit{and} \quad D_q^2 z(t) \le 0 \quad \textit{eventually and} \tag{2.9}$$

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} D_q z(t) = 0. \tag{2.10}$$

(*b*) *If* $x(t) < 0$ *eventually, then either*

$$z(t) > 0, \quad D_q z(t) > 0, \quad \textit{and} \quad D_q^2 z(t) \geq 0 \quad \textit{eventually and} \tag{2.11}$$

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} D_q z(t) = \infty, \tag{2.12}$$

or

$$z(t) > 0, \quad D_q z(t) < 0, \quad \textit{and} \quad D_q^2 z(t) \geq 0 \quad \textit{eventually and} \tag{2.13}$$

(2.10) *holds.*

Proof. We will prove (a); the proof of (b) is similar and thus omitted. Since $x(t) > 0$ eventually and $r(t) \geq 0$, it follows from (2.4) that $D_q^2 z(t) \leq 0$ eventually and $D_q z$ is an eventually nonincreasing function. Then either there exists an $L := \lim_{t\to\infty} D_q z(t) \in \mathbb{R}$, or $\lim_{t\to\infty} D_q z(t) = -\infty$. If $\lim_{t\to\infty} D_q z(t) = -\infty$, then $\lim_{t\to\infty} z(t) = -\infty$ and (2.7) and (2.8) hold. So, let $L := \lim_{t\to\infty} D_q z(t) \in \mathbb{R}$; then one of the following three cases holds: (i) $L < 0$; (ii) $L > 0$; (iii) $L = 0$.

(i) If $L < 0$, then $\lim_{t\to\infty} z(t) = -\infty$. From (2.3) it follows that the inequality

$$z(t) > p(t)x(q^{-k}t) \overset{(2.6)}{\geq} px(q^{-k}t)$$

holds. Thus $\lim_{t\to\infty} x(t) = \infty$. From (2.2) and (2.5) we see that $\lim_{t\to\infty} D_q z(t) = -\infty$, a contradiction.

(ii) If $L > 0$, we arrive at a contradiction analogous to (i).

(iii) Assume $L = 0$. Since $D_q z$ is an eventually decreasing function, $D_q z(t) > 0$ eventually and z is an eventually increasing function. Thus either $\lim_{t\to\infty} z(t) = M \in \mathbb{R}$, or $\lim_{t\to\infty} z(t) = \infty$. If $M > 0$, then $x(t) > z(t) > M/2$ for large $t \in (0,\infty)_q$, and from assumption (2.2) and equation (2.5) it follows that $\lim_{t\to\infty} D_q z(t) = -\infty$, a contradiction. Using a similar argument we reach a contradiction if $\lim_{t\to\infty} z(t) = \infty$. Therefore we assume there exists a finite limit, $\lim_{t\to\infty} z(t) = M \leq 0$. If $M < 0$, then

$$M > z(t) > p(t)x(q^{-k}t) \geq px(q^{-k}t).$$

Thus for large t we have

$$M/p < x(q^{-k}t),$$

and again from assumption (2.2) and equation (2.5) we have that $\lim_{t\to\infty} D_q z(t) = -\infty = L$, a contradiction of $L = 0$. Consequently $\lim_{t\to\infty} z(t) = 0$, and since z is an eventually increasing function, $z(t) < 0$ eventually and (2.9) and (2.10) hold. □

Lemma 2.3. *Assume x is a solution of* (2.1)*, r satisfies* (2.2)*, z is given by* (2.3)*, and*

$$-1 \le p(t) \le 0, \quad t \in (0, \infty)_q. \tag{2.14}$$

Then the following assertions are valid.

(a) *If $x(t) < 0$ eventually, then relations* (2.10) *and* (2.13) *hold.*

(b) *If $x(t) > 0$ eventually, then relations* (2.9) *and* (2.10) *hold.*

Proof. We will prove (a); the proof of (b) is similar and thus omitted. From (2.4) it follows that $D_q^2 z(t) \ge 0$ eventually, and $D_q z$ is an eventually nondecreasing function. Assumption (2.2) implies that $r(t) \ne 0$ eventually, and thus either $D_q z(t) > 0$ eventually or $D_q z(t) < 0$. Suppose that $D_q z(t) > 0$. Since $D_q z$ is a nondecreasing function, there exists a constant $c > 0$ such that $D_q z(t) \ge c$ eventually. Then $\lim_{t\to\infty} D_q z(t) = \infty$. From (2.3) we obtain the inequality

$$z(t) < p(t)x(q^{-k}t) \le -x(q^{-k}t)$$

and therefore $\lim_{t\to\infty} x(t) = -\infty$. On the other hand, from (2.3) again and from the inequality $z(t) > 0$ there follows the estimate

$$x(t) > -p(t)x(q^{-k}t) \ge x(q^{-k}t).$$

The inequalities $x(t) < 0$ and $x(t) > x(q^{-k}t)$ eventually imply that x is a bounded function, a contradiction of the condition $\lim_{t\to\infty} x(t) = -\infty$ proved above. Thus $D_q z(t) < 0$, and z is an eventually decreasing function. Let $L = \lim_{t\to\infty} D_q z(t)$. Then $\lim_{t\to\infty} z(t) = -\infty$. From the inequality $x(t) < z(t)$ it follows that $\lim_{t\to\infty} x(t) = -\infty$, and then (2.5) implies the relation $\lim_{t\to\infty} D_q z(t) = \infty$. The contradiction obtained shows that $L = 0$, that is $\lim_{t\to\infty} D_q z(t) = 0$. Suppose that $z(t) < 0$ eventually. Since z is a decreasing function, there exists a constant $c < 0$ such that $z(t) \le c$ eventually. The inequality $z(t) > x(t)$ implies that $x(t) \le c$ eventually. From (2.5) it follows that $\lim_{t\to\infty} D_q z(t) = \infty$. The contradiction obtained shows that $z(t) > 0$, and since z is an eventually decreasing function, then there exists a finite limit $M = \lim_{t\to\infty} z(t)$. If $M > 0$, then $z(t) > M$ eventually. From (2.3) it follows that

$$M < z(t) < p(t)x(q^{-k}t) \le -x(q^{-k}t),$$

that is $x(q^{-k}t) < -M$. From (2.5) we obtain that $\lim_{t\to\infty} D_q z(t) = \infty$, a contradiction. Hence, $M = 0$, in other words $\lim_{t\to\infty} z(t) = 0$. Since z is a decreasing function, $z(t) > 0$ eventually, and we have shown that if x is an eventually negative solution of (2.1), then (2.10) and (2.13) are valid. □

Lemma 2.4. *The function x is an eventually negative solution of* (2.1) *if and only if $-x$ is an eventually positive solution of the equation*

$$D_q^2\Big(y(t) + p(t)y(q^{-k}t)\Big) + r(t) \min_{s\in\{0,\cdots,\ell\}} y\left(q^{-s}t\right) = 0.$$

Lemma 2.4 is readily verified.

3. MAIN RESULTS

In this section we present the main results on the oscillatory and asymptotic behavior of solutions to (2.1).

Theorem 3.1. *Assume r satisfies* (2.2)*, and*

$$-1 < p \le p(t) \le 0, \quad t \in (0,\infty)_q. \tag{3.1}$$

If x is a nonoscillatory solution of (2.1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t) > 0$ eventually. Then Lemma 2.3 implies that $z(t) < 0$ eventually and $\lim_{t\to\infty} z(t) = 0$. From (3.1) we have that

$$x(t) < -p(t)x(q^{-k}t) < x(q^{-k}t),$$

so that x is bounded. Let $c = \limsup_{t\to\infty} x(t)$, and suppose that $c > 0$. Choose an increasing quantum sequence of points $\{t_i\}$ from $(0,\infty)_q$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} x(t_i) = c$. Set $d = \limsup_{i\to\infty} x(q^{-k}t_i)$, and note that $d \le c$. Choose a subsequence of points $\{t_j\} \subset \{t_i\}$ such that $d = \lim_{j\to\infty} x(q^{-k}t_j)$, and pass to the limit in the inequality $z(t_j) \ge x(t_j) + px(q^{-k}t_j)$ as $j \to \infty$. We then see that

$$0 \ge c + pd \ge c + pc = c(1+p) > 0,$$

a contradiction. Thus $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. The case where $x(t) < 0$ eventually is similar and is omitted. □

Theorem 3.2. *Assume r satisfies* (2.2)*, and condition* (2.6) *holds. If x is a bounded nonoscillatory solution of* (2.1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t) > 0$ eventually; the case where $x(t) < 0$ eventually is similar and is omitted. Since x is bounded, it follows from (2.3) that z is also bounded. Since (2.6) holds, Lemma 2.2 implies that $z(t) < 0$ eventually and $\lim_{t\to\infty} z(t) = 0$. As in the proof of Theorem 3.1, let $c = \limsup_{t\to\infty} x(t)$, and suppose that $c > 0$. Choose an increasing quantum sequence of points $\{t_i\}$ from $(0,\infty)_q$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} x(t_i) = c$. Set $d = \limsup_{i\to\infty} x(q^{-k}t_i)$, and note that $d \le c$. Choose a subsequence of points $\{t_j\} \subset \{t_i\}$ such that $d = \lim_{j\to\infty} x(q^{-k}t_j)$, and pass to the limit in the inequality

$$z(t_j) \le x(t_j) + Px(q^{-k}t_j)$$

as $j \to \infty$. We then see a contradiction, so that $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.3. *Assume condition* (2.6) *holds, and the coefficient function* r *satisfies*

$$0 < r \le r(t) \le R, \quad t \in [t_0, \infty)_q. \tag{3.2}$$

If x *is an eventually positive solution of* (2.1)*, then either* $\lim_{t\to\infty} x(t) = \infty$ *or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Lemma 2.2 implies that either $\lim_{t\to\infty} z(t) = -\infty$ or $\lim_{t\to\infty} z(t) = 0$. First, we consider $\lim_{t\to\infty} z(t) = -\infty$. Then

$$z(t) > p(t)x(q^{-k}t) \ge px(q^{-k}t),$$

so that $\lim_{t\to\infty} x(t) = \infty$. Next, we consider $\lim_{t\to\infty} z(t) = 0$. In this case Lemma 2.2 implies that z is an eventually negative increasing function. If the solution x does not vanish at infinity, then there exist a constant $c > 0$ and an increasing quantum sequence of points $\{t_i\}$ from $(t_0, \infty)_q$ such that $t_{i+1} > q^\ell t_i$ and $x(t_i) > c/2$ for each $i \in \mathbb{N}$. Then, we have

$$\max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}t\right) > c/2, \quad t \in [t_i, t_{i+\ell}]_q.$$

From this last inequality and (3.2) we obtain the estimate

$$(q-1) \sum_{\eta\in[t_i,q^\ell t_i]_q} \eta r(\eta) \max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}\eta\right) > (q-1)(\ell+1)t_0 rc/2. \tag{3.3}$$

It then follows from (3.3) and the choice of the quantum sequence $\{t_i\}$ that

$$\begin{aligned}(q-1) \sum_{\eta\in[t_0,\infty)_q} \eta r(\eta) \max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}\eta\right) &\ge (q-1)\sum_{i=1}^{\infty} \sum_{\eta\in[t_i,q^\ell t_i]_q} \eta r(\eta) \max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}\eta\right) \\ &> (q-1)\sum_{i=1}^{\infty}(\ell+1)t_0 rc/2 = \infty.\end{aligned}$$

From (2.5) we then see that $\lim_{t\to\infty} D_q z(t) = -\infty$. On the other hand, Lemma 2.2 implies that $D_q z(t) > 0$ eventually, a contradiction. Thus $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.4. *Assume condition* (3.2) *holds, and* $p(t) \equiv -1$. *If* x *is an eventually positive solution of* (2.1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Lemma 2.3 implies that $\lim_{t\to\infty} z(t) = 0$, where z is an eventually increasing negative function. Suppose that the solution x does not vanish at infinity. From (2.3) and the fact that $z(t) < 0$, it follows that $x(t) < x(q^{-k}t)$ eventually, so that x is bounded. Let $c = \limsup_{t\to\infty} x(t) > 0$. Choose an increasing quantum sequence of points $\{t_i\}$ from $(0,\infty)_q$ such that such that $t_{i+1} > q^\ell t_i$ and $x(t_i) > c/2$ for each $i \in \mathbb{N}$. Then, we have

$$\max_{s\in\{0,\cdots,\ell\}} x\left(q^{-s}t\right) > c/2, \quad t \in [t_i, t_{i+\ell}]_q.$$

The proof is then completed in a way identical to the proof of Theorem 3.3. □

We now present a few sufficient conditions for the oscillation of all solutions of (2.1).

Theorem 3.5. *Assume r satisfies* (2.2)*, and at least one of the following conditions*

$$1 < p \le p(t) \le P, \tag{3.4}$$

$$0 \le p(t) \le P < 1, \tag{3.5}$$

$$p(t) \equiv 1, \tag{3.6}$$

holds for all $t \in [t_0,\infty)_q$. Then each solution of (2.1) *oscillates.*

Proof. Assume to the contrary that x is a nonoscillatory solution of (2.1). Let $x(t) > 0$ eventually; the case where $x(t) < 0$ eventually is similar and is omitted.

First, let (3.4) hold. By (2.4), $D_q^2 z(t) \le 0$ eventually and $D_q z(t)$ is nonincreasing. From (2.2) we know that $D_q z(t) \ne 0$ eventually, and since $x(t) > 0$ and $p(t) > 0$ in this case, $z(t) > 0$ and $D_q z(t) > 0$ eventually. Suppose that $\lim_{t\to\infty} z(t) = c < \infty$; we will show that $\liminf_{t\to\infty} x(t) > 0$. To this end, assume instead that $\liminf_{t\to\infty} x(t) = 0$. Choose an increasing quantum sequence of points $\{t_i\}$ from $(0,\infty)_q$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} x(q^{-k}t_i) = 0$. It then follows from (2.3) that $\lim_{i\to\infty} x(t_i) = c$. Using (2.3) and (3.4) we have that

$$z(q^k t_i) = x(q^k t_i) + p(q^k t_i)x(t_i) > p(q^k t_i)x(t_i) \ge p x(t_i);$$

letting $i \to \infty$ we see that $c \ge pc > c$, a contradiction. Thus $\liminf_{t\to\infty} x(t) > 0$, so that there exists a positive constant d with $x(t) \ge d > 0$ eventually. From (2.2) and (2.5) it follows that $\lim_{t\to\infty} D_q z(t) = -\infty$, a contradiction of $D_q z(t) > 0$ eventually. Consequently, $\lim_{t\to\infty} z(t) = \infty$. By (2.3) and (3.4), we must have $\lim_{t\to\infty} x(t) = \infty$, which again implies by (2.2) and (2.5) that $\lim_{t\to\infty} D_q z(t) = -\infty$, a contradiction. We conclude that if (3.4) holds, then (2.1) has no eventually positive solutions.

Next, let (3.5) hold. As in the previous case, through two contradictions we arrive at the result.

Finally, let (3.6) hold. As in the first case, $D_q^2 z(t) \le 0$, $D_q z(t) > 0$, and $z(t) > 0$ eventually. Using (2.3) twice, we see that

$$x(q^k t) - x(q^{-k} t) = z(q^k t) - z(t);$$

as z is eventually increasing, it follows that $x(q^k t) > x(q^{-k} t)$ eventually. Thus $\liminf_{t\to\infty} x(t) > 0$. As in the first case, this leads to a contradiction and the result follows. □

4. EXAMPLE

In this section we offer an example related to the results of the previous section. Note that in Theorem 3.1, in the case where $p(t) < 0$ eventually, we do not consider the oscillatory behavior of solutions of (2.1) because there always exists a nonoscillatory solution. This is shown in the following example.

Example 4.1. *Consider the quantum equation*

$$D_q^2\Big(x(t) + p(t)x(q^{-k}t)\Big) + r(t) \max_{s\in\{0,\cdots,\ell\}} x(q^{-s}t) = 0, \qquad t \in (t_0,\infty)_q, \tag{4.1}$$

where $q = 2$, $k = 2$, $r(t) = 1/t$, $t_0 = 8$, ℓ *is a positive integer, and*

$$p(t) = \frac{1 - 6t + 4t^2 + 4t^3}{-8t^2(4 - 6t + t^2)} \in \left[\frac{-2257}{10240}, 0\right), \quad t \in [t_0,\infty)_q.$$

Then (4.1) *has a negative solution* x *that satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Since $r(t) = 1/t$,

$$\sum_{\eta\in[t_0,\infty)_q} \eta r(\eta) = \sum_{\eta\in[2^3,\infty)_2} 1 = \infty,$$

so that (2.2) is satisfied. Using a computer algebra system, one can verify that

$$x(t) = -\sqrt{t}\exp\left(\frac{-\ln^2(t)}{2\ln(2)}\right)$$

is a negative, increasing solution of (4.1) that vanishes at infinity, as guaranteed by Theorem 3.1. □

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