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Triple Positive Solutions to Initial-boundary Value Problems of Nonlinear Delay Differential Equations

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Abstract. In this paper, we consider the existence of triple positive solutions to the boundary value problem of nonlinear delay differential equation

$$\begin{cases} (\phi(x'(t)))' + a(t)f(t, x(t), x'(t), x_t) = 0, & 0 < t < 1, \\ x_0 = 0, \\ x(1) = 0, \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\phi(0) = 0$, and x_t is a function in $C([-\tau, 0], \mathbb{R})$ defined by $x_t(\sigma) = x(t + \sigma)$ for $-\tau \le \sigma \le 0$. By using a fixed-point theorem in a cone introduced by Avery and Peterson, we provide sufficient conditions for the existence of triple positive solutions to the above boundary value problem. An example is also presented to demonstrate our result. The conclusions in this paper essentially extend and improve the known results.

Keywords: Boundary value problems; Positive solutions; Delay differential equations; Increasing homeomorphism and positive homomorphism.

AMS(2000) Mathematical Subject Classification: $34B15$.

1 Introduction

Throughout this paper, for any intervals I and J of $\mathbb{R}$, we denote by $C(I, J)$ the set of all continuous functions defined on I with values in J. Let τ be a nonnegative real number and $t \in [0, 1]$, and let x be a continuous real-valued function defined at least on $[t - \tau, t]$. We define $x_t \in C([-\tau, 0], \mathbb{R})$ by

$$x_t(\sigma) = x(t + \sigma), \quad -\tau \le \sigma \le 0.$$

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In this paper we consider the nonlinear delay differential equation

$$(\phi(x'(t)))' + a(t)f(t, x(t), x'(t), x_t) = 0, \quad 0 < t < 1, \tag{1.1}$$

with the conditions

$$x_0 = 0 \tag{1.2}$$

and

$$x(1) = 0. \tag{1.3}$$

Note that, according to our notation, (1.2) means $x(\sigma) = 0$ for $-\tau \le \sigma \le 0$. Also, $f(t, u, v, \mu)$ is a nonnegative real-valued continuous function defined on $[0, 1] \times [0, \infty) \times \mathbb{R} \times C([-\tau, 0), \mathbb{R})$, $a(t)$ is a nonnegative continuous function defined on $(0, 1)$, and $\phi : \mathbb{R} \to \mathbb{R}$ is an *increasing homeomorphism and positive homomorphism* (defined below) with $\phi(0) = 0$.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an *increasing homeomorphism and positive homomorphism* if the following conditions are satisfied:

(1) for all $x, y \in \mathbb{R}$, $\phi(x) \le \phi(y)$ if $x \le y$;

(2) ϕ is a continuous bijection and its inverse mapping ϕ^{-1} is also continuous;

(3) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in [0, +\infty)$.

In the above definition, we can replace condition (3) by the following stronger condition:

(4) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

If conditions (1), (2) and (4) hold, then ϕ is homogeneously generating a p-Laplacian operator, i.e., $\phi(x) = |x|^{p-2}x$ for some $p > 1$.

In recent years, the existence of solutions to nonlinear boundary values problems of delay differential equations has been extensively studied, see [1], [4], [8], [10]-[25], and the references therein. However, there is little research on problems involving the increasing homeomorphism and positive homomorphism operator. In [18], Liu and Zhang study the existence of positive solutions of the quasilinear differential equation

$$\begin{cases} (\phi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x(0) - \beta x'(0) = 0, \quad x(1) + \delta x'(1) = 0, \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\phi(0) = 0$. They obtain the existence of one or two positive solutions by using a fixed-point theorem in cones. However, in the literature, the existence of *three* positive solutions has never been established for boundary value problems of delay differential equations with increasing homeomorphism and positive homomorphism operators. Therefore, the aim of this paper is to offer some criteria for the existence of triple positive solutions to the boundary value problem (1.1)–(1.3). We also emphasize the generality of the nonlinear term f considered in (1.1) which involves the first-order derivative. Our main tool is a fixed-point theorem in cones introduced by Avery and Peterson [2]. Note that existence of triple solutions to many other boundary value problems of ordinary differential equations have been tackled in the literature; see for example [5, 6, 22, 23] and the references cited therein.

By a solution of boundary value problem (1.1)–(1.3), we mean a function $x \in C[-\tau,1] \cap C^1[0,1]$ that satisfies (1.1) when $0 < t < 1$, while $x(t) = 0$ for $-\tau \le t \le 0$, and $x(1) = 0$. Throughout, we shall assume that

(H_1) $f \in C([0,1] \times [0,\infty) \times \mathbb{R} \times C([-\tau,0),\mathbb{R}),[0,\infty))$;

(H_2) $a \in C(0,1) \cap L^1[0,1]$ with $a(t) > 0$ on $(0,1)$.

The plan of the paper is as follows. In Section 2, for the convenience of the readers we give some definitions and the fixed-point theorem of Avery and Peterson [2]. The main results are developed in Section 3. As an application, we also include an example.

2 Preliminaries

In this section, we provide some background materials cited from cone theory in Banach spaces. The following definitions can be found in the books by Deimling [7] and by Guo and Lakshmikantham [9].

Definition 2.1 *Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:*

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;

(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\le$, i.e., $x \le y$ if and only if $y - x \in P$.

Definition 2.2 *A map α is said to be a nonnegative continuous concave functional in a cone P of a real Banach space E if $\alpha : P \to [0,\infty)$ is continuous and for all $x, y \in P$ and $t \in [0,1]$,*

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y).$$

A map γ is said to be a nonnegative continuous convex functional in a cone P of a real Banach space E if $\gamma : P \to [0,\infty)$ is continuous and for all $x, y \in P$ and $t \in [0,1]$,

$$\gamma(tx + (1-t)y) \le t\gamma(x) + (1-t)\gamma(y).$$

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. For positive numbers $a,\ b,\ c,\ d$, we define the following convex sets of P :

$$\begin{aligned}
&P(\gamma,d) = \{x \in P : \gamma(x) < d\},\\
&P(\gamma,\alpha,b,d) = \{x \in P : b \le \alpha(x),\ \gamma(x) \le d\},\\
&P(\gamma,\theta,\alpha,b,c,d) = \{x \in P : b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\},
\end{aligned}$$

and the closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \le \psi(x),\ \gamma \le d\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proof of our main result.

Theorem 2.1 *([2]) Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for all $0 \le \lambda \le 1$, and for some positive numbers $M,\ d$,*

$$\alpha(x) \le \psi(x), \quad \|x\| \le M\gamma(x), \quad \textit{for all } x \in \overline{P(\gamma, d)}. \tag{2.1}$$

Suppose that $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a < b$ such that

(S_1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \ne \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S_2) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;

(S_3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \le d, \quad i = 1, 2, 3; \quad b < \alpha(x_1); \quad a < \psi(x_2) \quad \textit{with} \quad \alpha(x_2) < b; \quad \psi(x_3) < a.$$

3 Main Results

In this section, we impose growth conditions on f which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions to the boundary value problem (1.1)–(1.3).

Let $E = C^1[-\tau, 1]$ be a Banach space equipped with the norm

$$\|x\| = \max\left\{\max_{t\in[-\tau,1]} |x(t)|,\ \max_{t\in[-\tau,1]} |x'(t)|\right\}.$$

From (1.1) we have $(\phi(x'(t)))' = -a(t)f(t, x(t), x'(t), x_t) \le 0$; thus x is concave on $[0, 1]$. Consequently, we define a cone $P \subset E$ by

$$P = \{x \in E : x(t) \ge 0 \text{ for } t \in [-\tau, 1],\ x_0 = 0, x(1) = 0,\ x \text{ is concave on } [0, 1]\}. \tag{3.1}$$

For $x \in P$, define

$$u(t) = \int_0^t \phi^{-1}\left(\int_s^t a(r)f(r, x(r), x'(r), x_r)dr\right) ds - \int_t^1 \phi^{-1}\left(\int_t^s a(r)f(r, x(r), x'(r), x_r)dr\right) ds,$$

where $0 < t < 1$. Clearly, $u(t)$ is continuous and strictly increasing in $(0, 1)$ and $u(0^+) < 0 < u(1^-)$. Thus, $u(t)$ has a unique zero in $(0, 1)$. Let $t_0 = t_x$ (i.e., t_0 is dependent on x) be the zero of $u(t)$ in $(0, 1)$. It follows that

$$\int_0^{t_0} \phi^{-1}\left(\int_s^{t_0} a(r)f(r, x(r), x'(r), x_r)dr\right) ds = \int_{t_0}^1 \phi^{-1}\left(\int_{t_0}^s a(r)f(r, x(r), x'(r), x_r)dr\right) ds. \tag{3.2}$$

To apply Theorem 2.1, we define the nonnegative continuous concave functional α_1, the nonnegative continuous convex functionals θ_1, γ_1, and the nonnegative continuous functional ψ_1 on the cone P by

$$\alpha_1(x) = \min_{t\in[1/k,(k-1)/k]} |x(t)|, \quad \gamma_1(x) = \max_{t\in[0,1]} |x'(t)|, \quad \psi_1(x) = \theta_1(x) = \max_{t\in[0,1]} |x(t)|,$$

where $x \in P$ and k (≥ 3) is an integer.

We will need the following lemmas in deriving the main result.

Lemma 3.1 *For $x \in P$, there exists a constant $M \geq 1$ such that*

$$\max_{t\in[-\tau,1]} |x(t)| \leq M \max_{t\in[-\tau,1]} |x'(t)|.$$

Proof. By Lemma 3.1 of [3], there exists a constant $L > 0$ such that

$$\max_{t\in[0,1]} |x(t)| \leq L \max_{t\in[0,1]} |x'(t)|. \tag{3.3}$$

Since $x_0 = 0$ implies $x(t) = 0 = x'(t)$ for $t \in [-\tau, 0]$, it is then clear that

$$\max_{t\in[-\tau,0]} |x(t)| = \max_{t\in[-\tau,0]} |x'(t)|. \tag{3.4}$$

Now let $M = \max\{L, 1\}$; thus (3.3) and (3.4) yield that

$$\max_{t\in[-\tau,1]} |x(t)| \leq M \max_{t\in[-\tau,1]} |x'(t)|. \qquad \square \tag{3.5}$$

With Lemma 3.1 and the concavity of x for all $x \in P$, the functionals defined above satisfy

$$\frac{1}{k}\theta_1(x) \leq \alpha_1(x) \leq \theta_1(x), \quad \|x\| = \max\{\theta_1(x), \gamma_1(x)\} \leq M\gamma_1(x), \quad \alpha_1(x) \leq \psi_1(x). \tag{3.6}$$

Therefore, condition (2.1) of Theorem 2.1 is satisfied.

Let the operator $T : P \to E$ be defined by

$$Tx(t) := \begin{cases} 0, & -\tau \leq t \leq 0, \\ \displaystyle\int_0^t \phi^{-1}\left(\int_s^{t_0} a(r)f(r, x(r), x'(r), x_r)dr\right) ds, & 0 \leq t \leq t_0, \\ \displaystyle\int_t^1 \phi^{-1}\left(\int_{t_0}^s a(r)f(r, x(r), x'(r), x_r)dr\right) ds, & t_0 \leq t \leq 1, \end{cases} \tag{3.7}$$

where t_0 is defined by (3.2). It is well known that boundary value problem (1.1)–(1.3) has a positive solution x if and only if $x \in P$ is a fixed point of T.

Lemma 3.2 *Suppose that (H_1) and (H_2) hold. Then $TP \subset P$ and $T : P \to P$ is completely continuous.*

Proof. By (3.7), we have for $x \in P$,

$$Tx(t) \geq 0, \quad t \in [-\tau, 1], \quad Tx(\sigma) = 0, \quad \sigma \in [-\tau, 0], \quad Tx(1) = 0. \tag{3.8}$$

Moreover, $Tx(t_0)$ is the maximum value of Tx on $[0,1]$, since

$$(Tx)'(t) := \begin{cases} 0, & -\tau \le t \le 0, \\ \phi^{-1}\left(\int_t^{t_0} a(r)f(r,x(r),x'(r),x_r)dr\right) \ge 0, & 0 \le t \le t_0, \\ -\phi^{-1}\left(\int_{t_0}^{t} a(r)f(r,x(r),x'(r),x_r)dr\right) \le 0, & t_0 \le t \le 1, \end{cases} \tag{3.9}$$

is continuous and nonincreasing in $[0,1]$ and $(Tx)'(t_0) = 0$. So Tx is concave on $[0,1]$, which together with (3.8) shows that $TP \subset P$. Using a similar argument as in [3], one can show that $T : P \to P$ is completely continuous. $\square$

We can further prove the following result: for $x \in P$,

$$\min_{t\in[1/k,(k-1)/k]} Tx(t) \ge \frac{1}{k} \max_{t\in[0,1]} Tx(t). \tag{3.10}$$

In fact, from (3.7) we have

$$Tx(t) \ge \begin{cases} \dfrac{Tx(t_0)}{t_0} t \ge \max\limits_{t\in[0,1]} Tx(t)t, & 0 \le t \le t_0, \\ \dfrac{Tx(t_0)}{1-t_0}(1-t) \ge \max\limits_{t\in[0,1]} Tx(t)(1-t), & t_0 \le t \le 1, \end{cases}$$

which implies that (3.10) holds.

Let

$$\delta = \min\left\{\int_{1/k}^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)dr\right) ds,\ \int_{1/2}^{(k-1)/k} \phi^{-1}\left(\int_{1/2}^{s} a(r)dr\right) ds\right\},$$

$$\rho = \phi^{-1}\left(\int_0^1 a(r)dr\right),$$

$$N = \max\left\{\int_{1/k}^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)dr\right) ds,\ \int_{1/2}^{(k-1)/k} \phi^{-1}\left(\int_{1/2}^{s} a(r)dr\right) ds\right\}.$$

We are now ready to apply the Avery-Peterson fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to boundary value problem (1.1)–(1.3).

Theorem 3.1 *Suppose that* (H_1) *and* (H_2) *hold. Let* $0 < a < b \le \frac{Md}{k}$ *and suppose that* f *satisfies the following conditions:*

(A_1) $f(t,u,v,\mu) < \phi(\frac{d}{\rho})$ *for* $(t,u,v) \in [0,1] \times [0,Md] \times [-d,d]$, $\|\mu\| \le Md$;

(A_2) $f(t,u,v,\mu) > \phi(\frac{kb}{\delta})$ *for* $(t,u,v) \in [\frac{1}{k},\frac{k-1}{k}] \times [b,kb] \times [-d,d]$, $\|\mu\| \le kb$;

(A_3) $f(t,u,v,\mu) < \phi(\frac{a}{N})$ *for* $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$, $\|\mu\| \le kb$.

Then the boundary value problem (1.1)–(1.3) has at least three positive solutions x_1, x_2 *and* x_3 *such that*

$$\max_{t\in[0,1]} |x_i'(t)| \le d,\ i = 1,2,3,$$

$$b < \min_{t\in[1/k,(k-1)/k]} |x_1(t)|, \qquad a < \max_{t\in[0,1]} |x_2(t)|$$

with

$$\min_{t\in[1/k,(k-1)/k]} |x_2(t)| < b,$$

and

$$\max_{t\in[0,1]} |x_3(t)| < a.$$

Proof. The boundary value problem (1.1)–(1.3) has a solution x if and only if x solves the operator equation $x = Tx$. Thus, we set out to verify that the operator T satisfies the Avery-Peterson fixed point theorem, which then implies the existence of three fixed points of T.

For $x \in \overline{P(\gamma_1, d)}$, we have $\gamma_1(x) = \max_{t\in[0,1]} |x'(t)| \le d$, and, by Lemma 3.1, $\max_{t\in[0,1]} |x(t)| \le Md$ for $t \in [0,1]$. Then condition (A_1) implies that $f(t, x(t), x'(t), x_t) \le \phi(d/\rho)$. On the other hand, $x \in P$ implies that $Tx \in P$, so Tx is concave on $[0,1]$ and

$$\max_{t\in[0,1]} |(Tx)'(t)| = \max\{|(Tx)'(0)|,\ |(Tx)'(1)|\}.$$

Thus,

$$\begin{aligned}
\gamma_1(Tx) &= \max_{t\in[0,1]} |(Tx)'(t)| \\
&= \max\left\{\phi^{-1}\left(\int_0^{t_0} a(r) f(r, x(r), x'(r), x_r)dr\right)ds,\ \ \phi^{-1}\left(\int_{t_0}^1 a(r) f(r, x(r), x'(r), x_r)dr\right)ds\right\} \\
&\le \frac{d}{\rho}\phi^{-1}\left(\int_0^1 a(r)dr\right) \ = \ \frac{d}{\rho}\rho \ = \ d.
\end{aligned}$$

Therefore, $T : \overline{P(\gamma_1, d)} \to \overline{P(\gamma_1, d)}$.

To check condition (S_1) of Theorem 2.1, choose

$$x_0(t) = -4k^2 b\left(t - \frac{1}{2k}\right)^2 + kb, \quad 0 \le t \le 1.$$

It is easy to see that $x_0 \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d)$ and $\alpha_1(x_0) > b$, and so

$$\{x \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d) : \alpha_1(x) > b\} \ne \emptyset.$$

Now, for $x \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d)$ with $\alpha_1(x) > b\}$ and $t \in [1/k, (k-1)/k]$, we have

$$b \le x(t) \le kb, \quad |x'(t)| \le d.$$

Thus, for $t \in [1/k, (k-1)/k]$, it follows from condition (A_2) that

$$f(t, x(t), x'(t), x_t) > \phi(kb/\delta).$$

By definition of α_1 and P, by (3.10) we have

$$
\begin{aligned}
\alpha_1(Tx) &= \min_{t\in[1/k,(k-1)/k]} |(Tx)(t)| \ \ge\ \frac{1}{k}\max_{[0,1]} Tx(t) \ =\ \frac{1}{k}(Tx)(t_0)\\
&= \frac{1}{k}\int_0^{t_0} \phi^{-1}\left(\int_s^{t_0} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\\
&= \frac{1}{k}\int_{t_0}^{1} \phi^{-1}\left(\int_{t_0}^{s} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\\
&\ge \frac{1}{k}\left\{\min\{\int_0^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)f(r,x(r),x'(r),x_r)dr\right) ds,\right.\\
&\qquad \left.\int_{1/2}^{1} \phi^{-1}\left(\int_{1/2}^{s} a(r)f(r,x(r),x'(r),x_r)dr\right) dsr\right\}\\
&\ge \frac{1}{k}\min\left\{\int_{1/k}^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)f(r,x(r),x'(r),x_r)dr\right) ds,\right.\\
&\qquad \left.\int_{1/2}^{(k-1)/k} \phi^{-1}\left(\int_{1/2}^{s} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\right\}\\
&> \frac{1}{k}\frac{kb}{\delta}\delta \ =\ b,
\end{aligned}
$$

i.e., $\alpha_1(Tx) > b$ for all $x \in P(\gamma_1,\theta_1,\alpha_1,b,kb,d)$. This shows that condition (S_1) of Theorem 2.1 is satisfied.

Moreover, by (3.6) we have

$$
\alpha_1(Tx) \ge \frac{1}{k}\theta_1(Tx) > \frac{1}{k}kb = b, \tag{3.11}
$$

for all $x \in P(\gamma_1,\theta_1,\alpha_1,b,kb,d)$ with $\theta_1(Tx) > kb$. Hence, condition (S_2) of Theorem 2.1 is fulfilled.

Finally, we show that condition (S_3) of Theorem 2.1 holds as well. Clearly, $0 \notin R(\gamma_1,\psi_1,a,d)$ since $\psi_1(0) = 0 < a$. Suppose that $x \in R(\gamma_1,\psi_1,a,d)$ with $\psi_1(x) = a$. Then, by condition (A_3), we obtain that

$$
\begin{aligned}
\psi_1(Tx) &= \max_{t\in[0,1]} |(Tx)(t)| \ =\ (Tx)(t_0)\\
&= \int_0^{t_0} \phi^{-1}\left(\int_s^{t_0} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\\
&= \int_{t_0}^{1} \phi^{-1}\left(\int_{t_0}^{s} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\\
&\le \max\left\{\int_0^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)f(r,x(r),x'(r),x_r)dr\right) ds,\right.\\
&\qquad \left.\int_{1/2}^{1} \phi^{-1}\left(\int_{1/2}^{s} a(r)f(r,x(r),x'(r),x_r)dr\right) ds\right\}\\
&< \frac{a}{N}\max\left\{\int_0^{1/2} \phi^{-1}\left(\int_s^{1/2} a(r)dr\right) ds,\ \ \int_{1/2}^{1} \phi^{-1}\left(\int_{1/2}^{s} a(r)dr\right) ds\right\}\\
&= \frac{a}{N}N \ =\ a.
\end{aligned}
$$

Hence, we have $\psi_1(Tx) = \max_{t\in[0,1]} |Tx(t)| < a$. So condition (S_3) of Theorem 2.1 is met.

Since (3.6) holds for $x \in P$, all the conditions of Theorem 2.1 are satisfied. Therefore, the boundary value problem (1.1)–(1.3) has at least three positive solutions x_1, x_2 and x_3 such that

$$\max_{t\in[0,1]} |x_i'(t)| \le d,\ i = 1, 2, 3,$$

$$b < \min_{t\in[1/k,(k-1)/k]} |x_1(t)|, \qquad a < \max_{t\in[0,1]} |x_2(t)|,$$

with

$$\min_{t\in[1/k,(k-1)/k]} |x_2(t)| < b,$$

and

$$\max_{t\in[0,1]} |x_3(t)| < a.$$

The proof is complete. $\square$

To illustrate our main results, we shall now present an example.

Example 3.1 Consider the boundary value problem with increasing homeomorphism and positive homomorphism

$$\begin{cases} (\phi(x'(t)))' + f(t, x(t), x'(t), x_t) = 0, & 0 \le t \le 1, \\ x(\sigma) = 0, & -\tau \le \sigma \le 0, \\ x(1) = 0, & \end{cases} \tag{3.12}$$

where $\phi(x') = |x'(t)|x'(t)$,

$$f(t,u,v,\mu) = \begin{cases} \dfrac{e^t}{4} + 2306u^{10} + \dfrac{v}{15000} + \left(\dfrac{\mu}{30000}\right)^3, & u \le 4, \\ \dfrac{e^t}{4} + 2306(5-u)u^{10} + \dfrac{v}{15000} + \left(\dfrac{\mu}{30000}\right)^3, & 4 \le u \le 5, \\ \dfrac{e^t}{4} + 2306(u-5)u^{10} + \dfrac{v}{15000} + \left(\dfrac{\mu}{30000}\right)^3, & 5 \le u \le 6, \\ \dfrac{e^t}{4} + 2306 \cdot 6^{10} + \dfrac{v}{15000} + \left(\dfrac{\mu}{30000}\right)^3, & u \ge 6. \end{cases}$$

Choose $a = 1/2,\ b = 1,\ k = 4,\ M = 1,\ d = 30000$. We note that $\delta = \frac{1}{12},\ \rho = 1,\ N = \frac{\sqrt{2}}{6}$. Consequently, $f(t,u,v,\mu)$ satisfies

$$f(t,u,v,\mu) < \phi\left(\frac{d}{\rho}\right) = 9 \times 10^8$$

for $0 \le t \le 1,\ 0 \le u \le 30000,\ -30000 \le v \le 30000,\ \|\mu\| \le 30000$, and

$$f(t,u,v,\mu) > \phi\left(\frac{4b}{\delta}\right) = 2304$$

for $1/4 \le t \le 3/4,\ 1 \le u \le 4,\ -30000 \le v \le 30000,\ \|\mu\| \le 4$, and

$$f(t,u,v,\mu) < \phi\left(\frac{a}{N}\right) = 4.5$$

for $0 \le t \le 1,\ 0 \le u \le 1/2,\ -30000 \le v \le 30000,\ \|\mu\| \le 4$.

All the conditions of Theorem 3.1 are satisfied. Hence, the boundary value problem (3.12) has at least three positive solutions $x_1,\ x_2$ and x_3 such that

$$\max_{t\in[0,1]} |x_i'(t)| \le 30000,\ i = 1,2,3,$$

$$1 < \min_{t\in[1/4,3/4]} |x_1(t)|, \qquad \frac{1}{2} < \max_{t\in[0,1]} |x_2(t)|,$$

with

$$\min_{t\in[1/4,3/4]} |x_2(t)| < 1,$$

and

$$\max_{t\in[0,1]} |x_3(t)| < \frac{1}{2}.$$

Remark 3.1 The same conclusions of Theorem 3.1 hold when ϕ fulfills conditions (1), (2) and (4). In particular, for the p-Laplacian operator $\phi(x) = |x|^{p-2}x$, for some $p > 1$, our conclusions are true and new.

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