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THE GENERALIZED APPROXIMATION METHOD AND NONLINEAR HEAT TRANSFER EQUATIONS

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ABSTRACT. Generalized approximation technique for a solution of one-dimensional steady state heat transfer problem in a slab made of a material with temperature dependent thermal conductivity, is developed. The results obtained by the generalized approximation method (GAM) are compared with those studied via homotopy perturbation method (HPM). For this problem, the results obtained by the GAM are more accurate as compared to the HPM. Moreover, our (GAM) generate a sequence of solutions of linear problems that converges monotonically and rapidly to a solution of the original nonlinear problem. Each approximate solution is obtained as the solution of a linear problem. We present numerical simulations to illustrate and confirm the theoretical results.

1. INTRODUCTION

Fins are extended surfaces and are frequently used in various industrial engineering applications to enhance the heat transfer between a solid surface and its convective, radiative environment. For surfaces with constant heat transfer coefficient and constant thermal conductivity, the governing equation describing temperature distribution along the surfaces are linear and can be easily solved analytically. But most metallic materials have variable thermal properties, usually, depending

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on temperature. The governing equations for the temperature distribution along the surfaces are nonlinear. In consequence, exact analytic solutions of such nonlinear problems are not available in general and scientists use some approximation techniques such as perturbation method [1], [2], homotopy perturbation method [3], [4], [5] etc., to approximate the solutions of nonlinear equations as a series solution. These methods have the drawback that the series solution may not always converges to the solution of the problem and hence produce inaccurate and meaningless results.

2. HEAT TRANSFER PROBLEM: HPM METHODS

When using perturbation methods, small parameter should be exerted into the equation to produce accurate results. But the exertion of a small parameter in to the equation means that the nonlinear effect is small and almost negligible. Hence, the perturbation method can be applied to a restrictive class of nonlinear problems and is not valid for general nonlinear problems.

It is claimed that the homoptopy perturbation method does not require the existence of a small parameter and gives excellent results compared to the perturbation method for all values of the parameter, see for example [6, 7, 8]. In these papers, the authors discussed the solutions of temperature distributions in a slab with variable thermal conductivity and the two methods are compared in the field of heat transfer.

However, the claim that the homoptopy perturbation method is independent of the choice of a parameter and gives excellent results compared to the perturbation method for all values of the parameter, is not true. In fact, the solution obtained by the homotopy perturbation method may not converge to the solution of the problem in some cases.

In this paper, we introduce a new analytical method (GAM - Generalized approximation method) for the solution of nonlinear heat flow problems that produce excellent results and is independent of the choice of a parameter. Hence our method can be applied to a much larger class of nonlinear boundary value problems. This method generates a bounded monotone sequence of solutions of linear problems that converges uniformly and rapidly to the solution of the original problem. The results obtained via GAM are compared to those via HPM. For this problem, it is found that GAM produces excellent results compare to homotopy perturbation. We use the computer programme, Mathematica.

Consider one-dimensional conduction in a slab of thickness L made of a material with temperature dependent thermal conductivity $k = k(T)$. The two faces are maintained at uniform temperatures T_1 and T_2 with $T_1 > T_2$. The governing equation describing the temperature distribution

$$\begin{aligned} &\frac{d}{dx}(k\frac{dT}{dx}) = 0,\ x \in [0, L], \\ &T(0) = T_1, \quad T(L) = T_2. \end{aligned} \tag{2.1}$$

see [8]. The thermal conductivity k is assumed to vary linearly with temperature, that is, $k = k_2[1 + \eta(T - T_2)]$, where η is a constant and k_2 is the thermal conductivity at temperature T_2. After introducing the dimensionless quantities

$$\theta = \frac{T - T_2}{T_1 - T_2},\ y = \frac{x}{L},\ \epsilon = \eta(T_1 - T_2) = \frac{k_1 - k_2}{k_2},$$

where k_1 is the thermal conductivity at temperature T_1, the problem (2.1) reduces to

$$\begin{aligned} &-\frac{d^2\theta}{dy^2} = \frac{\epsilon(\frac{d\theta}{dy})^2}{(1 + \epsilon\theta)} = f(\theta, \theta'),\ y \in [0, 1] = I, \\ &\theta(0) = 1, \quad \theta(1) = 0. \end{aligned} \tag{2.2}$$

Three term expansion of the approximate solution of (2.2) by homotopy perturbation method is given by

$$\theta(y) = 1 - y + \frac{\epsilon}{2}(y - y^2) + \epsilon^2(y^2 - \frac{y^3}{2} - \frac{y}{2}),\ y \in I \tag{2.3}$$

see [8].

Results obtained for different values of ϵ via HPM (2.3) are presented in Table 1 and Fig. 1. Clearly, for small value for ϵ ($\epsilon \leq 1$), (2.3) is a good approximation to the solution. However, as ϵ increases, (2.3) deviates from the actual solution of the problem (2.2) and produce inaccurate results.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\epsilon = 0.5$	0.912375	0.824	0.734125	0.642	0.546875	0.448	0.344625	0.236	0.121375
$\epsilon = 0.8$	0.91008	0.82304	0.73696	0.64992	0.56	0.46528	0.36384	0.25376	0.13312
$\epsilon = 1$	0.9045	0.816	0.7315	0.648	0.5625	0.472	0.3735	0.264	0.1405
$\epsilon = 1.5$	0.876375	0.776	0.692125	0.618	0.546875	0.472	0.386625	0.284	0.157375
$\epsilon = 2$	0.828	0.704	0.616	0.552	0.5	0.448	0.384	0.296	0.172
$\epsilon = 2.5$	0.759375	0.6	0.503125	0.45	0.421875	0.4	0.365625	0.3	0.184375
$\epsilon = 3$	0.6705	0.464	0.3535	0.312	0.3125	0.328	0.3315	0.296	0.1945
$\epsilon = 3.5$	0.561375	0.296	0.167125	0.138	0.171875	0.232	0.281625	0.284	0.202375
$\epsilon = 4$	0.432	0.096	-0.056	-0.072	0.	0.112	0.216	0.264	0.208
$\epsilon = 4.5$	0.282375	-0.136	-0.315875	-0.318	-0.203125	-0.032	0.134625	0.236	0.211375

Table 1-Approximate solutions of (2.2) via HPM for different values of ϵ

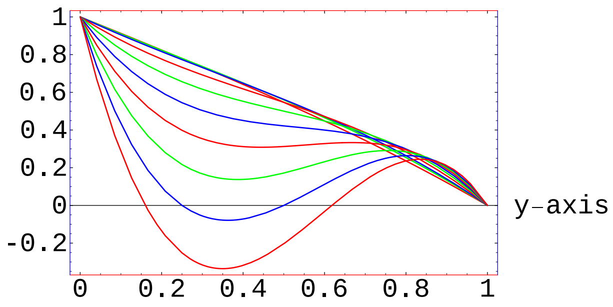

Fig.1; Graphical results obtained via HPM for different values of ϵ.

3. HEAT TRANSFER PROBLEM: INTEGRAL FORMULATION

We write (2.2) as an equivalent integral equation,

(3.1)

$$\theta(y) = (1-y) + \int_0^1 G(y,s) f(\theta(s), \theta'(s)) ds = (1-y) + \int_0^1 G(y,s) \frac{\epsilon(\theta')^2}{(1+\epsilon\theta)} ds,$$

where,

$$G(y,s) = \begin{cases} (1-s)y, & 0 \le y \le s \le 1, \\ (1-y)s, & 0 \le s \le y \le 1, \end{cases}$$

is the Green's function. Clearly, $G(y,s) > 0$ on $(0,1)\times(0,1)$ and since $\frac{\epsilon(\theta')^2}{(1+\epsilon\theta)} \geq 0$, hence, any solution θ of the BVP (2.2) is positive on I. We recall the concept of lower and upper solutions.

Definition 3.1. A function α is called a lower solution of the BVP (2.2), if $\alpha \in C^1(I)$ and satisfies

$$\begin{aligned} &-\alpha''(y) \leq f(\alpha(y), \alpha'(y)), \quad y \in (0,1) \\ &\alpha(0) \leq 1,\ \alpha(1) \leq 0. \end{aligned}$$

An upper solution $\beta \in C^1(I)$ of the BVP (2.2) is defined similarly by reversing the inequalities.

For example, $\alpha = 1-y$ and $\beta = 2-\frac{y^2}{2}$ are lower and upper solutions of the BVP (2.2).

Definition 3.2. A continuous function $h : (0,\infty) \to (0,\infty)$ is called a Nagumo function if

$$\int_\lambda^\infty \frac{sds}{h(s)} = \infty,$$

for $\lambda = \max\{|\alpha(0)-\beta(1)|, |\alpha(1)-\beta(0)|\}$. We say that $f \in C[\mathbb{R}\times\mathbb{R}]$ satisfies a Nagumo condition relative to α, β if for $y \in [\min\alpha, \max\beta] = [0,2]$, there exists a Nagumo function h such that $|f(y,y')| \leq h(|y'|)$.

Clearly,

$$|f(\theta,\theta')| = |\frac{\epsilon(\frac{d\theta}{dy})^2}{(1+\epsilon\theta)}| \leq \epsilon|\theta'^2| = h(|\theta'|) \text{ for } \theta \in [0,2]$$

and since $\int_\lambda^\infty \frac{sds}{h(s)} = \int_\lambda^\infty \frac{sds}{\epsilon s^2} = \infty$, where $\lambda = 2$ in this case. Hence f satisfies a Nagomo condition. Existence of solution to the BVP (2.2) is guaranteed by the following theorem. The proof is on the same line as given in [9, 10] for more general problems.

Theorem 3.3. *Assume that there exist lower and upper solutions $\alpha, \beta \in C^1(I)$ of the BVP (2.2) such that $\alpha \leq \beta$ on I. Assume that $f : \mathbb{R}\times\mathbb{R} \to (0,\infty)$ is continuous, satisfies a Nagumo condition and is non-increasing with respect to θ'. Then the BVP (2.2) has a unique $C^1(I)$*

positive solution θ such that $\alpha(y) \le \theta(y) \le \beta(y)$, $y \in I$. Moreover, there exists a constant C depending on α, β and h such that $|\theta'(y)| \le C$.

Using the relation $\int_\lambda^C \frac{sds}{h(s)} \ge \max\beta - \min\alpha = 2$, we obtain $C \ge 2e^{2\epsilon}$. In particular, we choose $C = 2e^{2\epsilon}$.

4. Heat Transfer Problem: Generalized Approximation Method (GAM)

Observe that

$$\begin{aligned} &f_{\theta\theta}(\theta(s), \theta'(s)) = \frac{2\epsilon^3\theta'^2}{(1+\epsilon\theta)^3} \ge 0,\ f_{\theta'\theta'}(\theta(s), \theta'(s)) = \frac{2\epsilon}{1+\epsilon\theta} \ge 0, \\ &f_{\theta\theta'}(\theta(s), \theta'(s)) = \frac{-2\epsilon^2\theta'}{(1+\epsilon\theta)^2} \text{ and } f_{\theta\theta}f_{\theta'\theta'} = \frac{4\epsilon^4\theta'^2}{(1+\epsilon\theta)^4} = (f_{\theta\theta'})^2. \end{aligned} \tag{4.1}$$

Hence, the quadratic form

$$\begin{aligned} v^T H(f) v &= (\theta - z)^2 f_{\theta\theta} + 2(\theta - z)(\theta' - z') f_{\theta\theta'} + (\theta' - z')^2 f_{\theta'\theta'} \\ &= \left((\theta - z)\sqrt{\frac{\epsilon^3\theta'^2}{(1+\epsilon\theta)^3}} - (\theta' - z')\sqrt{\frac{2\epsilon}{(1+\epsilon\theta)}} \right)^2 \ge 0, \end{aligned} \tag{4.2}$$

where $H(f) = \begin{pmatrix} f_{\theta\theta} & f_{\theta\theta'} \\ f_{\theta\theta'} & f_{\theta'\theta'} \end{pmatrix}$ is the Hessian matrix and $v = \begin{pmatrix} \theta - z \\ \theta' - z' \end{pmatrix}$. Consequently,

$$f(\theta, \theta') \ge f(z, z') + f_\theta(z, z')(\theta - z) + f_{\theta'}(z, z')(\theta' - z'). \tag{4.3}$$

Define $g : \mathbb{R}^4 \to \mathbb{R}$ by

$$g(\theta, \theta';\, z, z') = f(z, z') + f_\theta(z, z')(\theta - z) + f_{\theta'}(z, z')(\theta' - z'), \tag{4.4}$$

then g is continuous and satisfies the following relations

$$\begin{cases} f(\theta, \theta') \ge g(\theta, \theta';\, z, z'), \\ f(\theta, \theta') = g(\theta, \theta';\, \theta, \theta'). \end{cases} \tag{4.5}$$

We note that for every θ, $z \in [\min_{y\in I}\alpha, \max_{y\in I}\beta]$ and $z' \in$ some compact subset of $\mathbb{R}$, g satisfies a Nagumo condition relative to α, β. Hence, there exists a constant C_1 such that any solution θ of the linear BVP

$$\begin{aligned} -\theta''(y) &= g(\theta, \theta';\, z, z'),\ y \in I, \\ \theta(0) &= 1, \quad \theta(1) = 0, \end{aligned}$$

with the property that $\alpha \leq \theta \leq \beta$ on I, must satisfies $|\theta'| < C_1$ on I.
To develop the iterative scheme, we choose $w_0 = \alpha$ as an initial approximation and consider the linear BVP

$$\begin{aligned} -\theta''(y) &= g(\theta, \theta';\, w_0, w_0'),\ y \in I, \\ \theta(0) = 1, &\quad \theta(1) = 0. \end{aligned} \tag{4.6}$$

In view of (4.5) and the definition of lower and upper solutions, we obtain

$$\begin{aligned} g(w_0, w_0';\, w_0, w_0') &= f(w_0, w_0') \geq -w_0'', \\ g(\beta, \beta';\, w_0, w_0') &\leq f(\beta, \beta') \leq -\beta'', \text{ on } I, \end{aligned}$$

which imply that w_0 and β are lower and upper solutions of (4.6). Hence, by Theorem 3.3, there exists a solution w_1 of (4.6) such that $w_0 \leq w_1 \leq \beta$, $|w_1'| < C_1$ on I. Using (4.5) and the fact that w_1 is a solution of (4.6), we obtain

$$-w_1''(y) = g(w_1, w_1';\, w_0, w_0') \leq f(w_1, w_1') \tag{4.7}$$

which implies that w_1 is a lower solution of (2.2). Similarly, we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned} -\theta''(y) &= g(\theta, \theta';\, w_1, w_1'),\ y \in I, \\ \theta(0) = 1, &\quad \theta(1) = 0. \end{aligned} \tag{4.8}$$

Hence, there exists a solution w_2 of (4.8) such that $w_1 \leq w_2 \leq \beta$, $|w_2'| < C_1$ on I.
Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$\alpha = w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_{n-1} \leq w_n \leq \beta,\ |w_n'| < C_1 \text{ on } I,$$

where w_n is a solution of the linear problem

$$\begin{aligned} -\theta''(y) &= g(\theta, \theta';\, w_{n-1}, w_{n-1}'),\ y \in I \\ \theta(0) &= 1,\ \theta(1) = 0 \end{aligned}$$

and is given by
(4.9)

$$w_n(y) = (1-y) + \int_0^1 G(y,s) g(w_n(s), w_n'(s);\, w_{n-1}(s), w_{n-1}'(s)) ds,\ y \in I.$$

The sequence is uniformly bounded and equicontinuous. The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit w on I. From the boundary conditions, we have

$$1 = w_n(0) \to w(0) \text{ and } 0 = w_n(1) \to w(1).$$

Hence w satisfies the boundary conditions. Moreover, by the dominated convergence theorem, for any $y \in I$,

$$\int_0^1 G(y,s)g(w_n(s), w_n'(s);\, w_{n-1}(s), w_{n-1}'(s))ds \to \int_0^1 G(y,s)f(w(s), w'(s))ds.$$

Passing to the limit as $n \to \infty$, we obtain

$$w(y) = (1-y) + \int_0^1 G(y,s)f(w(s), w'(s))ds,\ y \in I,$$

that is, w is a solution of (2.2).
Since $\alpha = 1 - y$, $\beta = 2 - \frac{y^2}{2}$ are lower and upper solutions of the problem (2.2). Hence, any solution θ of the problem satisfies $1 - y \leq \theta \leq 2 - \frac{y^2}{2}$, $y \in I$. In other words, any solution of the problem is positive and is bounded by 2.

5. Convergence Analysis

Define $e_n = w - w_n$ on I. Then, $e_n \in C^1(I)$, $e_n \geq 0$ on I and from the boundary conditions, we have $e_n(0) = 0 = e_n(1)$. In view of (4.5), we obtain

$$-e_n''(t) = f(w(t), w'(t)) - g(w_n(t), w_n'(t); w_{n-1}(t), w_{n-1}'(t)) \geq 0,\ t \in I,$$

which implies that e_n is concave on I and there exists $t_1 \in (0,1)$ such that

$$e_n'(t_1) = 0,\ e_n'(t) \geq 0 \text{ on } [0, t_1] \text{ and } e_n'(t) \leq 0 \text{ on } [t_1, 1]. \tag{5.1}$$

Using the definition of g and the non-increasing property of $f(\theta, \theta')$ with respect to θ, we have

$$
\begin{aligned}
-e_n''(t) &= f(w(t), w'(t)) - g(w_n(t), w_n'(t); w_{n-1}(t), w_{n-1}'(t)),\ t \in I \\
&= f(w_{n-1}(t), w_{n-1}'(t)) + f_\theta(w_{n-1}(t), w_{n-1}'(t))(w(t) - w_{n-1}(t)) \\
&+ f_{\theta'}(w_{n-1}(t), w_{n-1}'(t))(w'(t) - w_{n-1}'(t)) + \frac{1}{2} v^T H(f) v \\
&- g(w_n(t), w_n'(t); w_{n-1}(t), w_{n-1}'(t)) \\
&= f_\theta(w_{n-1}(t), w_{n-1}'(t))(w(t) - w_n(t)) + \\
&f_{\theta'}(w_{n-1}(t), w_{n-1}'(t))(w'(t) - w_n'(t)) + \frac{1}{2} v^T H(f) v \\
&\leq f_{\theta'}(w_{n-1}(t), w_{n-1}'(t)) e_n'(t) + \frac{1}{2} v^T H(f) v,
\end{aligned}
$$

where

$$
v^T H(f) v = \left((w - w_{n-1}) \sqrt{\frac{\epsilon^3 \xi_2^2}{(1 + \epsilon \xi_1)^3}} - (w' - w_{n-1}') \sqrt{\frac{2\epsilon}{(1 + \epsilon \xi_1)}} \right)^2,
$$

where $w_{n-1} \leq \xi_1 \leq w$ and ξ_2 lies between w_{n-1}' and w'.

$$
v^T H(f) v \leq \left(|e_{n-1}| C_2 \epsilon \sqrt{\epsilon} + |e_{n-1}'| \sqrt{2\epsilon} \right)^2 \leq \epsilon (C_2 \epsilon + \sqrt{2})^2 \|e_{n-1}\|_1^2 = d \|e_{n-1}\|_1^2,
$$

where $d = \epsilon(C_2\epsilon + \sqrt{2})^2$, $C_2 = \max\{C,\, C_1\}$ and $\|e_{n-1}\|_1 = \max\{\|e_{n-1}\|,\, \|e_{n-1}'\|\}$ is the C^1 norm. Hence,

$$
-e_n''(t) \leq f_{\theta'}(w_{n-1}(t), w_{n-1}'(t)) e_n'(t) + \frac{d}{2} \|e_{n-1}\|_1^2,\ t \in I,
$$

which implies that

$$
(c_1(t) e_n'(t))' \geq -\frac{d c_1(t)}{2} \|e_{n-1}\|_1^2,\ t \in I, \tag{5.2}
$$

where

$$
c_1(t) = e^{\int f_{\theta'}(w_{n-1}(t), w_{n-1}'(t)) dt} = (1 + \epsilon w_{n-1}(t))^2,\ t \in I.
$$

Clearly $1 \leq c_1(t) \leq (1+\epsilon)^2$ on I. Integrating (5.2) from t to t_1 ($t \leq t_1$), using $e_n'(t_1) = 0$, we obtain

$$
e_n'(t) \leq \frac{d \int_t^{t_1} c_1(s) ds}{2 c_1(t)} \|e_{n-1}\|_1^2,\ t \in [0, t_1] \tag{5.3}
$$

and integrating (5.2) from t_1 to t, we obtain

$$(5.4) \qquad e_n'(t) \geq -\frac{d\int_{t_1}^{t} c_1(s)ds}{2c_1(t)} \|e_{n-1}\|_1^2,\ t \in [t_1, 1].$$

From (5.3) and (5.4) together with (5.1), it follows that

$$(5.5) \quad e_n'(t) \leq \frac{d\int_{t}^{t_1} c_1(s)ds}{2c_1(t)} \|e_{n-1}\|_1^2 \leq \frac{d\int_{0}^{1} c_1(s)ds}{2c_1(t)} \|e_{n-1}\|_1^2,\ t \in [0, 1]$$

and
(5.6)

$$e_n'(t) \geq -\frac{d\int_{t_1}^{t} c_1(s)ds}{2c_1(t)} \|e_{n-1}\|_1^2 \geq -, \frac{d\int_{0}^{1} c_1(s)ds}{2c_1(t)} \|e_{n-1}\|_1^2 m,\ t \in [0, 1].$$

Hence,

$$(5.7) \qquad \|e_n'\| \leq d_1 \|e_{n-1}\|_1^2,$$

where $d_1 = \max\{\frac{d\int_0^1 c_1(s)ds}{2c_1(t)} : t \in I\}$. Integrating (5.5) from 0 to t, using the boundary condition $e_n'(0) = 0$ and taking the maximum over I, we obtain

$$(5.8) \qquad \|e_n\| \leq d_1 \|e_{n-1}\|_1^2,\ t \in I.$$

From (5.7) and (5.8), it follows that

$$\|e_n\|_1 \leq d_1 \|e_{n-1}\|_1^2$$

which shows quadratic convergence.

6. NUMERICAL RESULTS FOR THE GAM

Starting with the initial approximation $w_0 = 1 - y$, results obtained via GAM for $\epsilon = 0.5$, 0.8 and 1, are given in the Tables (Table 1, Table 2 and Table 3 respectively) and also graphically in Fig.2. Form the tables and graphs, it is clear that with only a few iterations it is possible to obtain good approximations of the exact solution of the problem. Moreover, the convergence is very fast. Even for larger values of ϵ, the GAM produces excellent results, see for example, Fig.3 and Fig.4 for ($\epsilon = 2$),($\epsilon = 2$), ($\epsilon = 3$), ($\epsilon = 4$) respectively. In fact, the GAM accurately approximate the actual solution of the problem independent of the choice of the parameters ϵ involved, see Fig.5 .

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.926897	0.850445	0.770062	0.685011	0.594355	0.496892	0.391076	0.274894	0.145703
w_2	0.927888	0.852043	0.771941	0.68693	0.596178	0.498601	0.392748	0.276598	0.147208
w_3	0.92791	0.852085	0.771998	0.686995	0.596245	0.498664	0.3928	0.276637	0.147228
w_4	0.927911	0.852086	0.772	0.686997	0.596247	0.498666	0.392802	0.276638	0.147229

Table 1; Results obtained via GAM for $\epsilon = 0.5$

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.923726	0.844367	0.761495	0.67454	0.582739	0.485077	0.38019	0.266234	0.140679
w_2	0.924981	0.846597	0.764396	0.677808	0.586103	0.48832	0.38316	0.268796	0.142518
w_3	0.925081	0.84679	0.764665	0.67813	0.586448	0.488658	0.38346	0.269028	0.14265
w_4	0.925091	0.846809	0.764692	0.678163	0.586484	0.488693	0.383491	0.269051	0.142664

Table 2; Results obtained via GAM for $\epsilon = 0.8$

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.921729	0.840541	0.756109	0.667965	0.575454	0.477674	0.373374	0.260812	0.137531
w_2	0.923296	0.843436	0.760015	0.672516	0.580267	0.482379	0.377637	0.264317	0.139835
w_3	0.923501	0.843836	0.760579	0.673195	0.581002	0.483104	0.378284	0.264818	0.140122
w_4	0.923533	0.843898	0.760666	0.6733	0.581117	0.483218	0.378386	0.264896	0.140167

Table 3; Results obtained via GAM for $\epsilon = 1$

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.913445	0.824728	0.733929	0.640986	0.545655	0.447457	0.345576	0.238689	0.124668
w_1	0.916286	0.830217	0.741648	0.65033	0.555869	0.457667	0.354822	0.245968	0.12894
w_1	0.917358	0.832312	0.744615	0.653932	0.559799	0.461569	0.358317	0.248672	0.130483
w_1	0.917797	0.833171	0.745834	0.655412	0.561412	0.463168	0.359745	0.249774	0.131109

Table 4; Results obtained via GAM for $\epsilon = 2$

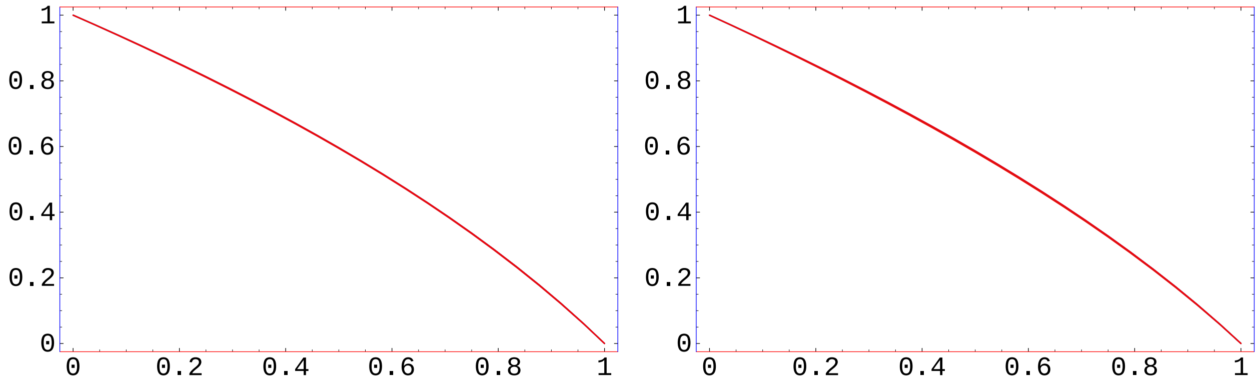

Fig.2. Results obtained by the GAM for $\epsilon = 0.5$ (left graph) and $\epsilon = 0.8$ (right graph).

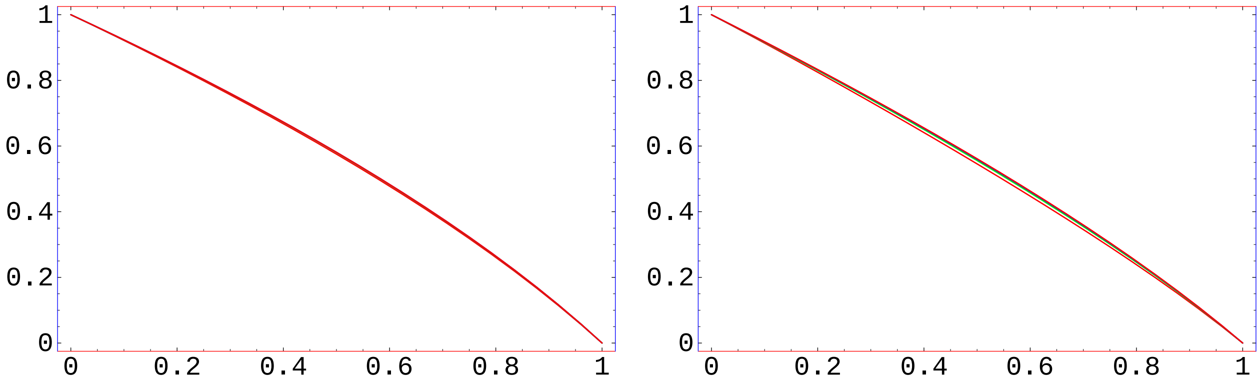

Fig.3. Results obtained by the GAM for $\epsilon = 1$ (left graph) and $\epsilon = 2$ (right graph).

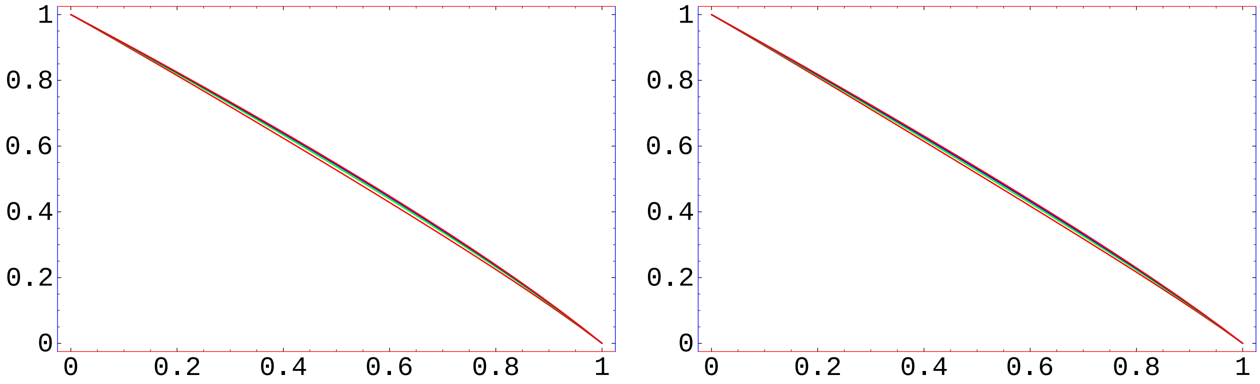

Fig.4. Results obtained by the GAM for $\epsilon = 3$ (left graph) and $\epsilon = 4$ (right graph).

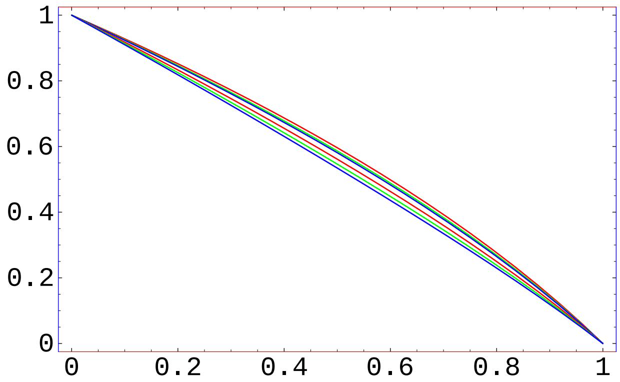

Fig. 5; Graph of the results obtained by the GAM for $\epsilon = 0.5, 0.8, 1, 2, 3, 4$

7. Comparison with homotopy perturbation method

Finally, we compare results via GAM (Red) to the corresponding results via HPM (Green), Fig.6, Fig.7 and Fig.8 for different values of ϵ. Clearly, GAM accurately approximate the solution for any value of ϵ, while for larger value of ϵ, the HPM diverges. This fact is also evident from Fig. 8.

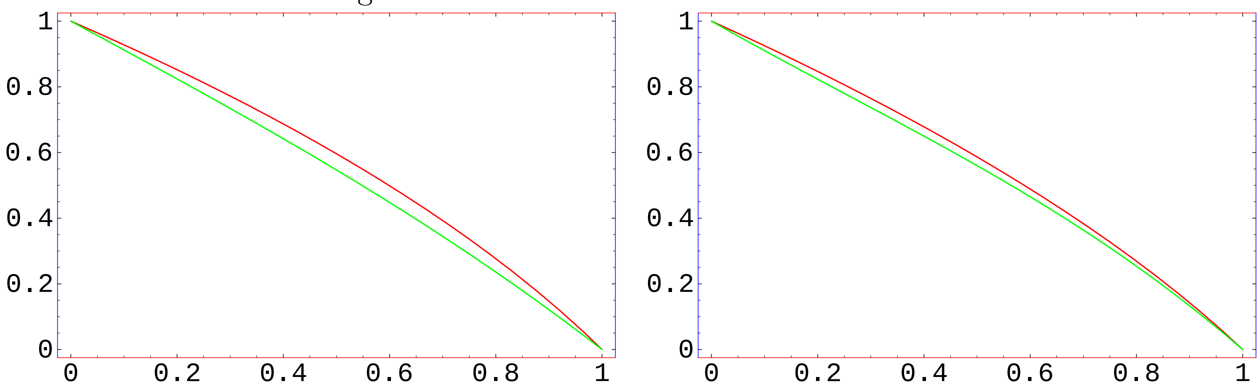

Fig.6. GAM and HPM for $\epsilon = 0.5$ (left graph) and $\epsilon = 0.8$ (right graph).

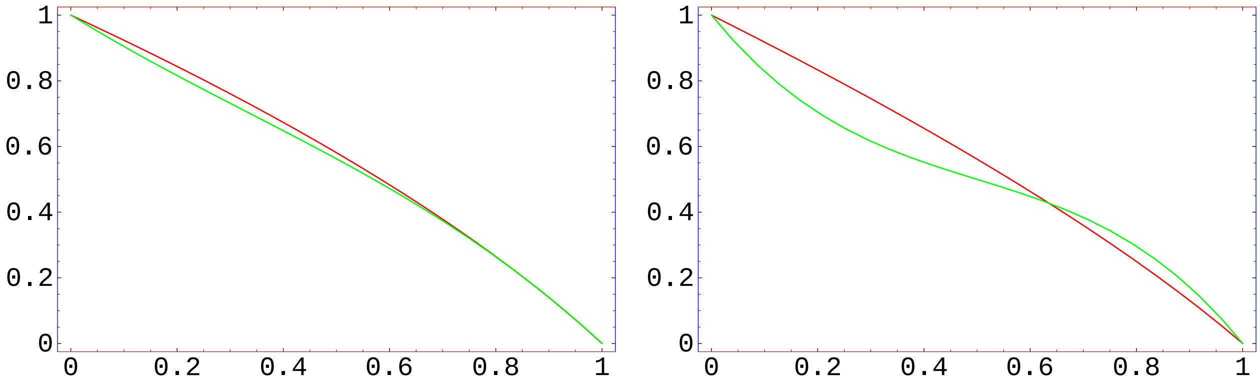

Fig.7. GAM and HPM for $\epsilon = 1$ (left graph) and $\epsilon = 2$ (right graph).

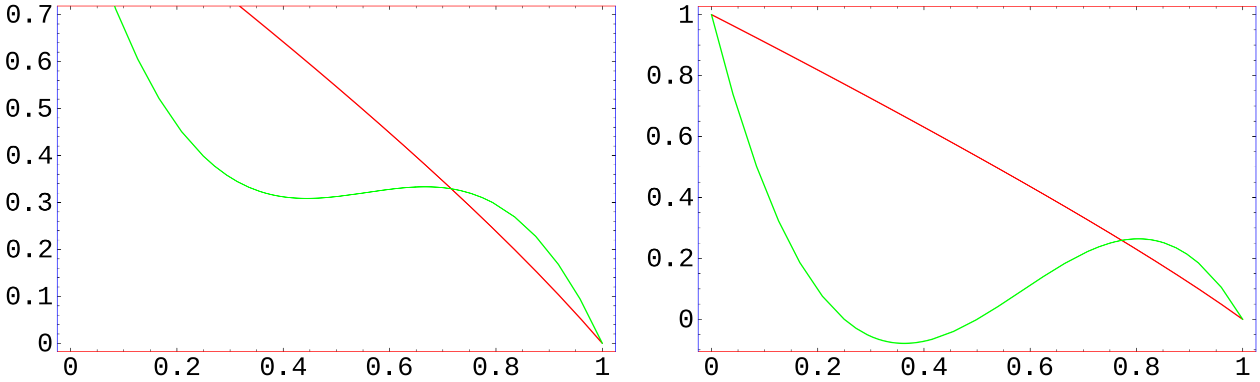

Fig.8. GAM and HPM for $\epsilon = 3$ (left graph) and $\epsilon = 4$ (right graph).

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