Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **8**, 1-7; **http://www.math.u-szeged.hu/ejqtde/**

Stability in discrete equations with variable delays

ERNEST YANKSON
Department of Mathematics and Statistics
University of Cape Coast
Ghana.
email: ernestoyank@yahoo.com

Abstract

In this paper we study the stability of the zero solution of difference equations with variable delays. In particular we consider the scalar delay equation

$$\Delta x(n) = -a(n)x(n - \tau(n))$$

and its generalization

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n - \tau_j(n)).$$

Fixed point theorems are used in the analysis.

Key words: Fixed point; Contraction mapping; Asymptotic stability
Mathematics subject classifications: 34K13,34C25, 34G20.

1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{Z}$ the integers, $\mathbb{Z}^-$ the negative integers, and $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x \geq 0\}$. In this paper we study the asymptotic stability of the zero solution of the scalar delay equation

$$\Delta x(n) = -a(n)x(n - \tau(n)) \tag{1.1}$$

and its generalization

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n - \tau_j(n)). \tag{1.2}$$

where $a, a_j : \mathbb{Z}^+ \to \mathbb{R}$ and $\tau, \tau_j : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n - \tau(n) \to \infty$ as $n \to \infty$.
For each n_0, define $m_j(n_0) = \inf\{s - \tau_j(s) \ : \ s \geq n_0\}$, $m(n_0) = \min\{m_j(n_0) \ : \ 1 \leq j \leq N\}$. Note that (1.2) becomes (1.1) for $N = 1$.

Recently, in [11], Raffoul studied the stability of the zero solution of (1.1) when

$\tau(n) = r$. Our objective in this research is to generalize the stability results in [11] to (1.2) for variable $\tau_j(n)$'s. For more on stability using fixed point theory we refer to [1],[7],[9],[11],[12] and for basic results on difference calculus we refer to [2] and [8]. We also refer to [3],[4],[5],[6] and [10] for other results on stability for difference equations.

Remark 1.1 *In [7], the author and Islam showed that the zero solution of the equation*

$$x(n+1) = b(n)x(n) + a(n)x(n-\tau(n))$$

is asymptotically stable with one of the assumptions being that

$$\prod_{s=0}^{n-1} b(s) \to 0 \text{ as } n \to \infty. \tag{1.3}$$

However, as pointed out in [11], condition (1.3) *cannot hold for* (1.2) *since* $b(n) = 1$*, for all* $n \in \mathbb{Z}$*. The results we obtain in this paper overcome the requirement of* (1.3).

Let $D(n_0)$ denote the set of bounded sequences $\psi : [m(n_0), n_0] \to \mathbb{R}$ with the maximum norm $||.||$. Also, let $(B, ||.||)$ be the Banach space of bounded sequences $\varphi : [m(n_0), \infty) \to \mathbb{R}$ with the maximum norm. Define the inverse of $n - \tau_i(n)$ by $g_i(n)$ if it exists and set

$$Q(n) = \sum_{j=1}^{N} b(g_j(n)),$$

where

$$\sum_{j=1}^{N} b(g_j(n)) = 1 - \sum_{j=1}^{N} a(g_j(n)).$$

For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, a solution of (1.2) through (n_0, ψ) is a function $x : [m(n_0), n_0 + \alpha) \to \mathbb{R}^n$ for some positive constant $\alpha > 0$ such that $x(t)$ satisfies (1.2) on $[n_0, n_0+\alpha)$ and $x(n) = \psi(n)$ for $n \in [m(n_0), n_0]$. We denote such a solution by $x(n) = x(n, n_0, \psi)$. For a fixed n_0, we define

$$||\psi|| = \max\{|\psi(n)| \ : \ m(n_0) \le n \le n_0\}.$$

2 Stability

In this section we obtain conditions for the zero solution of (1.2) to be asymptotically stable.

We begin by rewriting (1.2) as

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(g_j(n))x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s) \tag{2.1}$$

where Δ_n represents that the difference is with respect to n. If we let

$$\sum_{j=1}^{N} b_j(g_j(n)) = 1 - \sum_{j=1}^{N} a_j(g_j(n)),$$

then (2.1) is equivalent to

$$x(n+1) = \sum_{j=1}^{N} b_j(g_j(n))x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s) \tag{2.2}$$

Lemma 2.1 *Suppose that $Q(n) \neq 0$ for all $n \in \mathbb{Z}^+$ and the inverse function $g_j(n)$ of $n - \tau_j(n)$ exists. Then $x(n)$ is a solution of (2.2) if and only if*

$$x(n) = \Big(x(n_0) - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))x(s)\Big) \prod_{s=n_0}^{n-1} Q(s) + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))x(s)$$

$$-\sum_{s=n_0}^{n-1} \Big([1-Q(s)] \prod_{k=s+1}^{n-1} Q(k) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))x(u)\Big),\ n \geq n_0.$$

Proof. By the variation of parameters formula we obtain

$$x(n) \;=\; x(n_0) \prod_{s=n_0}^{n-1} Q(s) + \sum_{k=0}^{n-1} \Big(\prod_{s=k}^{n-1} Q(s) \Delta_k \sum_{j=1}^{N} \sum_{s=k-\tau_j(k)}^{k-1} a_j(g_j(s))x(s)\Big). \tag{2.3}$$

Using the summation by parts formula we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\Big(\prod_{s=k}^{n-1}Q(s)\Delta_k\sum_{j=1}^{N}\sum_{s=k-\tau_j(k)}^{k-1}a_j(g_j(s))x(s)\Big)\\
&=\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}a_j(g_j(s))x(s)\\
&\quad-\prod_{s=n_0}^{n-1}Q(s)\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}a_j(g_j(s))x(s)\\
&\quad-\sum_{s=n_0}^{n-1}\Big([1-Q(s)]\prod_{k=s+1}^{n-1}Q(k)\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}a_j(g_j(u))x(u)\Big). \qquad (2.4)
\end{aligned}
$$

Substituting (2.5) into (2.3) gives the desired result. This completes the proof of Lemma 2.1.

We next state and prove our main results.

Theorem 2.1 *Suppose that the inverse function $g_j(n)$ of $n-\tau_j(n)$ exists, and assume there exists a constant $\alpha\in(0,1)$ such that*

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|a_j(g_j(s))|+\\
&\sum_{s=n_0}^{n-1}\Big(|[1-Q(s)]|\Big|\prod_{k=s+1}^{n-1}Q(k)\Big|\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|a_j(g_j(u))|\Big)\le\alpha. \qquad (2.5)
\end{aligned}
$$

Moreover, assume that there exists a positive constant M such that

$$\Big|\prod_{s=n_0}^{n-1}Q(s)\Big|\le M.$$

Then the zero solution of (1.2) is stable.

Proof. Let $\epsilon>0$ be given. Choose $\delta>0$ such that

$$(M+M\alpha)\delta+\alpha\epsilon\le\epsilon.$$

Let $\psi\in D(n_0)$ such that $|\psi(n)|\le\delta$. Define

$$\mathbb{S}=\{\varphi\in B:\varphi(n)=\psi(n)\text{ if }n\in[m(n_0),n_0],\ ||\varphi||\le\epsilon\}.$$

Then $(\mathbb{S}, ||.||)$ is a complete metric space where, $||.||$ is the maximum norm.

Define the mapping $P : S \to S$ by

$$\big(P\varphi\big)(n) = \psi(n) \text{ for } n \in [m(n_0), n_0]$$

and

$$\begin{aligned}
\big(P\varphi\big)(n) &= \Big(\psi(n_0) - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} a_j(g_j(s))\psi(s)\Big) \prod_{s=n_0}^{n-1} Q(s) \\
&+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s))\varphi(s) \\
&- \sum_{s=n_0}^{n-1} \Big([1-Q(s)] \prod_{k=s+1}^{n-1} Q(k) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\varphi(u)\Big). \quad (2.6)
\end{aligned}$$

Clearly, $\mathbb{P}\varphi$ is continuous. We first show that P maps from S to S. By (2.6)

$$\begin{aligned}
|\big(P\varphi\big)(n)| &\leq M\delta + M\alpha\delta + \Big\{ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} a_j(g_j(s)) \\
&\quad + \sum_{s=n_0}^{n-1} \Big([1-Q(s)] \prod_{k=s+1}^{n-1} Q(k) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\varphi(u)\Big\} ||\varphi|| \\
&\leq (M + M\alpha)\delta + \alpha\epsilon \\
&\leq \epsilon.
\end{aligned}$$

Thus P maps from S into itself. We next show that P is a contraction. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}
|(P\zeta)(t) &- (P\eta)(t)| \leq \Big\{ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |a_j(g_j(s))| \\
&\quad + \sum_{s=n_0}^{n-1} \Big(|[1-Q(s)]| \Big| \prod_{k=s+1}^{n-1} Q(k) \Big| \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))|\Big)\Big\} ||\zeta - \eta|| \\
&\leq \alpha ||\zeta - \eta||
\end{aligned}$$

This shows that P is a contraction. Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (1.2) and for any $\varphi \in S, ||P\varphi|| \leq \epsilon$. This proves that the zero solution of (1.2) is stable.

Theorem 2.2 *Assume that the hypotheses of Theorem 2.1 hold. Also assume that*

$$\prod_{k=n_0}^{n-1} Q(k) \to 0 \text{ as } n \to \infty. \tag{2.7}$$

Then the zero solution of (1.2) *is asymptotically stable.*

Proof. We have already proved that the zero solution of (1.2) is stable. Let $\psi \in D(n_0)$ such that $|\psi(n)| \le \delta$ and define

$\mathbb{S}^* = \{\varphi \in B : \varphi(n) = \psi(n)$ if $n \in [m(n_0), n_0],\ ||\varphi|| \le \epsilon$ and $\varphi(n) \to 0,$ as $n \to \infty\}.$

Define $P : S^* \to S^*$ by (2.6). From the proof of Theorem 2.2, the map P is a contraction and for every $\varphi \in S^*$, $||(P\varphi)|| \le \epsilon$.

We next show that $(P\varphi)(n) \to 0$ as $n \to \infty$. The first term on the right side of (2.6) goes to zero because of condition (2.7). It is clear from (2.5) and the fact that $\varphi(n) \to 0$ as $n \to \infty$ that $\sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |a_j(g_j(s))||\varphi(s)| \to 0$ as $n \to \infty$.

Now we show that the last term on the right side of (2.6) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, for each $\epsilon_1 > 0$, there exists a $N_1 > n_0$ such that $s \ge N_1$ implies $|\varphi(s - \tau_j(s))| < \epsilon_1$ for $j = 1, 2, 3, ..., N$. Thus for $n \ge N_1$, the last term, I_3 in (2.6) satisfies

$$\begin{aligned}
|I_3| &= \Big|\sum_{s=n_0}^{n-1} \Big([1 - Q(s)] \prod_{k=s+1}^{n-1} Q(k) \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} a_j(g_j(u))\varphi(u)\Big)\Big| \\
&\le \sum_{s=n_0}^{N_1-1} \Big(|[1 - Q(s)]|\Big|\prod_{k=s+1}^{n-1} Q(k)\Big|\sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))||\varphi(u)|\Big) \\
&\quad + \sum_{s=N_1}^{n-1} \Big(|[1 - Q(s)]|\Big|\prod_{k=s+1}^{n-1} Q(k)\Big|\sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))||\varphi(u)|\Big) \\
&\le \max_{\sigma \ge m(n_0)} |\varphi(\sigma)| \sum_{s=n_0}^{N_1-1} \Big(|[1 - Q(s)]|\Big|\prod_{k=s+1}^{n-1} Q(k)\Big|\sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))|\Big) \\
&\quad + \epsilon_1 \sum_{s=N_1}^{n-1} \Big(|[1 - Q(s)]|\Big|\prod_{k=s+1}^{n-1} Q(k)\Big|\sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))|\Big)
\end{aligned}$$

By (2.7), there exists $N_2 > N_1$ such that $n \ge N_2$ implies

$$\max_{\sigma \ge m(n_0)} |\varphi(\sigma)| \sum_{s=n_0}^{N_1-1} \Big(|[1 - Q(s)]|\Big|\prod_{k=s+1}^{n-1} Q(k)\Big|\sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))|\Big) < \epsilon_1.$$

Apply (2.5) to obtain $|I_3| \leq \epsilon_1 + \epsilon_1\alpha < 2\epsilon_1$. Thus, $I_3 \to 0$ as $n \to \infty$. Hence $(P\varphi)(n) \to 0$ as $n \to \infty$, and so $P\varphi \in S^*$.

By the contraction mapping principle, P has a unique fixed point that solves (1.2) and goes to zero as n goes to infinity. Therefore, the zero solution of (1.2) is asymptotically stable.

References

[1] T. Burton and T. Furumochi, Fixed points and problems in stability theory, *Dynam. Systems Appl.* **10** (2001), 89-116.

[2] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 1999.

[3] S. Elaydi, Periodicity and stability of linear Volterra difference systems, *J. Math. Anal. Appl.* **181** (1994), 483-492.

[4] S. Elaydi and S. Murakami, Uniform asymptotic stability in linear Volterra difference equations, *J. Differerence Equ. Appl.* **3** (1998), 203-218.

[5] P. Eloe, M. Islam and Y. Raffoul, Uniform asymptotic stability in nonlinear Volterra discrete systems, Special Issue on Advances in Difference Equations IV, *Computers Math. Appl.* **45** (2003), 1033-1039.

[6] M. Islam and Y. Raffoul, Exponential stability in nonlinear difference equations, *J. Differerence Equ. Appl.* **9** (2003), 819-825.

[7] M. Islam and E. Yankson, Boundedness and Stability in nonlinear delay difference equations employing fixed point theory, *Electron. J. Qual. Theory Differ. Equ* **26** (2005).

[8] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, Harcourt Academic Press, San Diego, 2001.

[9] Y. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed point theory, *Mathematical and Computer Modelling* **40** (2004), 691-700.

[10] Y. Raffoul, General theorems for stability and boundedness for nonlinear functional discrete systems, *J. Math. Anal. Appl.* **279** (2003), 639-650.

[11] Y. N. Raffoul, Stability and periodicity in discrete delay equations, *J. Math. Anal. Appl.* **324** (2006) 1356-1362.

[12] B. Zhang, Fixed points and stability in differential equations with variable delays, *Nonlinear Analysis* **63** (2005), 233-242.

(Received November 14, 2008)