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Positive solutions of second-order semipositone singular three-point boundary value problems

Yingxin Guo

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China

Abstract

In this paper we prove the existence of positive solutions for a class of second order semipositone singular three-point boundary value problems. The results are obtained by the use of a Guo-Krasnoselskii's fixed point theorem in cones.

Keywords. Singular boundary value problems, Semipositone, Fixed point, Positive solution

MSC: 34B15, 34B25.

1 Introduction

In this paper, we study the positive solutions for the following second-order semipositone singular boundary value problems (BVP):

$$\begin{cases} -u'' = \lambda h(t) f(t,u) + \lambda g(t,u), & 0 < t < 1, \\ u(0) = u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\eta \in (0,1), \alpha \in (0,1)$ is a constant, f, g may be singular at $t = 0, 1$.

The second-order boundary value problem arises in the study of draining and coating flows. Choi [1] obtained the following results in 1991.

Choi's Theorem. Let $f(t,u) = p(t)e^u, h(t) \equiv 1, g(t,u) \equiv 0, \alpha = 0$ and assume $p \in C^1(0,1), p(t) > 0$ in (0,1) and $p(t)$ can be singular at $t = 0$, but is at most $O(\frac{1}{t^{2-\delta}})$ as $t \to 0^+$ for some $\delta > 0$.

Then there exists a $\lambda^* > 0$ such that (1.1) has a positive solution for $0 < \lambda^* < \lambda$.

Wong [2] later obtained the similar results in 1993 when $f(t,u) = p(t)q(u), \alpha = 0, h(t) \equiv 1$ where $p(t) > 0$ is singular at 0 and at most $O(\frac{1}{t^\alpha})$ as $t \to 0^+$ for some $\alpha \in [0,2)$; h is locally Lipschitz continuous, increasing. Ha and Lee [3] obtained in 1997 the similar results. Recently Agarwal *et al.* [4] improved the above results and obtained the results when $0 < f(t,u) \le M_\eta p(t), p(t) \in C([0,1],[0,\infty)), M_\eta$ is a positive constant for each given $\eta > 0$ and satisfying $\int_0^1 tp(t)dt < \infty$. But

Corresponding author. E-mail addresses: yxguo312@163.com
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in the existing literature, few people considered the BVP (1.1). Only a handful of papers [8-10] have appeared when the nonlinearity term is allowed to change sign; Moreover most of them treated with semipositone problems of the form $\alpha = 0, h(t) \equiv 1, f(t,u) + M \geq 0$ for some $M > 0$. It is value to point out that g may not to be nonnegative in this paper. we obtain an interval of λ which ensures the existence of at least one positive solution of BVP (1.1). Our results are new and different from those of [1-6]. Particularly, we do not use the method of lower and upper solutions which was essential for the technique used in [1-5] .

This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results. In section 3, by using Krasnoselskii's fixed point theorem in cones, we discuss the existence of positive solutions of the BVP(1.1). In each theorem, an interval of eigenvalues is determined to ensure the existence of positive solutions of the BVP(1.1)

2 Preliminaries and lemmas

Firstly, let us list the following assumptions that are used throughout the paper:

(A) $h(t) \in C((0,1),[0,+\infty)), h(t) \not\equiv 0$, and

$$\int_0^1 G(s,s)h(s)ds < +\infty.$$

(B) $f(t,u) \in C((0,1) \times [0,+\infty),[0,+\infty))$, and there exists constants $m_1 \geq m_2 \geq 1$ such that for any $t \in (0,1), u \in [0,+\infty)$,

$$c^{m_2} f(t,u) \leq f(t,cu) \leq c^{m_1} f(t,u), \forall\ c \geq 1.$$

(C) $g(t,u) \in C((0,1),(-\infty,+\infty))$, further, for any $t \in (0,1)$ and $u \in [0,+\infty)$, there exists a function $q(t) \in L^1((0,1),(0,+\infty))$ such that $|g(t,u)| \leq q(t)$.

(D)

$$0 < \int_0^1 t(1-t)[h(t)f(t,1) + q(t)]dt < +\infty.$$

Remark 2.1. *By (B), for any* $c \in [0,1], (t,x) \in (0,1) \times [0,+\infty)$, *we easily get*

$$c^{m_1} f(t,u) \leq f(t,cu) \leq c^{m_2} f(t,u).$$

Remark 2.2. *If* $f(t,x)$ *satisfies* (B), *then for any* $t \in (0,1), x \in [0,+\infty), f(t,x)$ *is increasing on* $x \in [0,+\infty)$, *and for any* $[m,n] \subset (0,1)$, *we have*

$$\lim_{x\to+\infty} \min_{t\in[m,n]} \frac{f(t,x)}{x} = +\infty.$$

Proof. The increasing property of $f(t,x)$ is obvious, and since that if we choose $x>1$, from (B), we have $f(t,x)\geq x^{m_2}f(t,1)$, so

$$\frac{f(t,x)}{x}\geq x^{m_2-1}f(t,1),\forall t\in(0,1),$$

the

$$\lim_{x\to+\infty}\min_{t\in[m,n]}\frac{f(t,x)}{x}=+\infty.$$

is obtained.

we consider the three-point BVP

$$\begin{cases} u''+h(t)=0, \qquad 0<t<1,\\ u(0)=u(1)=\alpha u(\eta). \end{cases} \tag{2.1}$$

where $\eta\in(0,1)$.

Lemma 2.1. *Let $\alpha\neq 1, h\in L^1[0,1]$, then the there-point* BVP

$$\begin{cases} u''+h(t)=0, \qquad 0<t<1,\\ u(0)=u(1)=\alpha u(\eta). \end{cases}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)h(s)ds,$$

where $G(t,s)=g(t,s)+\frac{\alpha}{1-\alpha}g(\eta,s)$, here

$$g(t,s)=\begin{cases} t(1-s), \qquad 0\leq t\leq s\leq 1,\\ s(1-t), \qquad 0\leq s\leq t\leq 1. \end{cases}$$

Proof. From $u''=-h(t)$ we have

$$u'(t)=-\int_0^t h(s)ds+B.$$

For $t\in[0,1]$, integrating from 0 to t we get

$$u(t)=-\int_0^t\Big(\int_0^x h(s)ds\Big)dx+Bt+A.$$

which means that

$$u(t)=-\int_0^t(t-s)h(s)ds+Bt+A.$$

So,

$$\begin{aligned} u(0)&=A\\ u(1)&=-\int_0^1(1-s)h(s)ds+B+A.\\ u(\eta)&=-\int_0^\eta(\eta-s)h(s)ds+B\eta+A. \end{aligned}$$

Combining this with $u(0) = u(1) = \alpha u(\eta)$ we conclude that

$$
\begin{aligned}
B &= \int_0^1 (1-s)h(s)ds, \\
A &= -\frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)h(s)ds + \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)h(s)ds \\
&= \frac{\alpha}{1-\alpha}\int_0^\eta (1-\eta)sh(s)ds + \frac{\alpha}{1-\alpha}\int_\eta^1 \eta(1-s)h(s)ds. \\
&= \frac{\alpha}{1-\alpha}\int_0^1 g(\eta, s)h(s)ds.
\end{aligned}
$$

Therefore, the three-point BVP has a unique solution

$$
\begin{aligned}
u(t) &= -\int_0^t (t-s)h(s)ds + t\int_0^1 (1-s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 g(\eta, s)h(s)ds \\
&= \int_0^t s(1-t)h(s)ds + \int_t^1 t(1-s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 g(\eta, s)h(s)ds \\
&= \int_0^1 g(t, s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 g(\eta, s)h(s)ds \\
&= \int_0^1 G(t, s)h(s)ds.
\end{aligned}
$$

This completes the proof.□

Remark 2.3. (i) It is obvious that the Green's function of BVP(2.1) $G(t, s)$ is continuous and $G(t, s) \geq 0$ for any $0 \leq t, s \leq 1$. Moreover we easily get $G(t, s) \leq G(s, s)$, and

$$
\begin{aligned}
G(s, s) &= g(s, s) + \frac{\alpha}{1-\alpha}g(\eta, s) \\
&\leq s(1-s) + \frac{\alpha}{1-\alpha}s(1-s) \\
&\leq \frac{1}{1-\alpha}s(1-s) \\
&\leq \frac{1}{4(1-\alpha)}.
\end{aligned}
$$

(ii) For any $t_0 \in (0, 1)$, the Green's function G(t,s) of BVP(2.1) has the following property:

$$\begin{aligned}
\frac{G(t,s)}{G(t_0,s)} &= \frac{g(t,s)+\frac{\alpha}{1-\alpha}g(\eta,s)}{g(t_0,s)+\frac{\alpha}{1-\alpha}g(\eta,s)} \\
&\geq \frac{g(t,s)}{g(t_0,s)} \\
&= \begin{cases} \dfrac{t}{t_0}, & t_0, t \leq s, \\ \dfrac{t(1-s)}{s(1-t_0)}, & t \leq s \leq t_0, \\ \dfrac{1-t}{1-t_0}, & s \leq t, t_0, \\ \dfrac{s(1-t)}{t_0(1-s)}, & t_0 \leq s \leq t, \end{cases} \geq t(1-t).
\end{aligned}$$

Let $X = C[0,1]$ be a real Banach space endowed with the norm $||x|| = \max_{t\in[0,1]} |x(t)|$. Let $P = \{x \in C[0,1] : x(t) \geq 0\}$ and $K = \{x \in P : x(t) \geq t(1-t)\|x\|\}$. Obviously, P, K are cones in $C[0,1]$ and $K \subset P$.

Define the function, for $y \in X$,

$$[y(t)]^* = \begin{cases} y(t), & y(t) \geq 0, \\ 0, & y(t) < 0, \end{cases}$$

and $\phi(t) = \lambda \int_0^1 G(t,s)q(s)ds$, which is the solution of the BVP

$$\begin{cases} x'' + \lambda q(t) = 0, & 0 < t < 1, \\ x(0) = x(1) = \alpha x(\eta). \end{cases}$$

We firstly consider the boundary value problem

$$\begin{cases} -x'' = \lambda[h(t)f(t,[x(t)-\phi(t)]^*) + g(t,[x(t)-\phi(t)]^*) + q(t)], & 0 < t < 1, \\ x(0) = x(1) = \alpha x(\eta), \end{cases} \tag{2.2}$$

We will show there exists a solutions x_1 for the BVP (2.2) with $x_1(t) \geq \phi(t), t \in [0,1]$. If this is true, then $u(t) = x_1(t) - \phi(t)$ is a nonnegative solutions (positive on (0,1)) of the BVP (2.2). In fact, since for any $t \in (0,1)$,

$$-u'' - \phi'' = \lambda[h(t)f(t,u) + g(t,u) + q(t)],$$

we have

$$-u'' = \lambda h(t)f(t,u) + \lambda g(t,u).$$

So we can only study the BVP(2.2). For any fixed $x \in P$, choose $0 < a < 1$ such that $a\|x\| < 1$, then $a[x(t)-\phi(t)]^* \leq ax(t) \leq a\|x\| < 1$, so by (B) and Remark 2.1, we have

$$f(t,[x(t)-\phi(t)]^*) \leq (\frac{1}{a})^{m_1} f(t, a[x(t)-\phi(t)]) \leq a^{(m_2-m_1)}\|x\|^{m_2} f(t,1).$$

Therefore, for any $t \in [0,1]$, we have

$$
\begin{aligned}
|Tx(t)| &\le \lambda \int_0^1 G(t,s)[|h(s)f(s,[x(s)-\phi(s)]^*)| + |g(s,[x(s)-\phi(s)]^*)| + q(s)]ds \\
&\le \lambda \int_0^1 G(s,s)[a^{(m_2-m_1)}\|x\|^{m_2} h(s)f(s,1) + 2q(s)]ds \\
&\le \frac{\lambda}{1-\alpha}(a^{(m_2-m_1)}\|x\|^{m_2} + 2) \int_0^1 s(1-s)[h(s)f(s,1) + q(s)]ds \\
&\le +\infty.
\end{aligned}
$$

Define an operator $T : P \to P$ by

$$Tx(t) = \lambda \int_0^1 G(t,s)[|h(s)f(s,[x(s)-\phi(s)]^*)| + |g(s,[x(s)-\phi(s)]^*)| + q(s)]ds, x \in P.$$

Lemma 2.2.[12] *Suppose that E is a Banach space, $T_n : E \to E$ $(n = 1, 2, \cdots)$ are completely continuous operators, $T : E \to E$, and*

$$\lim_{n\to\infty} \max_{\|u\|<r} \|T_n u - Tu\| = 0, \forall\ r > 0,$$

then T is a completely continuous operator.

Lemma 2.3. *Assume that (A), (B) hold, then $T(K) \subset K$ and $T : K \to K$ is completely continuous .*

Proof. For any $x \in K$, let $y(t) = Tx(t)$. By definition of the operator T, we have $x(0) = x(1)$ and $x'' \le 0$, so there exists a $t_0 \in (0,1]$, such that $\|y\| = y(t_0)$. By Remark 2.3 (ii), we have

$$
\begin{aligned}
y(t) &= \lambda \int_0^1 G(t,s)[h(s)f(s,[x(s)-\phi(s)]^*) + g(s,[x(s)-\phi(s)]^*) + q(s)]ds \\
&= \lambda \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)[h(s)f(s,[x(s)-\phi(s)]^*) + g(s,[x(s)-\phi(s)]^*) + q(s)]ds \\
&\ge t(1-t)y(t_0) = t(1-t)\|y\|, t \in [0,1].
\end{aligned}
$$

So $y \in K$, that is $T(K) \subset K$.

Define the function h_n for $n \ge 2$, by

$$
h_n(t) = \begin{cases}
\inf\{h(t), h(\frac{1}{n})\}, & 0 < t \le \frac{1}{n}, \\
h(t), & \frac{1}{n} \le t \le 1 - \frac{1}{n}, \\
\inf\{h(t), h(1-\frac{1}{n})\}, & 1 - \frac{1}{n} \le t \le 1.
\end{cases}
$$

Then $h_n : [0,1] \to [0,+\infty)$ is continuous and $h_n \leq h(t), t \in (0,1)$. Following, for $n \geq 2$, let

$$T_n x(t) = \lambda \int_0^1 G(t,s)[h_n(s)f(s,[x(s)-\phi(s)]^*)+g(s,[x(s)-\phi(s)]^*)+q(s)]ds, x \in P.$$

By the same method as in the beginning, we get $T_n(K) \subset K$. Obviously, T_n is also completely continuous on K for any $n \geq 2$ by an application of Ascoli Arzela theorem (see [11]). Define $D_r = \{x \in K : ||x|| \leq r\}$. Noticing $[x(t)-\phi(t)]^* \leq x(t) \leq \|x\| \leq r < r+1$. Then, for any $t \in [0,1]$, for each fixed $r > 0$ and $x \in D_r$,

$$\begin{aligned}
\|T_n x(t) - Tx(t)\| &\leq \lambda \lim_{n\to\infty} \max_{0\leq t\leq 1} \int_0^1 G(t,s)[h(s)-h_n(s)]f(s,[x(s)-\phi(s)]^*)ds \\
&\leq \lambda \lim_{n\to\infty} \max_{0\leq t\leq 1} \int_0^1 G(t,s)[h(s)-h_n(s)]f(s,r+1)ds \\
&\leq \lambda(r+1)^{m_1} \max_{0\leq t\leq 1} f(t,1) \lim_{n\to\infty} \int_0^1 G(s,s)[h(s)-h_n(s)]ds \\
&\leq \lambda(r+1)^{m_1} \max_{0\leq t\leq 1} f(t,1) \lim_{n\to\infty} \int_{e(n)} G(s,s)h(s)ds \\
&\to 0(n\to\infty),
\end{aligned}$$

where $e(n) = [0,1/n] \cup [(n-1)/n, 1]$. By Lemma 2.2, T_n converges uniformly to T as $n \to \infty$, and therefore T is completely continuous. This completes the proof.$\square$

Lemma 2.4.[7,13] *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open bounded subsets of K with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : K \to K$ be a completely continuous operator such that either*

$$\begin{aligned}
&(1)\ \|Tx\| \leq ||x||, x \in \partial\Omega_1, \mathit{and}\ \|Tx\| \geq ||x||, x \in \partial\Omega_2, \mathit{or} \\
&(2)\ \|Tx\| \geq ||x||, x \in \partial\Omega_1, \mathit{and}\ \|Tx\| \leq ||x||, x \in \partial\Omega_2.
\end{aligned}$$

then T has a fixed point in $\overline{\Omega_2} \setminus \Omega_1$.

3 Main results

In this section, we present and prove our main results.

Theorem 3.1. *Suppose that (A)-(D) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda < \lambda^*$, the BVP (1.1) has at least one $C[0,1] \cap C^2[0,1]$ positive solution.*

Proof. By Lemma 2.3, we know T is a completely continuous operator. Let $\Omega_1 = \{x \in C[0,1] : \|x\| < \frac{1}{1-\alpha} r\}$ where $r = \int_0^1 q(s)ds$. Choose

$$\lambda^* = \min\Big\{1, r\Big[[(\frac{1}{1-\alpha}r+1)^{m_1}+2]\int_0^1 s(1-s)[h(s)f(s,1)+q(s)]ds\Big]^{-1}\Big\}.$$

Then for any $x \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}\|Tx\| &\le \lambda \int_0^1 G(t,s)[h(s)f(s,[x(s)-\phi(s)]^*) + g(s,[x(s)-\phi(s)]^*) + q(s)]ds \\ &\le \lambda \int_0^1 G(s,s)[h(s)f(s,[x(s)-\phi(s)]^*) + 2q(s)]ds \\ &\le \lambda \frac{1}{1-\alpha} \int_0^1 s(1-s)[h(s)f(s,(\frac{1}{1-\alpha}r+1)) + 2q(s)]ds \\ &\le \lambda \frac{1}{1-\alpha}[(\frac{1}{1-\alpha}r+1)^{m_1}+2] \int_0^1 s(1-s)[h(s)f(s,1)+q(s)]ds \\ &\le \frac{1}{1-\alpha} r = \|x\|.\end{aligned}$$

Thus

$$\|Tx\| \le \|x\|, x \in K \cap \partial\Omega_1.$$

On the other hand, choose $[m,n] \subset (0,1)$ and a constant $L > 0$ such that

$$\frac{\lambda L}{\frac{1}{1-\alpha}+1} \min_{m\le t\le n}[h(t)t(1-t)] \min_{m\le t\le n} \int_m^n G(t,s)ds \ge 1.$$

By Remark 2.2, for any $t \in [m,n]$, there exists a constant $D > 0$ such that

$$\frac{f(t,x)}{x} > L, \quad x > D.$$

Choose

$$R = \max\left\{\lambda(\frac{1}{1-\alpha}+1)r, \frac{1}{1-\alpha}r+1, \frac{(\frac{1}{1-\alpha}+1)D}{\min\limits_{m\le t\le n}[t(1-t)]}\right\}$$

and let $\Omega_2 = \{x \in C[0,1] : \|x\| < R\}$, then for any $x \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}x(t)-\phi(t) &= x(t) - \lambda \int_0^1 G(t,s)q(s)ds \\ &\ge x(t) - \frac{\lambda}{1-\alpha}[t(1-t)] \int_0^1 q(s)ds \\ &\ge \left[1 - \frac{\lambda r}{(1-\alpha)R}\right]x(t) \\ &\ge \frac{1}{\frac{1}{1-\alpha}+1}x(t) \ge 0, t \in [0,1].\end{aligned}$$

Then

$$\begin{aligned}\min_{m\le t\le n} x(t) &\ge \min_{m\le t\le n} \frac{1}{\frac{1}{1-\alpha}+1}x(t) \ge \min_{m\le t\le n} \frac{\|x\|}{\frac{1}{1-\alpha}+1}[t(1-t)] \\ &= \frac{R}{\frac{1}{1-\alpha}+1} \min_{m\le t\le n}[t(1-t)] \ge D.\end{aligned}$$

Therefore

$$
\begin{aligned}
\min_{m\le t\le n} Tx(t) &= \min_{m\le t\le n} \lambda \int_0^1 G(t,s)[h(s)f(s,[x(s)-\phi(s)]^*)+g(s,[x(s)-\phi(s)]^*)+q(s)]ds\\
&\ge \min_{m\le t\le n} \lambda \int_0^1 G(t,s)h(s)f(s,[x(s)-\phi(s)]^*)ds\\
&\ge \min_{m\le t\le n} \lambda \int_m^n G(t,s)h(s)L[x(s)-\phi(s)]ds\\
&\ge \frac{\lambda L}{\frac{1}{1-\alpha}+1} \min_{m\le t\le n} \int_m^n G(t,s)h(s)x(s)ds\\
&\ge \frac{\lambda L}{\frac{1}{1-\alpha}+1} \min_{m\le t\le n} \int_m^n G(t,s)h(s)[s(1-s)]\|x\|ds\\
&\ge \frac{\lambda L}{\frac{1}{1-\alpha}+1} \min_{m\le t\le n}[h(t)t(1-t)] \min_{m\le t\le n} \int_m^n G(t,s)ds\|x\|\\
&\ge \|x\|.
\end{aligned}
$$

So

$$\|Tx\| \ge \|x\|, x\in K\cap\partial\Omega_2.$$

By Lemma 2.4, T has a fixed point x with $\frac{1}{1-\alpha}r < \|x\| < R$ such that

$$
\begin{cases}
-x''(t)=\lambda[h(t)f(t,[x(t)-\phi(t)]^*)+g(t,[x(t)-\phi(t)]^*)+q(t)], \quad 0<t<1,\\
u(0)=u(1)=\alpha u(\eta),
\end{cases}
$$

Since $\|x\| > \frac{1}{1-\alpha}r$,

$$
\begin{aligned}
x(t)-\phi(t) &\ge \|x\|t(1-t)-\lambda\int_0^1 G(t,s)q(s)ds\\
&\ge \|x\|t(1-t)-\frac{\lambda}{1-\alpha}[t(1-t)]\int_0^1 q(s)ds\\
&\ge \frac{(1-\lambda)r[t(1-t)]}{1-\alpha}\\
&\ge 0, t\in[0,1].
\end{aligned}
$$

Let $u(t)=x(t)-\phi(t)$, then $u(t)$ is a $C[0,1]\cap C^2[0,1]$ positive solution of the BVP(1.1). We complete the proof.□

In the end of this paper, we point out Theorem 3.2 which easily to be showed by the same method as in the proof of Theorem 3.1:

Theorem 3.2. *Suppose that* $(A),(D)$ *hold, and*

(B*) $f(t,u)\in C((0,1)\times[0,+\infty),[0,+\infty))$, *and there exists constants* $0<m_3\le m_4<1$ *such that for any* $t\in[0,1], x\in[0,+\infty)$,

$$c^{m_4}f(t,u)\le f(t,cu)\le c^{m_3}f(t,u), \forall\ c\in[0,1].$$

(C*) $g(t,u)\in C((0,1),(-\infty,+\infty))$, *further, for any* $t\in(0,1)$ *and* $u\in[0,+\infty)$, *there exists a function* $q(t)\in C([0,1],(0,+\infty))$ *such that* $|g(t,u)|\le q(t)$.

Then there exists a constant $\lambda^* > 0$ such that, for any $\lambda > \lambda^*$, the BVP(1.1) has at least one $C[0,1] \cap C^2[0,1]$ positive solution.

Example. Consider the following second-order semipositone singular boundary value problems (BVP):

$$\begin{cases} -u'' = \lambda\Big[\dfrac{u^{3/2}}{3t(1-t)} + \dfrac{1}{\sqrt{t}}\arctan u\Big], \quad 0 < t < 1, \\ u(0) = u(1) = \dfrac{1}{2}u(\dfrac{1}{2}), \end{cases}$$

Where $\alpha = \frac{1}{2}, \eta = \frac{1}{2}, h(t) = 1, f(t,u) = \frac{u^{3/2}}{3t(1-t)}, g(t,u) = \frac{1}{\sqrt{t}}\arctan u$. Then

$$f(t,cu) = \frac{(cu)^{3/2}}{3t(1-t)} = c^{3/2}\frac{u^{3/2}}{3t(1-t)} = c^{3/2}f(t,u),$$

$$|g(t,u)| = |\frac{1}{\sqrt{t}}\arctan u| \leq \frac{\pi}{2\sqrt{t}} = q(t),$$

$$\int_0^1 G(s,s)h(s)ds = \int_0^{1/2}[s(1-s) + \frac{1}{2}s]ds + \int_{1/2}^1 [s(1-s) + \frac{1}{2}(1-s)]ds = \frac{7}{24},$$

$$\int_0^1 t(1-t)[h(t)f(t,1) + q(t)]dt = \int_0^1 t(1-t)\Big(\frac{1}{3t(1-t)} + \frac{\pi}{2\sqrt{t}}\Big)dt = \frac{5+2\pi}{15}.$$

So (A)-(D) are satisfied. Therefore, by theorem 3.1, for any $0 < \lambda < \lambda^* = \frac{15\pi}{(5+2\pi)[2+(1+2\pi)^{3/2}]}$, the BVP (1.1) has at least one $C[0,1] \cap C^2[0,1]$ positive solution.

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