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RADIAL SOLUTIONS TO A SUPERLINEAR DIRICHLET PROBLEM USING BESSEL FUNCTIONS

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ABSTRACT. We look for radial solutions of a superlinear problem in a ball. We show that for if n is a sufficiently large nonnegative integer, then there is a solution u which has exactly n interior zeros. In this paper we give an alternate proof to that which was given in [1].

1. INTRODUCTION

In this paper we look for solutions $u : \mathbb{R}^N \to \mathbb{R}$ of the partial differential equation

$$\begin{cases} \Delta u + f(u) = g(|x|) \text{ for } x \in \Omega \\ u = 0 \text{ for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

for $N \geq 2$ and where Ω is the ball of radius $T > 0$ centered at the origin in $\mathbb{R}^N$, Δ is the Laplacian operator, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and where $g \in \mathbb{C}^1[0, T]$.

Motivation: A. Castro and A. Kurepa proved existence of solutions of (1.1) for a wide variety of nonlinearities, f. See [1]. In this paper we give an alternate and, in our estimation, a somewhat easier proof of this result by approximating solutions of (1.1) with appropriate linear equations. In a groundbreaking paper in 1979, B. Gidas, W. Ni, and L. Nirenberg [2] proved that if Ω is a ball then all positive solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

are *spherically symmetric*. K. McLeod, W.C. Troy and F.B. Weissler studied the radial solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega$$
$$\lim_{|x|\to\infty} u(x) = 0$$

for $\Omega \in \mathbb{R}^N$ in [3].

We assume the following hypotheses:

(H1) f is a locally Lipschitz continuous function, f is increasing for large $|u|$ and $f(0) = 0$.

(H2) $\displaystyle\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty$ (that is, f is superlinear).

Let $F(u) = \int_0^u f(s)ds$ and note that from **(H2)** it follows that

$$\lim_{|u|\to\infty} \frac{F(u)}{u^2} = \infty. \tag{1.2}$$

(H3) There exists a k with $0 < k \leq 1$, such that

$$\lim_{u\to\infty} \left(\frac{u}{f(u)}\right)^{\frac{N}{2}} \left(NF(ku) - \frac{(N-2)}{2}uf(u) - \frac{N+2}{2}||g||\ |u| - T||g'||\ |u|\right) = \infty$$

where $||\ ||$ is the supremum norm on $[0, T]$.

(H3*) There exists a k with $0 < k \leq 1$, such that

$$\lim_{u\to-\infty} \left(\frac{u}{f(u)}\right)^{\frac{N}{2}} \left(NF(ku) - \frac{(N-2)}{2}uf(u) - \frac{N+2}{2}||g||\ |u| - T||g'||\ |u|\right) = \infty.$$

(H4) There exists an $\mathbf{M} > 0$ such that

$$NF(u) - \frac{N-2}{2}uf(u) - \frac{N+2}{2}||g||\ |u| - T||g'||\ |u| > -M$$

for all u.

We assume that $u(x) = u(|x|)$ and let $r = |x|$. In this case (1.1) becomes the nonlinear ordinary differential equation

$$u'' + \frac{N-1}{r}u' + f(u) = g(r) \text{ for } 0 < r < T \tag{1.3}$$

$$u'(0) = 0, u(T) = 0. \tag{1.4}$$

Main Theorem: If **(H1)**-**(H4)** are satisfied then (1.1) has infinitely many radially symmetric solutions with $u(0) > 0$. If in place of **(H3)** we have **(H3*)** then (1.1) has infinitely many radially symmetric solutions with $u(0) < 0$.

2. Preliminaries

The technique used to solve (1.3) - (1.4) is the shooting method. That is, we first look at the initial value problem

$$u'' + \frac{N-1}{r}u' + f(u) = g(r) \quad \text{for } 0 < r < T \tag{2.1}$$

$$u(0) = d > 0, u'(0) = 0. \tag{2.2}$$

By varying d appropriately, we attempt to find a d such that $u(r, d)$ has exactly n zeros on $[0, T)$ and $u(T) = 0$.

Multiplying (2.1) by r^{N-1} and integrating on $(0, r)$ gives

$$u' = \frac{-1}{r^{N-1}} \int_0^r t^{N-1}[f(u) - g(t)]dt \tag{2.3}$$

Integrating (2.3) and applying the initial conditions we get

$$u(r) = d - \int_0^r \frac{1}{s^{N-1}} \left(\int_0^s t^{N-1}[f(u) - g(t)]\ dt \right) ds. \tag{2.4}$$

Let $\phi(u)$ be equal to the right hand side of (2.4). It is straightforward to show that $\phi(u)$ is a contraction mapping on $\mathcal{C}[0, \epsilon]$, the set of continuous functions with supremum norm on $[0, \epsilon]$, for some $\epsilon > 0$. Then by the contraction mapping principle there exists a $u \in \mathcal{C}[0, \epsilon]$ such that $\phi(u) = u$. Thus, u is continuous solution of (2.4). Then by **(H1)**, (2.2), and (2.3), we see that u' is continuous on $[0, \epsilon]$. From **(H1)** and (2.3) it follows that $\frac{u'}{r}$ is bounded, that $\lim_{r\to 0^+} \frac{u'}{r}$ exists, and so that $\frac{u'}{r}$ is continuous on $[0, \epsilon]$. Then it follows from (2.1) that u'' is continuous on $[0, \epsilon]$.

In order to show that $u \in \mathcal{C}^2[0, T]$, we define the energy equation of (2.1)-(2.2) as

$$E = \frac{u'^2}{2} + F(u). \tag{2.5}$$

Note that from (1.2) there exists a $J > 0$ such that

$$F(u) \geq -J \tag{2.6}$$

for all $u \in \mathbb{R}$.

From (2.5) and (2.6) we see that

$$u'^2 \leq 2(E + J). \tag{2.7}$$

Using (2.1) we see that

$$\begin{aligned} E' &= -\frac{N-1}{r}u'^2 - g(r)u' \\ &\leq ||g|||u'| \quad \text{(defined in } \textbf{(H3)}) \\ &\leq ||g||\sqrt{2}\sqrt{E+J} \qquad\qquad \text{(by (2.7))}. \end{aligned}$$

Dividing by $\sqrt{E+J}$ and integrating gives

$$\frac{1}{\sqrt{2}}|u'| \leq \sqrt{E(t)+J} \leq \sqrt{F(d)+J} + ||g||t \leq \sqrt{F(d)+J} + ||g||T.$$

Thus, from (2.7) it follows that $|u'|$ is uniformly bounded wherever it is defined and since $u(0) = d$, thus $|u|$ is uniformly bounded wherever it is defined. It follows from this that u and u' are defined on all of $[0, T]$ and from (2.1) it then follows that $u \in \mathcal{C}^2[0, T]$.

The next several arguments presented were essentially originally proved in [1] and are included here for completeness.

Since $f(u) > 0$ for sufficiently large $u > 0$ (by **(H2)**), we see from (2.3) that $u' < 0$ on $(0, r)$ for small $r > 0$ if d is sufficiently large. Let k be the number given by **(H3)**. Now for sufficiently large d it follows that $u' < 0$ on $(0, r_{kd})$ where r_{kd} is the smallest positive value of r such that $u(r_{kd}) = kd$.

Remark 1: First, we want to find a lower bound for r_{kd}. Since f is increasing for large u (by **(H1)**), we see from (2.3) that

$$\begin{aligned} -r^{N-1}u' &\leq [f(d) + ||g||] \int_0^r t^{N-1}dt \\ &= [f(d) + ||g||]\frac{r^N}{N}. \end{aligned}$$

Dividing by r^{N-1} and integrating on $[0, r_{kd}]$ we see that

$$(1-k)d = \int_0^{r_{kd}} -u'dt \leq \int_0^{r_{kd}} \frac{t[f(d) + ||g||]}{N}dt = \frac{t[f(d) + ||g||]}{2N}r_{kd}^2.$$

Thus,

$$r_{kd} \geq \sqrt{\frac{2N(1-k)d}{f(d) + ||g||}}.$$

For sufficiently large d we have $||g|| \leq f(d)$ (by **(H2)**), thus we obtain for sufficiently large d

$$r_{kd} \geq \sqrt{\frac{2N(1-k)d}{2f(d)}}.$$

So,

$$r_{kd} \geq \sqrt{\frac{N(1-k)d}{f(d)}} \tag{2.8}$$

for sufficiently large d.

Remark 2: Because of its appearance in Pohozaev's identity we will see that it will be important to find a lower bound on

$$\int_0^{r_{kd}} t^{N-1}\left(NF(u) - \frac{N-2}{2}u\ f(u) - \frac{N+2}{2}g(t)\ u - t\ g'(t)\ u\right) dt. \tag{2.9}$$

By hypothesis **(H2)**, $F' = f > 0$ for large u. Therefore, F is increasing for large u. Since for large d, u is decreasing for $0 \le t \le r_{kd}$, and $kd \le u(t) \le d$, this implies $F(kd) \le F(u) \le F(d)$. So on $[0, r_{kd}]$ we have

$$\int_0^{r_{kd}} t^{N-1} N F(u)\ dt \ge F(kd)\ r_{kd}^N \quad \text{for large } d \tag{2.10}$$

then by hypothesis **(H1)**, f is increasing for large u and using this we have

$$\int_0^{r_{kd}} t^{N-1} \frac{N-2}{2} u\ f(u)\ dt \le \frac{N-2}{2N} d\ f(d)\ r_{kd}^N \quad \text{for large } d$$

so,

$$-\int_0^{r_{kd}} t^{N-1} \frac{N-2}{2} u f(u) dt \ge -\frac{N-2}{2N} d f(d) r_{kd}^N. \tag{2.11}$$

Now using the estimates in (2.8), (2.10), (2.11) and using the fact that g and g' are bounded, we estimate (2.9) as follows:

(2.12)

$$\int_0^{r_{kd}} t^{N-1} \left(NF(u) - \frac{N-2}{2} u f(u) - \frac{N+2}{2} g(t) u - t g'(t) u \right) dt \ge \left(F(kd) - \frac{N-2}{2N} d f(d) - \frac{N+2}{2N} ||g|| d - \frac{1}{N} T ||g'|| d \right) r_{kd}^N$$

$$\ge \left(NF(kd) - \frac{N-2}{2} d f(d) - \frac{N+2}{2} ||g|| d - T ||g'|| d \right) \left(\frac{1}{N} \left(\sqrt{\frac{N(1-k)d}{f(d)}} \right)^N \right)$$

$$= C(N,k) \left(NF(kd) - \frac{N-2}{2} d f(d) - \frac{N+2}{2} ||g|| d - T ||g'|| d \right) \left(\frac{d}{f(d)} \right)^{\frac{N}{2}}$$

where $C(N,k) = \frac{1}{N} [N(1-k)]^{\frac{N}{2}}$.

Lemma 2.1. *If* ***(H1)*** *-* ***(H4)*** *are satisfied, then*

$$\lim_{d \to \infty} \inf_{[0,T]} E(r,d) = \infty. \tag{2.13}$$

Proof. Let us suppose $0 \le r \le T$. Consider Pohozaev's identity which states

$$\left[r^N E - r^N g(r) u + \frac{N-2}{2} r^{N-1} u u' \right]' = r^{N-1} \left[NF(u) - \frac{N-2}{2} u f(u) - \frac{N+2}{2} g(r) u - r g'(r) u \right].$$

This can be verified by simply differentiating and then using (2.1).

Integrating Pohozaev's identity on $[0, r]$, and using **(H4)** and (2.12) gives

$$r^N E(r,d) - r^N g(r) u + \frac{N-2}{2} r^{N-1} u u' = \int_0^r t^{N-1} \left[NF(u) - \frac{N-2}{2} u f(u) - \frac{N+2}{2} g(t) u - t g'(t) u \right] dt$$

$$= \int_0^{r_{kd}} t^{N-1} \left[NF(u) - \frac{N-2}{2} u f(u) - \frac{N+2}{2} g(t) u - t g'(t) u \right] dt$$

$$+ \int_{r_{kd}}^r t^{N-1} \left[NF(u) - \frac{N-2}{2} u f(u) - \frac{N+2}{2} g(t) u - t g'(t) u \right] dt$$

$$\ge C(N,k) \left(\frac{d}{f(d)} \right)^{\frac{N}{2}} \left[NF(kd) - \frac{N-2}{2} d f(d) - \frac{N+2}{2} ||g|| d - T ||g'|| d \right] - M \left(\frac{r^N - r_{kd}^N}{N} \right).$$

Ignoring the last term on the right hand side we get

(2.14)

$$r^N E(r,d) - r^N g(r) u + \frac{N-2}{2} r^{N-1} u u' \ge C(N,k) \left(\frac{d}{f(d)} \right)^{\frac{N}{2}} \left[NF(kd) - \frac{N-2}{2} d f(d) - \frac{N+2}{2} ||g|| d - T ||g'|| d \right] - \frac{MT^N}{N}$$

Now let us estimate uu'.

First note from (1.2) that there exists a B such that if $|u| \geq B$ then $\dfrac{u^2}{F(u)} \leq 1$. That is if $|u| \geq B$ then $u^2 \leq F(u) \leq F(u) + J$. On other hand if $|u| \leq B$ then $u^2 \leq B^2$. And since $F(u) + J \geq 0$ (by (2.6)) we see that for all u we have

$$u^2 \leq F(u) + J + B^2. \tag{2.15}$$

Using Young's inequality, (2.5), and (2.15) gives us the following:

$$\begin{aligned} uu' &\leq \frac{1}{2}u^2 + \frac{1}{2}u'^2 \\ &\leq (F(u) + J + B^2) + \frac{1}{2}u'^2 \\ &= \left(\frac{1}{2}u'^2 + F(u)\right) + J + B^2 \\ &= E(r,d) + J + B^2. \end{aligned}$$

Substituting this into the left hand side of (2.14), rewriting, and estimating we see that

$$\begin{aligned} r^N E - r^N g(r)u + \frac{N-2}{2}r^{N-1}uu' &\leq T^N E + T^N ||g||\ |u| + \frac{N-2}{2}T^{N-1}|uu'| \\ &\leq T^N E + T^N ||g||^2 + T^N u^2 + \frac{N-2}{2}T^{N-1}[E + J + B^2] \\ &\leq T^N E + T^N ||g||^2 + T^N[E + J + B^2] + \frac{N-2}{2}T^{N-1}[E + J + B^2] \\ &= \left(2T^N + \frac{N-2}{2}T^{N-1}\right)E + T^{N-1}\left(\left(T + \frac{N-2}{2}\right)(J + B^2) + ||g||^2\right) \\ &= C_1 E + C_2 \end{aligned}$$

where $C_1 > 0$ and $C_2 > 0$ depend only on T, N, J, B and $||g||$.

Thus, combining the above with (2.14) gives:

$$C(N,k)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}}\left[NF(kd) - \frac{N-2}{2}df(d) - \frac{N+2}{2}||g||d - T||g'||d\right] - \frac{MT^N}{N} \leq C_1 E + C_2.$$

Thus,

$$C_1 E \geq C(N,k)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}}\left[NF(kd) - \frac{N-2}{2}df(d) - \frac{N+2}{2}||g||d - T||g'||d\right] - C_3$$

where C_3 depends on $T, N, J, B, ||g||$ and M.

By assumption the right hand side of the above inequality goes to infinity as $d \to \infty$. Therefore,

$$\lim_{d\to\infty} \inf_{[0,T]} E(r,d) = \infty.$$

□

Lemma 2.2. *If d is sufficiently large and $u(r_0) = 0$, then $u'(r_0) \neq 0$.*

Proof. By Lemma 2.1, if d is sufficiently large then $\inf_{[0,T]} E(r,d) > 0$. So if $u(r_0) = 0$ then we have $\frac{1}{2}u'(r_0)^2 = E(r_0) \geq \inf_{[0,T]} E(r,d) > 0$. □

Lemma 2.3. *For d sufficiently large u has a finite number of zeros on $[0, T]$.*

Proof. Suppose there exists $0 < z_1 < z_2 < ... < z_n < ... < T$ and $u(z_i) = 0$. Then by the mean value theorem there exists $m_1 < m_2 < ...$ such that $u'(m_k) = 0$ and where $z_k < m_k < z_{k+1} < T$. So there exists $z = \lim_{n\to\infty} z_n$ and by continuity $u(z) = 0$. Also, $\lim_{k\to\infty} m_k = z$ and $u'(z) = 0$ but by the above Lemma 2.2, this cannot happen for sufficiently large d. □

3. Finding zeros

Now we want to show that if d is sufficiently large then $u(r, d)$ will have lots of zeros on $[0, T]$. From (1.2) we know that $F(u) \to \infty$ as $|u| \to \infty$. Therefore, since $\lim_{d\to\infty} \inf_{[0,T]} E(r, d) = \infty$ (by Lemma 2.1), and since $F(u)$ is increasing for large u and decreasing when u is a large negative number, then for sufficiently large d there are exactly two solutions of $F(u) = \frac{1}{2} \inf_{[0,T]} E(r, d)$ which we denote as $h_2(d) < 0 < h_1(d)$. For $d > 0$ sufficiently large we see from **(H2)** that $u''(0) = \frac{-f(d) + g(0)}{N} < 0$ and $u'(0) = 0$ so u is initially decreasing on $(0, r)$. Note that $h_1(d) \to \infty$ as $d \to \infty$. From (2.3) we see that u will be decreasing as long as $f(u) \geq ||g||$. So we see that there is a smallest $r > 0$, $r_1(d)$, such that $u(r_1(d)) = h_1(d)$ and $d \geq u > h_1(d)$ on $[0, r_1(d))$.

Let

$$C(d) = \frac{1}{2} \min_{r\in[0,r_1(d)]} \frac{f(u)}{u} = \frac{1}{2} \min_{u\in[h_1(d),d]} \frac{f(u)}{u}. \tag{3.1}$$

Then by **(H2)** we see that $C(d) \to \infty$ as $d \to \infty$.

Lemma 3.1. *$r_1(d) \to 0$ as $d \to \infty$.*

Proof. To show this we compare

$$u'' + \frac{N-1}{r}u' + \frac{f(u)}{u}u = g(r) \tag{3.2}$$

with initial conditions $u(0) = d > 0$ and $u'(0) = 0$ with

$$v'' + \frac{N-1}{r}v' + C(d)v = 0 \tag{3.3}$$

with initial conditions $v(0) = d$ and $v'(0) = 0$. Note from (3.1) that

$$\frac{f(u)}{u} \geq 2C(d) > C(d) \quad \text{on } [0, r_1(d)]. \tag{3.4}$$

Claim: $u < v$ on $(0, r_1(d)]$ for sufficiently large d.

Proof of the Claim: Since

$$u(0) = d = v(0)$$
$$u'(0) = 0 = v'(0)$$

then for large d we see from (3.4) that

$$u''(0) = \frac{-f(d)}{N} + \frac{g(0)}{N} < -\frac{C(d)}{N}d = v''(0).$$

Thus, $u < v$ on $(0, \epsilon)$ for some $\epsilon > 0$.

Multiplying (3.2) by $r^{N-1}v$, (3.3) by $r^{N-1}u$, and then taking the difference of the resultant equations gives

$$(r^{N-1}(u'v - uv'))' + r^{N-1}uv\left(\frac{f(u)}{u} - \frac{g(r)}{u} - C(d)\right) = 0.$$

Since g is bounded, for sufficiently large d we see from (3.4) that

$$\begin{aligned}\frac{f(u)}{u} - \frac{g(r)}{u} - C(d) &\geq 2C(d) - \frac{||g||}{u} - C(d) \quad \text{on } [0, r_1(d)] \\ &= C(d) - \frac{||g||}{u} \\ &\geq C(d) - \frac{||g||}{h_1(d)} \\ &> 0 \quad (\text{since } C(d) \to \infty \text{ as } d \to \infty \text{ and } h_1(d) \to \infty \text{ as } d \to \infty).\end{aligned}$$

Now integrating this from 0 to r where $0 < r \leq r_1(d)$ and using $u(0) = v(0) = d$ and $u'(0) = v'(0) = 0$ gives

$$u'(r)v(r) - v'(r)u(r) < 0 \quad \text{on } (0, r_1(d)].$$

Suppose now there is a first r_0 with $0 < r_0 \leq r_1(d)$ such that $0 < u(r_0) = v(r_0)$ and $u < v$ on $(0, r_0)$. Then we see from the above inequality that $u'(r_0) < v'(r_0)$. On other hand, $u(r) < v(r)$ on $(0, r_0)$ and $u(r_0) = v(r_0)$. So

$$u(r) - u(r_0) < v(r) - v(r_0) \quad \text{on } (0, r_1(d)].$$

Thus, for $r < r_0$ we have

$$\lim_{r \to r_0^-} \frac{u(r) - u(r_0)}{r - r_0} \geq \lim_{r \to r_0^-} \frac{v(r) - v(r_0)}{r - r_0}$$

which gives

$$u'(r_0) \geq v'(r_0).$$

This is a contradiction since $u'(r_0) < v'(r_0)$. Hence this proves the claim.

Now let $z(r) = \left(r/\sqrt{C(d)}\right)^{\frac{N-2}{2}} v\left(r/\sqrt{C(d)}\right)$. Then

$$z'' + \frac{z'}{r} + \left(1 - \frac{\left(\frac{N-2}{2}\right)^2}{r^2}\right) z = 0. \tag{3.5}$$

The above equation is Bessel's equation of order $\dfrac{N-2}{2}$. Thus, $z(r) = A_1 J_{\frac{N-2}{2}}(r) + A_2 Y_{\frac{N-2}{2}}(r)$ for constants A_1 and A_2 and where $J_{\frac{N-2}{2}}$ is the Bessel function of order $\frac{N-2}{2}$ which is bounded at $r = 0$ and $Y_{\frac{N-2}{2}}$ is unbounded at $r = 0$. Since z is bounded at $r = 0$ and $Y_{\frac{N-2}{2}}$ is not, it must be that $z(r) = A_1 J_{\frac{N-2}{2}}(r)$, and A_1 is a positive constant.

Denoting $\beta_{\frac{N-2}{2},1}$ as the first positive zero of $J_{\frac{N-2}{2}}(r)$, we see that the first positive zero of v is $\dfrac{\beta_{\frac{N-2}{2},1}}{\sqrt{C(d)}}$ and since $u < v$ on $[0, r_1(d)]$ (by the Claim) we see that

$$r_1(d) < \frac{\beta_{\frac{N-2}{2},1}}{\sqrt{C(d)}}.$$

Since $C(d) \to \infty$ as $d \to \infty$ (as mentioned after (3.1)) it then follows that $\lim_{d \to \infty} r_1(d) = 0$. □

Lemma 3.2. *For large d, u has a first positive zero, $z_1(d)$, and $z_1(d) \to 0$ as $d \to \infty$.*

Proof. First we show that u has a zero. We prove this by contradiction. Suppose $u > 0$ on $[0, T]$ and consider $r > r_1(d)$. Then $0 < u < u(r_1(d)) = h_1(d)$ so $F(u) < F(h_1(d))$. Also since $F(h_1(d)) = \dfrac{1}{2} \inf_{[0,T]} E(r, d)$ we obtain

$$\frac{u'^2}{2} + F(h_1(d)) > \frac{u'^2}{2} + F(u) \geq \inf_{[0,T]} E(r, d) = 2F(h_1(d))$$

for $r > r_1(d)$.

Thus,

$$u'^2 \geq 2F(h_1(d)) \quad \text{for } r > r_1(d)$$

and thus

$$-\int_{r_1(d)}^{r} u'(t)dt \geq \int_{r_1(d)}^{r} \sqrt{2F(h_1(d))}dt$$

and since u is decreasing and $u(r_1(d)) = h_1(d)$ this gives

$$h_1(d) - u(r) = u(r_1(d)) - u(r) \geq \sqrt{2F(h_1(d))}(r - r_1(d)) \tag{3.6}$$

so,

$$h_1(d) - \sqrt{2F(h_1(d))}(r - r_1(d)) \geq u(r) > 0.$$

Thus,

$$\frac{h_1(d)}{\sqrt{2F(h_1(d))}} \geq r - r_1(d). \tag{3.7}$$

Evaluating at $r = T$ gives

$$T - r_1(d) \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}}$$

for large d.

Since $h_1(d) \to \infty$ as $d \to \infty$, taking the limit of the above, using Lemma 3.1 and (1.2) we see that

$$0 < T = \lim_{d\to\infty} [T - r_1(d)] \leq \lim_{d\to\infty} \frac{h_1(d)}{\sqrt{2F(h_1(d))}} = 0.$$

This is impossible. Thus u has a first zero, $z_1(d)$. Then repeating the above argument on $[0, z_1(d)]$ and letting $r = z_1(d)$ in (3.7) we get

$$0 \leq z_1(d) - r_1(d) \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \to 0$$

as $d \to \infty$. Also, since $r_1(d) \to 0$ as $d \to \infty$ (by Lemma 3.1) we see that $z_1(d) \to 0$ as $d \to \infty$. □

We next show for sufficiently large d that u attains the value $h_2(d)$ at some $r_2(d)$ where $z_1(d) < r_2(d) < T$. So we suppose $u' < 0$ on a maximal interval $(z_1(d), r)$. Here $h_2(d) < u < 0$ and this implies $F(u) \leq F(h_2(d))$ for sufficiently large d. Then as in the beginning of the proof of Lemma 3.2

$$\frac{1}{2}u'^2 + F(h_2(d)) \geq \frac{1}{2}u'^2 + F(u) \geq \inf_{[0,T]} E(r, d) = 2F(h_2(d))$$

so,

$$u'^2 \geq 2F(h_2(d)) \quad \text{on } (z_1(d), r).$$

Then

$$\int_{z_1(d)}^{r} -u'dt = \int_{z_1(d)}^{r} |u'|dt \geq \int_{z_1(d)}^{r} \sqrt{2F(h_2(d))}dt$$

and since $u(z_1(d)) = 0$ this leads to

$$-u(r) \geq \sqrt{2F(h_2(d))}(r - z_1(d))$$

and therefore

$$u(r) \leq -\sqrt{2}\sqrt{F(h_2(d))}(r - z_1(d)). \tag{3.8}$$

Now suppose by the way of contradiction that $u > h_2(d)$ on $(z_1(d), T)$. Then from (3.8) we see that

$$h_2(d) \leq u(r) \leq -\sqrt{2}\sqrt{F(h_2(d))}(r - z_1(d))$$

$$-h_2(d) \geq \sqrt{2}\sqrt{F(h_2(d))}(r - z_1(d)).$$

Evaluating this at $r = T$ gives

$$T - z_1(d) \leq \frac{-h_2(d)}{\sqrt{2}\sqrt{F(h_2(d))}}$$

and now taking the limit, using Lemma 3.2, and (1.2) we see that

$$0 < T = \lim_{d\to\infty} [T - z_1(d)] \leq \lim_{d\to\infty} \frac{-h_2(d)}{\sqrt{2}\sqrt{F(h_2(d))}} = 0.$$

And again this is impossible. Therefore, there exists a smallest value of r, $r_2(d)$, such that $z_1(d) < r_2(d) < T$ with $u(r_2(d)) = h_2(d)$ and $u > h_2(d)$ on $[0, r_2(d))$. Now evaluating (3.8) at $r = r_2(d)$ and using that $u(r_2(d)) = h_2(d)$ we obtain

$$h_2(d) = u(r_2(d)) \leq -\sqrt{2}\sqrt{F(h_2(d))}(r_2(d) - z_1(d))$$

now taking the limit as $d \to \infty$ and (1.2) gives

$$\lim_{d\to\infty} \sqrt{2}[r_2(d) - z_1(d)] \leq \lim_{d\to\infty} \frac{-h_2(d)}{\sqrt{F(h_2(d))}} = 0.$$

Hence $r_2(d) - z_1(d) \to 0$ as $d \to \infty$ and since $z_1(d) \to 0$ as $d \to \infty$ (from Lemma 3.2) it follows that

$$r_2(d) \to 0 \quad \text{as } d \to \infty. \tag{3.9}$$

We next want to show that u has a minimum on $(r_2(d), T)$. Suppose again by contradiction that u is decreasing on $(r_2(d), T)$. We want to show that there exists an extremum of u at r where $r > r_2(d)$.

Let $C(d) = \frac{1}{2} \min_{(-\infty, h_2(d)]} \frac{f(u)}{u}$. Note that $C(d) \to \infty$ as $d \to \infty$ by **(H2)**. Now as in the proof of Lemma 3.1 we compare

$$u'' + \frac{N-1}{r}u' + \frac{f(u)}{u}u = g(r) \tag{3.10}$$

with

$$v'' + \frac{N-1}{r}v' + C(d)v = 0 \tag{3.11}$$

with initial conditions $v(r_2(d)) = u(r_2(d))$ and $v'(r_2(d)) = u'(r_2(d))$. With an argument similar to the Claim in Lemma 3.1 we can show that $u > v$ on $(r_2(d), T)$ for sufficiently large d. Let $z(r) = \left(r/\sqrt{C(d)}\right)^{\frac{N-2}{2}} v\left(r/\sqrt{C(d)}\right)$. Then again as earlier z solves Bessel's equation

$$z'' + \frac{z'}{r} + \left(1 - \frac{\left(\frac{N-2}{2}\right)^2}{r^2}\right) z = 0 \tag{3.12}$$

of order $\frac{N-2}{2}$.

Now it is a well known fact about Bessel functions (see [4], Page 165, Theorem C) that there exists a constant K such that every interval of length K has at least one zero of $z(r)$. This implies that every interval of length $\frac{K}{\sqrt{C(d)}}$ has a zero of v. Thus for large d, we see that v must have a zero on $(r_2(d), T)$. And since $u > v$ on $(r_2(d), T)$ we see that u gets positive which contradicts that u is decreasing on $(r_2(d), T)$. Thus we see that there exists an $m_1(d)$ with $r_2(d) < m_1(d) < T$ such that u decreases on $(r_2(d), m_1(d))$ and $m_1(d)$ is a local minimum of u. Also we see that

$$m_1(d) - r_2(d) \leq \frac{K}{\sqrt{C(d)}} \to 0$$

as $d \to \infty$. And since $r_2(d) \to 0$ as $d \to \infty$ (by (3.9)) we see that $m_1(d) \to 0$ as $d \to \infty$. Also, $F(u(m_1)) = E(m_1(d)) \geq \inf_{[0,T]} E(r,d) \to \infty$ as $d \to \infty$ (by Lemma 2.1). In a similar way we can show that for large d, u has a second zero, $z_2(d)$, with $m_1(d) < z_2(d) < T$ and $z_2(d) \to 0$ as $d \to \infty$ and u has a second extremum, $m_2(d)$, with $z_2(d) < m_2(d) < T$ and $m_2(d) \to 0$ as $d \to \infty$. Continuing in this way we can get as many zeros of $u(r,d)$ as desired on $(0,T)$ for large enough d.

4. Proof of the Main Theorem

To prove the Main Theorem we construct the following sets.

Let $\mathcal{S}_k = \{ \ d \mid u(r,d)$ has exactly k zeros for all $r \in [0,T)$ and $\inf_{[0,T]} E > 0 \ \}$.

Let us denote $k_0 \geq 0$ as the smallest value of k such that $\mathcal{S}_k \neq \emptyset$. Also, as we saw at the end of section 3, $u(r,d)$ has more and more zeros on $(0,T)$ provided d is chosen large enough. And also $\inf_{[0,T]} E > 0$ if d is chosen large enough (by Lemma 2.1). Hence it follows that $\mathcal{S}_{k_0}$ is bounded above and nonempty.

Let $d_{k_0} = \sup \mathcal{S}_{k_0}$.

Lemma 4.1. *$u(r,d_{k_0})$ has exactly k_0 zeros on $[0,T)$.*

Proof. By definition of k_0, $u(r,d_{k_0})$ has at least k_0 zeros on $[0,T)$. Suppose $u(r,d_{k_0})$ has more than k_0 zeros on $[0,T)$. Then for d close to d_{k_0} and $d < d_{k_0}$, by continuity with respect to initial conditions and by Lemma 2.2, $u(r,d)$ also has more than k_0 zeros on $[0,T)$. However, if $d \in \mathcal{S}_{k_0}$, then $u(r,d)$ has exactly k_0 zeros on $[0,T)$. This is a contradiction to the definition of d_{k_0}. Thus, $u(r,d_{k_0})$ has exactly k_0 zeros on $[0,T)$. ☐

Lemma 4.2. *$u(T,d_{k_0}) = 0$.*

Proof. If $u(T,d_{k_0}) \neq 0$ then by continuity with respect to initial conditions and Lemma 2.2, $u(r,d)$ has the same number of zeros as $u(r,d_{k_0})$ for d close to d_{k_0}. But if $d > d_{k_0}$ then $d \notin \mathcal{S}_{k_0}$ so $u(r,d)$ cannot have the same number of zeros as $u(r,d_{k_0})$. This is a contradiction. Thus, $u(T,d_{k_0}) = 0$. ☐

Let $\mathcal{S}_{k_0+1} = \{ \ d > d_{k_0} \mid u(r,d)$ has exactly k_0+1 zeros on $[0,T)$ and $\inf_{[0,T]} E > 0 \ \}$.

Lemma 4.3. *$\mathcal{S}_{k_0+1} \neq \emptyset$ and $\mathcal{S}_{k_0+1}$ is bounded above.*

Proof. By continuity with respect to initial conditions and Lemma 2.2, if $d > d_{k_0}$ and d close to d_{k_0} then $u(r,d)$ has at most k_0+1 zeros on $[0,T)$. Also, if $d > d_{k_0}$ then $d \notin \mathcal{S}_{k_0}$ so $u(r,d)$ does not have exactly k_0 zeros on $[0,T)$. Now $u(r,d)$ cannot have less than k_0 zeros because this would imply that $\mathcal{S}_{k_0} = \emptyset$ for some value of k smaller than k_0 which contradicts the definition of k_0. Thus, $u(r,d)$ has at least k_0+1 zeros on $[0,T)$. Since we already showed that $u(r,d)$ for $d > d_{k_0}$ and d close to d_{k_0} has at most k_0+1 zeros on $[0,T)$ therefore, for $d > d_{k_0}$ and d close to d_{k_0}, $u(r,d)$ has exactly k_0+1 zeros on $[0,T)$. Hence $\mathcal{S}_{k_0+1}$ is nonempty. Then by remarks at the end of section 3, $\mathcal{S}_{k_0+1}$ is bounded above. ☐

Define $d_{k_0+1} = \sup \mathcal{S}_{k_0+1}$.

As above we can show that $u(r,d_{k_0+1})$ has exactly k_0+1 zeros on $[0,T)$ and $u(T,d_{k_0+1}) = 0$. Proceeding inductively, we can find solutions that tend to zero at infinity and with any prescribed number, n, of zeros on $[0,T)$ where $n \geq k_0$. Hence, this completes the proof of the Main Theorem if **(H3)** holds.

If **(H3*)** holds instead of **(H3)** let $v(r) = -u(r)$. Then v satisfies

$$v'' + \frac{N-1}{r}v' + f_2(v) = g_2(r) \tag{4.1}$$

$$v(0) = -d \tag{4.2}$$

$$(4.3) \qquad v'(0) = 0$$

where

$$f_2(v) = -f(-v)$$
$$g_2(r) = -g(r)$$
$$F_2(v) = \int_0^v f_2(u)du = \int_0^v -f(-u)du = F(-v).$$

And, now we look for solutions of (4.1)-(4.3) with $-d > 0$ (that is $d < 0$) along with $v(T) = 0$. It is straightforward to show that **(H1)**, **(H2)** and **(H4)** are satisfied by f_2 (and F_2).

Then by **(H3*)**

$$\infty = \lim_{u\to-\infty} \left(\frac{u}{f(u)}\right)^{\frac{N}{2}} \left(NF(ku) - \frac{(N-2)}{2}uf(u) - \frac{N+2}{2}||g||\ |u| - T||g'||\ |u|\right)$$
$$= \lim_{u\to\infty} \left(\frac{-u}{f(-u)}\right)^{\frac{N}{2}} \left(NF(-ku) - \frac{(N-2)}{2}(-u)f(-u) - \frac{N+2}{2}||g||\ |u| - T||g'||\ |u|\right)$$
$$= \lim_{u\to\infty} \left(\frac{u}{f_2(u)}\right)^{\frac{N}{2}} \left(NF_2(ku) - \frac{(N-2)}{2}uf_2(u) - \frac{N+2}{2}||g_2||\ |u| - T||g_2'||\ |u|\right).$$

Thus **(H3)** is satisfied by g_2 and f_2 (and F_2).

Also defining

$$E_2(r,d) = \frac{1}{2}v'^2 + F_2(v)$$

we see that

$$E_2(r,d) = \frac{1}{2}u'^2 + F_2(-u)$$
$$= \frac{1}{2}u'^2 + F(u)$$
$$= E(r,d).$$

Therefore, **(H1)-(H4)** are satisfied by f_2 (and F_2) and so by the first part of the theorem we see that there are an infinite number of solutions of (4.1)-(4.3) with $v(0) = -d > 0$ and $v(T) = 0$. Thus, $u(r) = -v(r)$ satisfies (1.3)-(1.4) with $u(0) = -v(0) = d < 0$. This completes the proof of the Main Theorem.

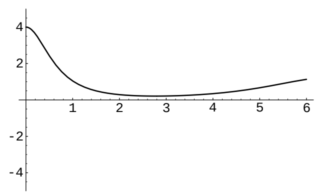
Here is an example of a u that satisfies the hypotheses (H1)-(H4):

$$(4.4) \qquad u'' + \frac{2}{r}u' + u^3 - u = 0$$

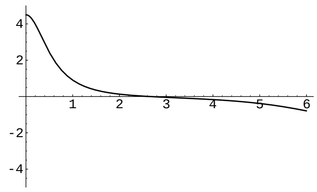
where $N = 3$, $f(u) = u^3 - u$ and $g(r) = 0$.

Here are some graphs of solutions of (4.4) for different values of d, all graphs are generated numerically using Mathematica:

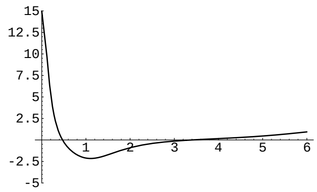
(a) Solution that remains positive when $d = 4$



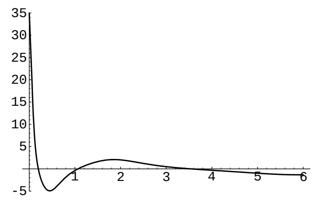
(b) Solution with exactly one zero when $d = 4.5$



(c) Solution with exactly two zeros when $d = 15$



(d) Solution with exactly three zeros when $d = 35$



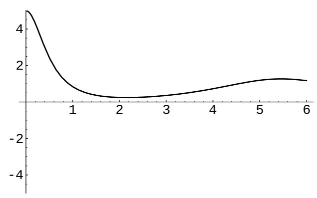
Now let us consider another example, here u satisfies the hypotheses (H1)-(H4):

$$u'' + \frac{2}{r}u' + u^3 - u = \frac{1}{r^2+1} \tag{4.5}$$

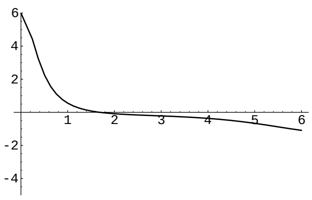
where $N = 3$, $f(u) = u^3 - u$ and $g(r) = \dfrac{1}{r^2+1}$.

Here are some graphs of solutions of (4.5) for different values of d, as above all graphs are generated numerically using Mathematica:

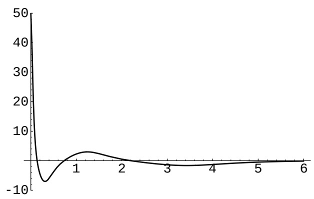
(a) Solution that remains positive when $d = 5$



(b) Solution with exactly one zero when $d = 6$



(c) Solution with exactly three zeros when $d = 50$



□

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