Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **7**, 1-8; **http://www.math.u-szeged.hu/ejqtde/**

Monotonic solutions of functional integral and differential equations of fractional order

El-Sayed, A.M.A and Hashem, H.H.G
Faculty of Science, Alexandria University, Alexandria, Egypt
e.mails : amasayed@hotmail.com and hendhghashem@yahoo.com

Abstract

The existence of positive monotonic solutions, in the class of continuous functions, for some nonlinear quadratic integral equations have been studied in [5]-[8]. Here we are concerned with a singular quadratic functional integral equations. The existence of positive monotonic solutions $x \in L_1[0,1]$ will be proved. The fractional order nonlinear functional differential equation will be given as a special case.

Keywords: Quadratic functional integral equations, Positive monotonic solutions, Measure of noncompactness fractional order integral.

1 Introduction

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For example, some problems considered in vehicular traffic theory, biology and queuing theory lead to the quadratic integral equations of this type (see [5] -[8]). J. Banas (see ([5]-[8]) proved the existence of solution of some equations in the class of continuous functions.

Here, we shall be concerned with the quadratic functional integral equation

$$x(t) \ = \ a(t) \ + \ f(t, x(t)) \int_0^t k\ (t,s)\ g(s, x(\phi(s)))\ ds \qquad t \ \in \ [0,1]. \tag{1}$$

We prove, under certain condition, the existence of positive monotonic solutions of Eq. (1) in the space of integrable functions.

The singular (fractional order) quadratic functional integral equation of the form

$$x(t) \ = \ a(t) \ + \ f(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ g(s, x(\phi(s)))\ ds \qquad t \ \in \ [0,1], \quad \beta \in (0,1), \tag{2}$$

will be studied as an application.

The existence of a monotonic positive integrable solution of the nonlinear functional differential equation of fractional order

$$D^{\beta} x(t) \ = \ g(s, x(\phi(s))), \ \ t \ \in \ (0,1] \quad \text{and} \quad I^{1-\beta} x(t)|_{t=0} \ = \ a, \tag{3}$$

where D^{β} is the Riemann-Liouville fractional order derivative, will be given as another application.
Also the results concerning the existence of monotonic positive integrable solution of the nonlinear functional equation

$$x(t) \;=\; g(t, x(\phi(t))), \tag{4}$$

which was proved in [3], will be given as a special case.

2 Preliminaries

Let $L_1 \;=\; L_1[0,1]$ be the class of Lebesgue integrable functions on $I \;=\; [0,1]$ with the standard norm. In this section we collect some definitions and results needed in our further investigations.
Assume that the function $f : (0,1) \times R \to R$ satisfies Caratheodory conditions i.e., it is measurable in t for any x and continuous in x for almost all t. Then to every function $x(t)$ being measurable on the interval $(0,1)$ we may assign the function

$$(Fx)(t) \;=\; f(t, x(t)), \quad t \in (0,1).$$

The operator F defined in such a way is called the superposition operator. This operator is one of the simplest and most important operators investigated in the nonlinear functional analysis. One of the most important result concerning the superposition operators contained in the below given theorem due to Krasnosel'skii [3].
Theorem 2.1 The superposition operator F maps L^1 into itself if and only if

$$|f(t,x)| \leq \; |c(t)| + k|x| \quad \text{for all} \;\; t \in (0,1)$$

and $x \in R$, where c(t) is a function from L^1 and k is a nonnegative constant.

Now let E be a Banach space with zero element 0 and X be a nonempty bounded subset of E, moreover denote by $B_r \;=\; B(0,r)$ the closed ball in E centered at 0 and with radius r.
In the sequel we shall need some criteria for compactness in measure. The complete description of compactness in measure was given by Fre'chet [11] , but the following sufficient condition will be more convenient for our purposes (see[3]).
Theorem 2.2 Let X be a bounded subset of L^1. Assume that there is a family of subsets $(\Omega_c)_{o \leq c \leq b-a}$ of the interval (a,b) such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X,\; x(t_1) \leq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$, then the set X is compact in measure.

The measure of weak noncompactness defined by De Blasi [5]-[6] is given by,

$$\beta(X) = inf(r > 0 : there\ exists\ a\ weakly\ copmact\ subset\ Y\ of\ E\ such\ that\ X \subset Y + K_r)$$

The function $\beta(X)$ possess several useful properties which may be found in [5].
The convenient formula for the function $\beta(X)$ in L^1 was given by Appell and De Pascale (see [1])

$$\beta(X) = \lim_{\epsilon \to 0}(sup_{x \in X}(sup[\int_D |x(t)|dt : D \subset [a,b], meas\ D \leq \epsilon])), \tag{5}$$

where the symbol meas D stands for Lebesgue measure of the set D.
Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see[2]) defined by

$$\chi(X) = inf(r > 0 : \ there\ exists\ a\ finite\ subset\ Y\ of\ E\ such\ that\ X \subset Y + K_r)$$

In the case when the set X is compact in measure, the Hausdorff and De Blasi measures of noncompactness will be identical. Namely we have (see[10])
Theorem 2.3 Let X be an arbitrary nonempty bounded subset of L^1. If X is compact in measure then $\beta(X) \ = \ \chi(X)$.

Finally we will recall the fixed point theorem du to Darbo [5]-[6].
Theorem 2.4 Let G be a nonempty, bounded, closed and convex subset of E and let $H : G \rightarrow G$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ, i.e.,there exists a constant $\alpha \in [0, 1)$ such that $\chi(HX) \ \leq \ \alpha\, \chi(X)$ for any nonempty subset X of G. Then H has at least one fixed point in the set G.

2.1 Fractional order integral operators

This section is devoted to study the definitions and some properties of the fractional order integral operators. Let β be a positive real number
Definition 1 The fractional order integral of order β of $f \in L_1$ is defined by (see [12])

$$I^{\beta} f(t) \ = \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \tag{6}$$

where (it is proved that)
(i) $I^{\gamma}\, I^{\beta}\, f(t) \ = \ I^{\gamma+\beta}\, f(t), \ \gamma, \beta > \ 0$.
(ii) The fractional order integral operator I^{β} maps L_1 into itself continuously.
(iii) $\lim_{\beta\rightarrow 1} I^{\beta}\, f(t) \ = \ \int_0^t \, f(s) ds.$
(iv) $\lim_{\beta\rightarrow 0} I^{\beta}\, f(t) \ = \ f(t).$
And the following Lemma was proved in ([9])
Lemma 2 Let the operator I^{β} be defined on L_1, then it maps the monotonic nondecrasing positive functions into functions of the same type.
Definition 2 The Riemann-Liouville fractional order derivative of order $\beta \in (0, 1)$ of the function f is given by

$$D^{\beta} f(t) \ = \ \frac{d}{dt} I^{1-\beta} f(t)$$

3 Existence of solutions of a quadratic functional integral equation

Assume that the following assumptions are satisfied

(i) $a : I \to R_+ = [\,0, +\infty\,)$ is integrable and nondecreasing function on I;

(ii) $f(t,x) = f : I \times R_+ \to R_+$ satisfy Carathéodory conditions (i.e. it is measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0,1]$) and there exists a function $m(t) \in L_1$ such that

$$f(t,x) \leq |m(t)|.$$

Moreover f is nondecreasing with respect to both variables separately;

(iii) $g(t,x) = g : I \times R_+ \to R_+$ satisfy Carathéodory conditions and there exists a function $a_1(t) \in L_1$ and a constant b such that

$$g(t,x) \leq |a_1(t)| + b\,|\,x\,| \quad \forall\,(t,x) \in I \times R_+.$$

Moreover, g is nondecreasing with respect to both variables separately;

(iv) $\phi : I \to I$ is increasing, absolutely continuous on I and there exists a constant $M_1 > 0$ such that $\phi'(t) \geq M_1$ on I ;

(v) $k : I \times I \to R_+$ is measurable in both variables and the operator K defined by

$$(Ky)(t) = \int_0^t k(t,s)\, y(s)\, ds, \quad t \in [0,1]$$

maps nondecreasing positive function $x \in L_1$ into function of the same type and

$$\int_s^1 |\,k(t,s)\,|\, m(t)\, dt < M, \qquad s \in [0,1].$$

Now, we are ready to prove the existence theorem.

Theorem 3.1 Let the assumptions (i)-(v) be satisfied. If $b\,M < M_1$, then the quadratic functional integral equation (1) has at least one positive nondecreasing solution $x \in L_1(I)$.

Proof. Let the operator H be defined by the formula

$$(Hx)(t) = a(t) + f(t,x(t)) \int_0^t k\,(t,s)\, g(s, x(\phi(s)))\, ds.$$

Let $x \in L_1$, then by assumptions (i),(ii) and (iii) we find that

$$|\,(Hx)(t)\,| \leq |\,a(t)\,| + |\,f(t,x(t))\,| \int_0^t k(t,s)(\,|a_1(s)| + b\,|\,x(\phi(s))\,|\,)\, ds$$

$$\leq |\,a(t)\,| + m(t) \int_0^t k(t,s)(\,|a_1(s)| + b\,|\,x(\phi(s))\,|\,)\, ds.$$

This implies that

$$||\,Hx\,|| = \int_0^1 |\,(Hx)(t)\,|\, dt$$

$$\leq \ || \ a \ || \ + \ \int_0^1 \ m(t) \ \int_0^t \ k(t,s)(\ |a_1(s)| \ + \ b \ | \ x(\phi(s)) \ | \ ) \ ds \ dt$$

$$= \ || \ a \ || \ + \ \int_0^1 \ \int_s^1 \ k(t,s) \ m(t) \ dt \ (\ |a_1(s)| \ + \ b \ | \ x(\phi(s)) \ | \ ) \ ds$$

$$\leq \ || \ a \ || \ + \ M \ \int_0^1 \ (\ |a_1(s)| \ + \ b \ | \ x(\phi(s)) \ | \ ) \ ds$$

$$\leq \ || \ a \ || \ + \ M \ \int_0^1 \ |a_1(s)| \ ds \ + \ M \ b \ \int_0^1 \ | \ x(\phi(s)) \ | \ ds$$

$$\leq \ || \ a \ || \ + \ M \ || \ a_1 \ || \ + \ M \ b \ {M_1}^{-1} \ \int_{\phi(0)}^{\phi(1)} \ | \ x(u) \ | \ du$$

$$\leq \ || \ a \ || \ + \ M \ || \ a_1 \ || \ + \ M \ b \ {M_1}^{-1} \ \int_0^1 \ | \ x(u) \ | \ du$$

which gives

$$|| \ Hx \ || \ \leq \ || \ a \ || \ + \ M \ || \ a_1 \ || \ + \ M \ b \ {M_1}^{-1} \ || \ x \ || \tag{7}$$

and proves that $Hx \in L_1$.
Moreover the estimate (7) shows that the operator H maps B_r into it self, where

$$r \ = \ [\ || \ a \ || \ + \ \ M \ || \ a_1 \ || \ ] \ [\ 1 \ - \ M \ b \ {M_1}^{-1} \ ]^{-1}.$$

Let $Q_r \subset B_r$ consisting of all functions positive and nondecreasing on I.
Clearly Q_r is nonempty, bounded, closed and convex ([3]). Now Q_r is a bounded subset of L_1 consisting of all functions positive and nondecreasing on I, then Theorem 2.2 shows that Q_r is compact in measure [3].
Now, we shall show that the operator H transforms positive and nondecreasing function into function of the same type.
First if $x \in Q_r$ is an arbitrary function, then $x(\phi(t))$ is positive and nondecreasing function on I. Therefore

$$\int_0^t \ k(t,s) \ g(s, x(\phi(s))) \ ds, \quad t \in (0,1)$$

is positive and nondecreasing functions on I.
Thus the operator H maps Q_r into itself. By using assumption (ii),(iii) and (v), the operator H is continuous on Q_r (see [13] and [4]).
Let X be a nonempty subset of Q_r. Fix $\epsilon > 0$ and take a measurable subset $D \subset I$ such that $meas. \ D \ \leq \ \epsilon$. Then for any $x \in X$, we get

$$|| \ Hx \ ||_D \ = \ \int_D \ | \ (Hx)(t) \ | \ dt \ \leq \ \int_D \ | \ a(t) \ | \ dt$$

$$+ \ \int_D \ m(t) \ \int_0^t \ k(t,s)(\ a_1(s) \ + \ b \ | \ x(\phi(s)) \ | \ ) \ ds \ dt$$

$$\leq \ || \ a \ ||_D \ + \ M \ || \ a_1 \ ||_D \ + \ b \ {M_1}^{-1} \ M \ \int_{\phi(D)} \ | \ x(u) \ | \ ds.$$

But

$$\lim_{\epsilon \to 0} \{ sup\{ \int_D | a(t) | dt : D \subset I, \quad meas.D < \epsilon \} \} = 0.$$

and

$$\lim_{\epsilon \to 0} \{ sup\{ \int_D | a_1(t) | dt : D \subset I, \quad meas.D < \epsilon \} \} = 0.$$

Then we obtain

$$\beta(Hx(t)) \leq M\ b\ {M_1}^{-1}\ \beta(x(t))$$

and

$$\beta(HX) \leq M\ b\ {M_1}^{-1}\ \beta(X).$$

which by Theorem 2.3 implies that

$$\chi\ (HX) \leq M\ b\ {M_1}^{-1}\ \chi(X).$$

Since $M\ b\ {M_1}^{-1} < 1$, it follows, from Theorems 2.3 and 2.4, that H is contraction and has at least one fixed point in Q_r which proves that the nonlinear quadratic functional integral equation (1) has at least one positive nondecreasing solution $x \in L_1[0,1]$. ■

4 Fractional order quadratic functional integral equation

Consider now the quadratic functional integral equation (2). Now we have the following theorem.

Theorem 4.1 Let $\frac{1}{\Gamma(\beta)} \int_s^1 (t-s)^{\beta-1}\ m(t)\ dt < M$. Then under the assumptions of Theorem 3.1. The nonlinear quadratic functional integral equation of fractional order

$$x(t) = a(t) + f(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ g(s, x(\phi(s)))\ ds \qquad t \in [0,1], \quad \beta \in (0,1),$$

has at least one positive nondecreasing solution $x \in L_1$.

Proof. Let $k(t,s) = \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}$. Then from the properties of the fractional order integral operator (subsection 2.1) we deduce that the operator

$$Kx(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ x(s)\ ds, \quad \beta \in (0,1)$$

satisfies the assumption (v) of Theorem 3.1 and the results follows from the results of Theorem 3.1.

Corollary 4.1 Under the assumptions of Theorem 3.1, with $a(t) = 0$ and $f(t, x(t)) = 1$, the nonlinear functional equation (4) has at least one positive nondecreasing solution $x \in L_1$.

Proof. Putting $a(t) = 0$ and $f(t,x) = 1$. Letting $\beta \to 0$ the quadratic functional integral equation (1) will be the functional equation (4) and the results follows from Theorem 3.1.

5 Fractional order functional differential equations

For the initial value problem of the nonlinear fractional order differential equation (3) we have the following theorem.

Theorem 5.1 Under the assumptions of Theorem 3.1, with $a(t) = a\,\frac{t^{\beta-1}}{\Gamma(\beta)}$ and $f(t, x(t) = 1$, the Cauchy type problem (3) has at least one positive nondecreasing solution $x \in L_1$.

Proof. Integrating (3) we obtain the integral equation

$$x(t) = a\,\frac{t^{\beta-1}}{\Gamma(\beta)} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, g(s, x(\phi(s)))\, ds \qquad t \in [0,1] \tag{8}$$

which by Theorem 3.1 has the desired solution.

Operating by D^β on (8) we obtain the problem (3). So the equivalence between problem (3) and integral equation (8) is proved and then the results follow from Theorem 3.1.

References

[1] J. Appel and E. De Pascale, Su alcuni parameteri connssi con la misura dinoncompattezza di Hausdorff in spazi di functioni misurabili, *Boll. Un. Mat. Ital.* (6), 3-B (1984), 497-515.

[2] J. Banas and K. Goebel, *Measure of noncompactness in Banach spaces*, Lect. Notes in pure and applied Math. 60, M. Dekker, New York and Basel 1980.

[3] J. Banas, On the superposition operator and integrable solutions of some functional equations, *Nonlin. Analysis T.M.A.* Vol. 12(1988), 777-784.

[4] J. Banas, Integrable solutions of Hammerstein and Urysohn integral equations *J. Austral Math. Soc.* (series A) 46 (1989), 51-68.

[5] J. Banas, M. Lecko, W. G. El-Sayed, Existence theorems of some quadratic integral equation, *J.Math. Anal. Appl.* 227 (1998), 276 - 279.

[6] J. Banas, A. Martininon, Monotonic solutions of a quadratic integral equation of Volterra type, *Comput. Math. Appl.* 47 (2004), 271 - 279.

[7] J. Banas, J. R. Martin and K. Sadarangani, On the solution of a quadratic integral equation of Hammerstein tupe, *Mathematical and Computer Modelling,* 43 (2006), 97-104.

[8] J. Banas, B. Rzepka, Monotonic solutions of a quadratic integral equations of fractional order *J.Math. Anal. Appl.* 332 (2007), 1371 -1378.

[9] A. M. A El-Sayed, W. G. El-Sayed and O. L Moustafa On some fractional functional equations *Pure Math. and Appl.* Vol. 6,No.4 (1995), 321-332.

[10] F. S. De Blasi, On a property of the unit sphere in Banach space, *Bull. Math. Soc. Sci. Math. R.S. Roum.(N.S.)* 21 (1977), 259-262.

[11] N. Dunford and J. Schwartz, *Linear operators I*, Int. Publ., Leyden (1963)

[12] I. Podlubny, *Fractional differential equations*, Academic Press, (1999).

[13] P.P. Zabrejko, A.I. Koshelev, M.A. Krasnoselskii, S.G.Mikhlin, L.S. Rakovshchik and V.J. Stecenko *Integral equations* Noordhoff, Leyden (1975).

(Received June 18, 2008)