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Forced Oscillation of Second Order Nonlinear Dynamic Equations on Time Scales

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Abstract: By means of the Kartsatos technique and generalized Riccati transformation techniques, we establish some new oscillation criteria for a second order nonlinear dynamic equations with forced term on time scales in terms of the coefficients.

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1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D.Thesis in 1988 in order to unify continuous and discrete analysis (see [6]). A time scale $\mathbb{T}$, is an arbitrary nonempty closed subset of the reals. Many authors have expounded on various aspects of this new theory; see the book by Bohner and Peterson [2] which summarizes and organizes much of the time scale calculus. For the notions used below we refer to the next section that provides some basic facts on time scale extracted from [2].

There are many interesting time scales and they give rise to plenty of applications, the cases when the time scale is equal to reals or the integers represent the classical theories of differential and of difference equations. Another useful time scale is $\mathbb{P}_{a,b} = \bigcup_{n=0}^{+\infty}[n(a+b), n(a+b)+a]$ which is widely used to study population in biological communities, electric circuit and so on.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solution of various equations on time scales. We refer the reader to paper [1, 3, 4, 5, 9, 10, 11] and references cited therein.

In [3], Bohner and Saker considered the perturbed nonlinear dynamic equation

$$\big(\alpha(t)(x^{\Delta})^{\gamma}\big)^{\Delta} + F(t, x^{\sigma}) = G(t, x^{\sigma}, x^{\Delta}), \quad t \in [a, b]. \tag{1.1}$$

They assumed that $\dfrac{F(t,u)}{f(u)} \geq q(t), \dfrac{G(t,u,v)}{f(u)} \leq p(t)$ and changed (1.1) into the following inequality

$$(\alpha(t)(x^{\Delta})^{\gamma})^{\Delta} + (q(t) - p(t))f(x^{\sigma}) \leq 0. \tag{1.2}$$

Using Riccati transformation techniques, they obtained some sufficient conditions for the solution to be oscillatory or converge to zero.

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In [11], Saker considered the second order forced nonlinear dynamic equation

$$(a(t)x^{\Delta})^{\Delta} + p(t)f(x^{\sigma}) = r(t), \quad t \in [t_0, +\infty), \tag{1.3}$$

and supposed that $\int_{t_0}^{+\infty} |r(s)|\Delta s < +\infty$, that is the forced term is must be *small* enough for all large $t \in \mathbb{T}$. Some additional assumptions have to be imposed on the unknown solutions. He got some sufficient condition, which imposed on the forced terms directly, for solution to be oscillatory or converge to zero.

Following this trend, in this paper, we consider a second order nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(t)) = e(t), \tag{1.4}$$

on time scale interval $[a, +\infty) = \{t \in \mathbb{T}, t \geq a\}$, the following conditions are assumed to hold

(H_1) $e, p \in C_{rd}(\mathbb{T}, \mathbb{R}), p(t) \geq 0$ and $p(t) \not\equiv 0$ for all large t;

(H_2) $f \in C^1(\mathbb{R}, \mathbb{R}), f^{'}(t) \geq 0, xf(x) > 0, x \neq 0.$

By a solution of (1.4), we mean a nontrivial real-valued function x satisfying (1.4) for $t \geq a$. A solution x of (1.4) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Eq.(1.4) is called oscillatory if all solutions are oscillatory. Our attention is restricted to those solution x of Eq.(1.4) which exist on half line $[t_x, +\infty)$ with $\sup\{|x(t)| : t \geq t_0\} \neq 0$ for any $t_0 \geq t_x$.

The equation that is considered in this paper is different from all the papers mentioned in the references. Since in all these papers the author considered the equations of the forms

$$(a(t)x^{\Delta})^{\Delta} + p(t)f(x^{\sigma}) = e(t),$$

so the results in this paper are completely different from the results given for this equation or to its extension. In difference equations there is a big difference between the oscillation of the equation

$$\Delta(a(t)\Delta x(t)) + p(t)f(x(t)) = e(t),$$

and the equation

$$\Delta(a(t)\Delta x(t)) + p(t)f(x(t+1)) = e(t).$$

This means that the results in this paper are completely new for the equation under consideration.

To the best of our knowledge, nothing is known regarding the oscillatory behavior of second order nonlinear dynamic equations with forced terms by Kartsatos technique[7, 8] on time scale up to now. To develop the qualitative theory of dynamic equation on time scales, in this paper, we shall consider the forced oscillatory behavior of the second order nonlinear dynamic equation (1.4) and extend the Kartsatos technique to time scales. When (1.4) is homogeneous, i.e. the forced term $e(t) = 0$, we got some new oscillatory results for it.

2 Some preliminaries

On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \phi = \sup \mathbb{T}, \sup \phi = \inf \mathbb{T}$, and ϕ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) < t$. The graininess μ of the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \to \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $b \in \mathbb{X}$ such that for any $\varepsilon > 0$, there exists a neighborhood $\mathbf{U}$ of t satisfying $|[f(\sigma(t)) - f(s)] - b[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$, for all $s \in U$. We say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^{\Delta}(t)$ exist for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-continuous$ provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.1}$$

Let f be a differentiable function on $\mathbb{T}$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $\mathbb{T}$ if $f^{\Delta} > 0, f^{\Delta} < 0, f^{\Delta} \geq 0$ and $f^{\Delta} \leq 0$ for all $t \in \mathbb{T}$, respectively.

We will make use of the following product fg and quotient $\dfrac{f}{g}$ rules for derivative of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \tag{2.2}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}, \tag{2.3}$$

where $f^{\sigma} = f \circ \sigma, gg^{\sigma} \neq 0$. By using the product rule, the derivative of $f(t) = (t - \alpha)^m$ for $m \in \mathbb{N}$ and $\alpha \in \mathbb{T}$ can be calculated as

$$f^{\Delta}(t) = \sum_{v=0}^{m-1} (\sigma(t) - \alpha)^v (t - \alpha)^{m-v-1}. \tag{2.4}$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a). \tag{2.5}$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t)|_a^b - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{2.6}$$

and infinite integrals are defined by

$$\int_a^{\infty} f^{\Delta}(s)\Delta s = \lim_{t\to\infty} \int_a^t f^{\Delta}(s)\Delta s. \tag{2.7}$$

Lemma 2.1 *(Hölder inequality) Let $f, g, k \in C_{rd}([a, b], R)$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\int_a^b |k(x)||f(x)g(x)|\Delta x \leq \left(\int_a^b |k(x)||f(x)|^p \Delta x\right)^{\frac{1}{p}} \left(\int_a^b |k(x)||g(x)|^q \Delta x\right)^{\frac{1}{q}}. \tag{2.8}$$

If $0 < p < 1$, then the inequality is reversed.

See Wong et al.[13]

3 Oscillation Criteria

In this section, we extend the Kartsatos technique to time scale and give some new oscillation criteria for Eq.(1.4). Since we are interested in oscillatory behavior, we suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e. it is a time scale interval of the form $[a, +\infty)$. In order to use the Kartsatos technique, we assume that the following condition holds

(H_3) There exists an $h \in C^2_{rd}([a, +\infty), \mathbb{R})$ such that $h^{\Delta\Delta}(t) = e(t)$ and h is oscillatory.

Let $x(t) = y(t) + h(t)$, then (1.4) can be rewritten as

$$y^{\Delta\Delta}(t) + p(t)f(y(t) + h(t)) = 0. \tag{3.1}$$

In order to prove our main results, we need the following auxiliary result

Lemma 3.1 *Suppose that $(H_1) - (H_3)$ hold and $x(t) > 0, t \geq t_0 \geq a$ is a nonoscillatory solution of (1.4). Then the solution y of (3.1) satisfies*

$$y(t) > 0, y^{\Delta}(t) > 0, y^{\Delta\Delta}(t) \leq 0, \quad t \geq t_4 \geq a.$$

Proof. Since $p(t) \geq 0$, from (3.1) we note that $y^{\Delta\Delta}(t) = -p(t)f(x(t)) \leq 0$ on $[t_0, +\infty)$ for some $t_0 \geq a$. Next, we show that $y^{\Delta}(t) \geq 0$ on $[t_1, +\infty)$ for some $t_1 \geq t_0$. If not, say $y^{\Delta}(t_2) < 0$ for some $t_2 \geq t_0$, since $y^{\Delta\Delta}(t) \leq 0$, we get $y^{\Delta}(t) \leq y^{\Delta}(t_2) < 0$ for all $t \geq t_2$, hence $y(t) \leq y(t_2) + \int_{t_2}^{t} y^{\Delta}(t_2)\Delta s \to -\infty$ as $t \to +\infty$, but this together with $h(t)$ being oscillatory contradicts the assumption that $x(t) > 0$. In fact, given that $y^{\Delta\Delta}(t) \leq 0, y^{\Delta}(t) \geq 0$, we must have $y^{\Delta}(t) > 0$ for all $t \geq t_3 \geq t_0$. Suppose that $y^{\Delta}(t_3) \equiv 0$, we have $y^{\Delta\Delta}(t) \equiv 0, t \geq t_3$, from (3.1) we get $p(t) \equiv 0, t \geq t_3$, which contradicts the assumption (H_1).

Next, we show that $y(t)$ is eventually positive. Since $x(t) > 0$ and $h(t)$ is oscillatory, so $y(t) = x(t) - h(t)$ cannot be eventually negative nor can it be identically zero. On the other hand, since $y^{\Delta}(t) > 0$ for all $t \geq t_3 \geq t_0$, thus $y(t)$ certainly cannot be oscillatory. Hence, we must have that $y(t) > 0, t \geq t_4 \geq t_0$.

If $x > 0$ is a nonoscillatory solution of (1.4), for simplicity, we conclude that the solution y of (3.1) satisfies

$$y(t) > 0, y^{\Delta}(t) > 0, y^{\Delta\Delta}(t) \leq 0, \quad t \geq a. \tag{3.2}$$

Lemma 3.2 *Suppose that $(H_1) - (H_3)$ hold and $x(t) < 0, t \geq t_0 \geq a$ is a nonoscillatory solution of (1.4). Then the solution y of (3.1) satisfies*

$$y(t) < 0, y^{\Delta}(t) < 0, y^{\Delta\Delta}(t) \geq 0, \quad t \geq t_4^{'} \geq a.$$

Proof. The proof is similar to that of Lemma 3.1 and we omit it.

If $x < 0$ is a nonoscillatory solution of (1.4), for simplicity, we conclude that the solution y of (3.1) satisfies

$$y(t) < 0, y^{\Delta}(t) < 0, y^{\Delta\Delta}(t) \geq 0, \quad t \geq a. \tag{3.3}$$

By means of generalized Riccati transformation techniques, we establish some new oscillation criteria for (1.4) in terms of the coefficients.

Theorem 3.3 *Suppose that $(H_1)-(H_3)$ hold. Furthermore, assume that there exist a constant M and two $\Delta-$differentiable function $g(t)>0$ and r such that $g(t)r(t)\leq M$ for all large t. If for any $t_0\geq a$, there exists a $t_1\geq t_0$ and a constant $d<0$ such that*

$$\limsup_{t\to+\infty} \left\{\left[\int_{t_1}^t \left(p(s)g^\sigma(s)\frac{f(h_+(s))}{\sigma(s)}\right)^d \Delta s\right]^{\frac{1}{d}} + \int_{t_1}^t [\lambda(s)r^2(s)+r^\Delta(s) - \frac{1}{4\lambda(s)}\left(\frac{g^\Delta(s)-2\lambda(s)r(s)g^\sigma(s)}{g^\sigma(s)}\right)^2]g^\sigma(s)\Delta s\right\} = +\infty, \tag{3.4}$$

and

$$\limsup_{t\to+\infty} \left\{\left[\int_{t_1}^t \left(p(s)g^\sigma(s)\frac{f(|h_-(s)|)}{\sigma(s)}\right)^d \Delta s\right]^{\frac{1}{d}} + \int_{t_1}^t [\lambda(s)r^2(s)+r^\Delta(s) - \frac{1}{4\lambda(s)}\left(\frac{g^\Delta(s)-2\lambda(s)r(s)g^\sigma(s)}{g^\sigma(s)}\right)^2]g^\sigma(s)\Delta s\right\} = +\infty, \tag{3.5}$$

where $0<\lambda(t)=\frac{t-t_0}{\mu(t)+t-t_0}$ and $h_+(t)=\max\{h(t),0\}, h_-(t)=\min\{h(t),0\}$. Then (1.4) is oscillatory.

Proof. Suppose that $x(t)>0, t\geq t_0\geq a$ is a nonoscillatory solution of (1.4). By lemma 3.1, we get that the solution y of (3.1) satisfies (3.2). Make the generalized Riccati substitution for Eq.(3.1)

$$w(t)=g(t)\left[-\frac{y^\Delta(t)}{y(t)}+r(t)\right], \tag{3.6}$$

we use the rules (2.2) and (2.3) to find

$$\begin{aligned} w^\Delta(t) &= g^\Delta(t)\left[-\frac{y^\Delta(t)}{y(t)}+r(t)\right]+g^\sigma(t)\left[-\frac{y^\Delta(t)}{y(t)}+r(t)\right]^\Delta \\ &= \frac{g^\Delta(t)}{g(t)}w(t)+p(t)g^\sigma(t)\frac{f(y(t)+h(t))}{y^\sigma(t)}+g^\sigma(t)\frac{y(t)}{y^\sigma(t)}\left(\frac{y^\Delta(t)}{y(t)}\right)^2+g^\sigma(t)r^\Delta(t) \\ &= \frac{g^\Delta(t)}{g(t)}w(t)+p(t)g^\sigma(t)\frac{f(y(t)+h(t))}{y^\sigma(t)}+g^\sigma(t)\frac{y(t)}{y^\sigma(t)}\left(r(t)-\frac{w(t)}{g(t)}\right)^2+g^\sigma(t)r^\Delta(t). \end{aligned}$$

Since $y(t_0)>0$ and $y^{\Delta\Delta}(t)\leq 0$, we obtain

$$y(t)\geq y(t)-y(t_0)=\int_{t_0}^t y^\Delta(s)\Delta s\geq y^\Delta(t)(t-t_0),$$

Therefore, we get $0<\frac{y^\Delta(t)}{y(t)}\leq\frac{1}{t-t_0}, t\geq t_0\geq a$, then

$$\frac{y^\sigma(t)}{y(t)}=\frac{y(t)+\mu(t)y^\Delta(t)}{y(t)}\leq 1+\frac{\mu(t)}{t-t_0}=\frac{\mu(t)+t-t_0}{t-t_0}=\frac{1}{\lambda(t)}. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\begin{aligned}
w^{\Delta}(t) \quad & \geq \frac{g^{\Delta}(t)}{g(t)} w(t) + p(t) g^{\sigma}(t) \frac{f(y(t)+h(t))}{y^{\sigma}(t)} + \lambda(t) g^{\sigma}(t) \left(r(t) - \frac{w(t)}{g(t)} \right)^2 + g^{\sigma}(t) r^{\Delta}(t) \\
& = \lambda(t) g^{\sigma}(t) \frac{w^2(t)}{g^2(t)} + \frac{g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t)}{g(t)} w(t) + \lambda(t) g^{\sigma}(t) r^2(t) + g^{\sigma}(t) r^{\Delta}(t) \\
& + p(t) g^{\sigma}(t) \frac{f(y(t)+h(t))}{y^{\sigma}(t)} \\
& = \left[\frac{\sqrt{\lambda(t) g^{\sigma}(t)}}{g(t)} w(t) + \frac{g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t)}{2\sqrt{\lambda(t) g^{\sigma}(t)}} \right]^2 - \frac{(g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t))^2}{4\lambda(t) g^{\sigma}(t)} \\
& + \lambda(t) g^{\sigma}(t) r^2(t) + g^{\sigma}(t) r^{\Delta}(t) + p(t) g^{\sigma}(t) \frac{f(y(t)+h(t))}{y^{\sigma}(t)} \\
& \geq p(t) g^{\sigma}(t) \frac{f(y(t)+h(t))}{y^{\sigma}(t)} + \lambda(t) g^{\sigma}(t) r^2(t) + g^{\sigma}(t) r^{\Delta}(t) - \frac{(g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t))^2}{4\lambda(t) g^{\sigma}(t)}.
\end{aligned} \tag{3.8}$$

Since $y^{\Delta\Delta}(t) \leq 0, y^{\Delta}(t) > 0$, there exists a constant $M_1^{'} > 0$ such that $y^{\Delta}(t) \leq M_1^{'}, t \geq t_0$. Integrating it from t_0 to t, we get $y(t) \leq y(t_0) + M_1^{'}(t - t_0)$, hence there exists $M_1 > 0$ (M_1 is a finite constant) such that $0 < y(t) \leq M_1 t$ and $0 < y^{\sigma}(t) \leq M_1 \sigma(t), t \geq \sigma(t_0)$,

We note that for all $t \geq t_0, y(t) + h(t) > h_+(t)$. To see this, we write $y(t) + h(t) = y(t) + h_+(t) + h_-(t)$ and observe that

(I) for $h_-(t) = 0, y(t) + h(t) = y(t) + h_+(t) > h_+(t)$ (since $y(t) > 0$) and

(II) for $h_+(t) = 0, y(t) + h(t) = y(t) + h_-(t) = x(t) > 0 = h_+(t)$.

Since f is nondecreasing, we have that $f(y(t) + h(t)) \geq f(h_+(t))$ and

$$\frac{f(y(t)+h(t))}{y^{\sigma}(t)} \geq \frac{f(h_+(t))}{y^{\sigma}(t)} \geq \frac{f(h_+(t))}{M_1 \sigma(t)}.$$

From (3.8), we get

$$w^{\Delta}(t) \geq p(t) g^{\sigma}(t) \frac{f(h_+(t))}{M_1 \sigma(t)} + \lambda(t) g^{\sigma}(t) r^2(t) + g^{\sigma}(t) r^{\Delta}(t) - \frac{(g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t))^2}{4\lambda(t) g^{\sigma}(t)}. \tag{3.9}$$

Let $t_1 \geq t_0$ be as in the statement of this theorem, integrating (3.9) from t_1 to $t \geq t_1$, we get

$$w(t) - w(t_1) \geq \int_{t_1}^{t} p(s) g^{\sigma}(s) \frac{f(h_+(s))}{M_1 \sigma(s)} \Delta s + \int_{t_1}^{t} \varphi(s) \Delta s, \tag{3.10}$$

where $\varphi(t) = \lambda(t) g^{\sigma}(t) r^2(t) + g^{\sigma}(t) r^{\Delta}(t) - \dfrac{(g^{\Delta}(t) - 2\lambda(t) r(t) g^{\sigma}(t))^2}{4\lambda(t) g^{\sigma}(t)}$. Using Lemma 2.1(Hölder inequality) with $k(t) = 1$ and $0 < b < 1, \frac{1}{b} + \frac{1}{d} = 1$, we obtain

$$\begin{aligned}
\int_{t_1}^{t} p(s) g^{\sigma}(s) \frac{f(h_+(s))}{M_1 \sigma(s)} \Delta s \quad & \geq \left(\int_{t_1}^{t} \left(\frac{1}{M_1} \right)^b \Delta s \right)^{\frac{1}{b}} \left(\int_{t_1}^{t} \left(p(s) g^{\sigma}(s) \frac{f(h_+(s))}{\sigma(s)} \right)^d \Delta s \right)^{\frac{1}{d}} \\
& = \frac{(t - t_1)^{\frac{1}{b}}}{M_1} \left(\int_{t_1}^{t} \left(p(s) g^{\sigma}(s) \frac{f(h_+(s))}{\sigma(s)} \right)^d \Delta s \right)^{\frac{1}{d}}.
\end{aligned} \tag{3.11}$$

Since M_1 is a finite constant, let t be large enough such that $\dfrac{(t-t_1)^{\frac{1}{b}}}{M_1} \geq 1$, then using (3.10) and (3.11), we obtain

$$w(t) - w(t_1) \geq \left[\int_{t_1}^{t} \left(p(s)g^{\sigma}(s)\frac{f(h_+(s))}{\sigma(s)}\right)^d \Delta s\right]^{\frac{1}{d}} + \int_{t_1}^{t} \varphi(s)\Delta s. \tag{3.12}$$

Now by (3.6) and $g(t)r(t) \leq M$ for all large t, we get

$$w(t) = g(t)\left[-\frac{y^{\Delta}(t)}{y(t)} + r(t)\right] \leq g(t)r(t) \leq M.$$

Taking $\limsup$ on both sides of (3.12) and letting $t \to +\infty$, we obtain the desired contradiction. (3.5) is required when we assume that the nonoscillatory solution x to be eventually negative and we can prove it in a way similar to that of x is eventually positive.

From theorem 3.3, we can obtain different sufficient conditions for the oscillation of all solutions of (1.4) by different choices of g and r. For instance, let $g(t) = 1, r(t) = 0$, then

Corollary 3.4 *Suppose that $(H_1)-(H_3)$ hold, If for any $t_0 \geq a$, there exists a $t_1 \geq t_0$ and a constant $d < 0$ such that*

$$\limsup_{t\to+\infty}\left(\int_{t_1}^{t}\left(p(s)\frac{f(h_+(s))}{\sigma(s)}\right)^d \Delta s\right)^{\frac{1}{d}} = \limsup_{t\to+\infty}\left(\int_{t_1}^{t}\left(p(s)\frac{f(|h_-(s)|)}{\sigma(s)}\right)^d \Delta s\right)^{\frac{1}{d}} = +\infty, \tag{3.13}$$

Then (1.4) is oscillatory.

For example, let $d = -1$. If $(H_1)-(H_3)$ hold, $f(0) > 0$ and

$$\liminf_{t\to+\infty}\int_{t_1}^{t}\frac{\sigma(s)}{p(s)f(h_+(s))}\Delta s = \liminf_{t\to+\infty}\int_{t_1}^{t}\frac{\sigma(s)}{p(s)f(|h_-(s)|)}\Delta s = 0, \tag{3.14}$$

then (1.4) is oscillatory.

Theorem 3.5 *Suppose that $(H_1)-(H_3)$ hold. Assume that there exist two positive $\Delta-$differentiable functions g and r such that for any finite constant $M_1 > 0$ and $M_2 < 0$*

$$\limsup_{t\to+\infty}\frac{1}{t^m}\int_a^t (t-s)^m u_1(s)\Delta s = \limsup_{t\to+\infty}\frac{1}{t^m}\int_a^t (t-s)^m u_2(s)\Delta s = +\infty, \tag{3.15}$$

where

$$u_1(t) = \left\{p(t)\frac{f(h_+(t))}{M_1\sigma(t)} + \lambda(t)r^2(t) + r^{\Delta}(t) - \frac{1}{4\lambda(t)}\left(\frac{g^{\Delta}(t) - 2\lambda(t)r(t)g^{\sigma}(t)}{g^{\sigma}(t)}\right)^2\right\}g^{\sigma}(t),$$

$$u_2(t) = \left\{p(t)\frac{f(h_-(t))}{M_2\sigma(t)} + \lambda(t)r^2(t) + r^{\Delta}(t) - \frac{1}{4\lambda(t)}\left(\frac{g^{\Delta}(t) - 2\lambda(t)r(t)g^{\sigma}(t)}{g^{\sigma}(t)}\right)^2\right\}g^{\sigma}(t),$$

and $0 < \lambda(t) = \frac{t-t_0}{\mu(t)+t-t_0}$. Assume that one of the following condition holds

(I) m is an even integer;

(II) m is an odd integer and $\frac{1}{t^m}\int_a^t g^{\sigma}(s)r^{\sigma}(s)\sum_{v=0}^{m-1}(\sigma(t)-s)^v(t-s)^{m-v-1}\Delta s$ is bounded.

Then (1.4) is oscillatory.

Proof. Similar to the proof of theorem 3.3, we may assume that (1.4) has a nonoscillatory solution $x(t) > 0, t \geq t_0 \geq a$ such that the solution y of (3.1) satisfies (3.2), i.e. $y(t) > 0, y^{\Delta}(t) > 0, y^{\Delta\Delta}(t) \leq 0, t \geq t_0 \geq a$. From (3.9), we get $u_1(t) \leq w^{\Delta}(t)$. Therefore, multiply both sides of $u_1 \leq w^{\Delta}$ with $(t-s)^m$ and integrate by parts, the right hand side leads to

$$\int_{t_0}^{t}(t-s)^m w^{\Delta}(s)\Delta s = (t-s)^m w(s)|_{t_0}^{t} + (-1)^{m+1}\int_{t_0}^{t}\sum_{v=0}^{m-1}(\sigma(t)-s)^v(t-s)^{m-v-1}w^{\sigma}(s)\Delta s. \quad (3.16)$$

Then from (3.16) and $w(t) = g(t)[-\frac{y^{\Delta}(t)}{y(t)} + r(t)] \leq g(t)r(t)$, we get

$$\begin{aligned}\int_{t_0}^{t}(t-s)^m u_1(s)\Delta s &\leq -(t-t_0)^m w(t_0)\\ &+(-1)^{m+1}\int_{t_0}^{t}\sum_{v=0}^{m-1}(\sigma(t)-s)^v(t-s)^{m-v-1}w^{\sigma}(s)\Delta s\\ &\leq -(t-t_0)^m w(t_0)\\ &+(-1)^{m+1}\int_{t_0}^{t} g^{\sigma}(s)r^{\sigma}(s)\sum_{v=0}^{m-1}(\sigma(t)-s)^v(t-s)^{m-v-1}\Delta s.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m u_1(s)\Delta s \leq -\left(\frac{t-t_0}{t}\right)^m w(t_0)\\ &+(-1)^{m+1}\frac{1}{t^m}\int_{t_0}^{t} g^{\sigma}(s)r^{\sigma}(s)\sum_{v=0}^{m-1}(\sigma(t)-s)^v(t-s)^{m-v-1}\Delta s.\end{aligned} \quad (3.17)$$

If (I) holds, we obtain

$$\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m u_1(s)\Delta s \leq -\left(\frac{t-t_0}{t}\right)^m w(t_0). \quad (3.18)$$

Taking lim sup on both sides of (3.18) and letting $t \to +\infty$, we obtain the desired contradiction.

If (II) holds, taking lim sup on both sides of (3.17) and letting $t \to +\infty$, we obtain the desired contradiction.

Remark 1. When $\mathbb{T} = \mathbb{R}$, equation (1.4) changes to

$$x''(t) + p(t)f(x(t)) = e(t).$$

Our results of Theorem 3.3 and Theorem 3.5 is new for the above equations.

Following the ideas of Wong [12], we establish the following three theorems which are the extension of Wong [12] Theorem [1, 2, 4] to time scales.

Theorem 3.6 *Suppose that $(H_1)-(H_3)$ hold and that h satisfies*

$$\liminf_{t\to+\infty}\frac{h(t)}{t} = -\infty, \quad \limsup_{t\to+\infty}\frac{h(t)}{t} = +\infty. \quad (3.19)$$

Then (1.4) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.4) has a nonoscillatory solution x, without loss of generality, we may assume that x is eventually positive solution of (1.4), i.e. $x(t) > 0, t \geq t_0 \geq a \in \mathbb{T}$. From lemma 3.1, we have that (3.2) holds. Since $y^{\Delta\Delta}(t) \leq 0, y^{\Delta}(t) > 0$, there exists a constant $M > 0$ such that $y^{\Delta}(t) \leq M, t \geq t_0$. Integrating it from t_0 to t, we get

$y(t) \leq y(t_0) + M(t - t_0)$, hence there exists $M' > 0$ (M' is a finite constant) such that $0 < y(t) \leq M't$, or

$$\limsup_{t \to +\infty} \frac{y(t)}{t} \leq M'. \tag{3.20}$$

On the other hand, we have that $y(t) + h(t) = x(t) > 0$ for large t or $y(t) > -h(t)$. Dividing by t and taking *limsup* on both sides for $y > -h$, we get

$$\limsup_{t \to +\infty} \frac{y(t)}{t} \geq \limsup_{t \to +\infty} \frac{-h(t)}{t} = -\liminf_{t \to +\infty} \frac{h(t)}{t} = +\infty, \tag{3.21}$$

which is contradicts to (3.20). The other part of hypothesis (3.19) is required when we assume the nonoscillatory solution x to be eventually negative and used a similar equation to (3.21) in that case.

Theorem 3.7 *Assume that* $(H_1) - (H_3)$ *hold and that* $h(t)$ *satisfies*

$$\int_a^{+\infty} p(t)f(h_+(t))\Delta t = \int_a^{+\infty} p(t)f(h_-(t))\Delta t = +\infty. \tag{3.22}$$

Then (1.4) is oscillatory.

Proof. As before, we may suppose $x(t) > 0, t \geq t_0 \geq a$ be a nonoscillatory solution of (1.4), by lemma 3.1, we have (3.2) holds. Integrating (3.1) from t_0 to t, we obtain

$$y^{\Delta}(t) - y^{\Delta}(t_0) + \int_{t_0}^{t} p(s)f(y(s) + h(s))\Delta s = 0. \tag{3.23}$$

For $y^{\Delta\Delta}(t) \leq 0, y^{\Delta}(t) > 0, \lim_{t \to +\infty} y^{\Delta}(t)$ exists and is finite, hence the integral in (3.23) converges as $t \to +\infty$.

Similar to the proof of Theorem 3.3, we note that for all $t \geq t_0, y(t) + h(t) > h_+(t)$. Since f is nondecreasing, we have that $f(y(t) + h(t)) \geq f(h_+(t))$. With $p(t) \geq 0$, we obtain

$$\int_{t_0}^{t} p(s)f(h_+(s))\Delta s \leq \int_{t_0}^{t} p(s)f(y(s) + h(s))\Delta s < +\infty, \tag{3.24}$$

for all $t \geq t_0$ hold. By applying (3.22) to (3.24), we obtain the desired contradiction.

Theorem 3.8 *Assume that* $(H_1) - (H_3)$ *hold and* $\int_a^{+\infty} p(t)\Delta t = +\infty$. *Suppose, in addition, that* $h(t)$ *satisfies* (H_4), *where*

(H_4) *There exist sequence* $\{s_n\}, \{s_n'\}$ *such that* $\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} s_n' = +\infty$ *and* $h(s_n) = \inf\{h(t) : t \geq s_n\}, h(s_n') = \sup\{h(t) : t \geq s_n'\}$.

Then (1.4) is oscillatory.

Proof. As before, we may assume $x(t) > 0, t \geq t_0 \geq a$ is a nonoscillatory solution of (1.4), by lemma 3.1, we get that the solution y of (3.1) satisfies (3.2). Note that there exists n_0 such that $s_{n_0} \geq t_0$ and for $t \geq s_{n_0} \geq t_0$

$$y(t) + h(t) \geq y(s_{n_0}) + h(s_{n_0}) = x(s_{n_0}) > 0. \tag{3.25}$$

Substituting (3.25) into (3.23) and using the fact that f is nondecreasing and $\int_a^{+\infty} p(t)\Delta t = +\infty$, we find that $y^{\Delta}(t) \to -\infty$ as $t \to +\infty$, which clearly contradicts $y^{\Delta}(t) > 0$ on $[t_0, +\infty) \subset \mathbb{T}$.

Corollary 3.9 *Suppose that $(H_1)-(H_3)$ hold and $\int_a^{+\infty} p(t)\Delta t = +\infty$. Suppose, in addition, that $h(t)$ satisfies $\lim_{t\to+\infty} h(t) = 0$ or $h(t)$ is periodic in t. Then (1.4) is oscillatory.*

If $h(t) \to 0$ as $t \to +\infty$ or $h(t)$ is periodic in t, then it is easy to see that the conditions of theorem 3.8 hold, hence corollary 3.9 follows from theorem 3.8.

Since the time scale $\mathbb{P}_{a,b} = \bigcup_{n=0}^{+\infty}[n(a+b), n(a+b)+a]$ can be used to study many models of real world, for instance, population in biological communities, electric circuit and so on, we give an example in such a time scale to demonstrate how the theory may be applied to specific problems.

Example 1 Consider the following second order dynamic equation

$$x^{\Delta\Delta}(t) + tsintx(t) = t^2cost, \quad for \;\; t \in \mathbb{P}_{\pi,\pi} = \bigcup_{n=0}^{+\infty}[2n\pi, (2n+1)\pi], \tag{3.26}$$

with the transition condition

$$x(2n\pi) = x((2n-1)\pi), \quad n \geq 1.$$

Then, we can choose $h(t) = -t^2cost + 4tsint + 6cost, t \in \mathbb{P}_{\pi,\pi}$ such that $h^{\Delta\Delta}(t) = e(t) = t^2cost$ and $p(t) = tsint > 0$ for $t \in \mathbb{P}_{\pi,\pi}$. Furthermore, we have

$$\liminf_{t\to+\infty} \frac{h(t)}{t} = -\infty, \quad and \quad \limsup_{t\to+\infty} \frac{h(t)}{t} = +\infty.$$

From Theorem 3.6, we obtain that (3.26) is oscillatory. Moreover, using Theorem 3.6, we can obtain that all solutions of $x^{\Delta\Delta}(t) + tsintx(t) = t^\gamma cost$ for $\gamma > 1$ and $t \in \mathbb{P}_{\pi,\pi}$ is oscillatory. But Sake [11] and Bohner and Saker [3] cannot judge the oscillations of (3.26) and the more gerenal equation.

As a special example, let $\mathbb{T} = \mathbb{N}$, we use our results to difference equations with forced terms.

Example 2 Consider the following second order difference equation

$$\Delta^2 x(n) + n^\alpha x(n) = 2(2n^2 + 4n + 3)(-1)^n, \tag{3.27}$$

where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$.

Then, we can choose $h(n) = n^2(-1)^n$ is oscillatory, such that

$$\Delta^2 h(n) = 2(2n^2 + 4n + 3)(-1)^{n+2} = 2(2n^2 + 4n + 3)(-1)^n.$$

Moreover, we get $\frac{h(n)}{n} = n(-1)^n$. So

$$\liminf_{n\to+\infty} \frac{h(n)}{n} = -\infty, \quad and \quad \limsup_{n\to+\infty} \frac{h(n)}{n} = +\infty.$$

By Theorem 3.6, we obtain that (3.27) is oscillatory.

4 Application to equations without forced terms

Our aim is to apply the results in Section 3, to give some sufficient conditions for oscillation of all solutions of the dynamic equations (1.4) without forced terms. For Eq.(1.4) in unforced case, i.e. $e(t) = 0$, our results are also new. Let $e(t) = 0$, Eq.(1.4) change into

$$x^{\Delta\Delta}(t) + p(t)f(x(t)) = 0. \tag{4.1}$$

Let $h(t) = 0$ and we obtain some new oscillatory criteria for (4.1).

$(H_2^{'})$ $f \in C^1(\mathbb{R}, \mathbb{R}), f^{'}(t) \geq 0, \dfrac{f(x)}{x} \geq k > 0.$

Theorem 4.1 *Assume $(H_1), (H_2^{'})$ hold. Suppose there exist a constant M and two $\Delta-$differentiable function $g(t) > 0$ and r such that $g(t)r(t) \leq M$ for all large t. If for any $t_0 \geq a$, there exists a $t_1 \geq t_0$ such that*

$$\limsup_{t\to+\infty} \int_{t_1}^{t} \left\{ kp(s)\lambda(s) + \lambda(s)r^2(s) + r^{\Delta}(s) - \frac{1}{4\lambda(s)} \left(\frac{g^{\Delta}(s) - 2\lambda(s)r(s)g^{\sigma}(s)}{g^{\sigma}(s)} \right)^2 \right\} g^{\sigma}(s)\Delta s = +\infty, \tag{4.2}$$

where $0 < \lambda(t) = \dfrac{t - t_0}{\mu(t) + t - t_0}$. Then (4.1) is oscillatory.

From theorem 4.1, we can obtain different sufficient conditions for the oscillation of all solutions of (4.1) by different choices of g and r. For instance, let $g(t) = 1, r(t) = 0$, we get the following well known result

Corollary 4.2 *Assume that $(H_1), (H_2^{'})$ hold, if for any $t_0 \geq a$, there exists a $t_1 \geq t_0$ such that*

$$\limsup_{t\to+\infty} \int_{t_1}^{t} \frac{s - t_0}{\mu(s) + s - t_0} p(s)\Delta s = +\infty. \tag{4.3}$$

Then (4.1) is oscillatory.

If $\mu(t) \leq k^{'}t$ ($k^{'}$ is a constant), then we get

Corollary 4.3 *(Leighton - Wintner theorem) Assume that $(H_1), (H_2^{'})$ hold, if*

$$\int_{a}^{+\infty} p(s)\Delta s = +\infty. \tag{4.4}$$

Then (4.1) is oscillatory.

Theorem 4.4 *Assume that $(H_1), (H_2^{'})$ hold. Suppose there exist two positive $\Delta-$differentiable function g and r such that*

$$\begin{aligned} \limsup_{t\to+\infty} \quad & \frac{1}{t^m} \int_a^t (t-s)^m \{kp(s)\lambda(s) + \lambda(s)r^2(s) + r^{\Delta}(s) \\ & - \frac{1}{4\lambda(s)} (\frac{g^{\Delta}(s) - 2\lambda(s)r(s)g^{\sigma}(s)}{g^{\sigma}(s)})^2\} g^{\sigma}(s)\Delta s = +\infty, \end{aligned} \tag{4.5}$$

where $0 < \lambda(t) = \dfrac{t - t_0}{\mu(t) + t - t_0}$. Assume further that one of the following condition holds

(I) m is an even integer;

(II) m is an odd integer and $\frac{1}{t^m} \int_a^t g^{\sigma}(s)r^{\sigma}(s) \sum_{v=0}^{m-1} (\sigma(t) - s)^v (t-s)^{m-v-1} \Delta s$ is bounded.

Then (4.1) is oscillatory.

When $g(t) = 1, r(t) = 0$, we get

Corollary 4.5 *Assume that $(H_1), (H_2^{'})$ hold, if for any $t_0 \geq a$, there exists a $t_1 \geq t_0$ such that*

$$\limsup_{t\to+\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \frac{s-t_0}{\mu(s)+s-t_0} p(s)\Delta s = +\infty. \tag{4.6}$$

Then (4.1) is oscillatory.

If $\mu(t) \leq k^{'}t$ ($k^{'}$ is a constant), then we get

Corollary 4.6 *(Kamenev theorem)* *Assume that $(H_1), (H_2^{'})$ hold, if*

$$\limsup_{t\to+\infty} \frac{1}{t^m} \int_{a}^{t} (t-s)^m p(s)\Delta s = +\infty. \tag{4.7}$$

Then (4.1) is oscillatory.

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