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EXISTENCE OF S^2-ALMOST PERIODIC SOLUTIONS TO A CLASS OF NONAUTONOMOUS STOCHASTIC EVOLUTION EQUATIONS

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Abstract. The paper studies the notion of Stepanov almost periodicity (or S^2-almost periodicity) for stochastic processes, which is weaker than the notion of quadratic-mean almost periodicity. Next, we make extensive use of the so-called Acquistapace and Terreni conditions to prove the existence and uniqueness of a Stepanov (quadratic-mean) almost periodic solution to a class of nonautonomous stochastic evolution equations on a separable real Hilbert space. Our abstract results will then be applied to study Stepanov (quadratic-mean) almost periodic solutions to a class of n-dimensional stochastic parabolic partial differential equations.

1. Introduction

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t : t \in \mathbb{R}\}$, that is, a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$.

The impetus of this paper comes from two main sources. The first source is a paper by Bezandry and Diagana [2], in which the concept of quadratic-mean almost periodicity was introduced and studied. In particular, such a concept was, subsequently, utilized to study the existence and uniqueness of a quadratic-mean almost periodic solution to the class of stochastic differential equations

$$dX(t) = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; \mathbb{H})$ is a densely defined closed linear operator, and $F : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; \mathbb{H})$, $G : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; L_2^0)$ are jointly continuous functions satisfying some additional conditions.

The second sources is a paper Bezandry and Diagana [3], in which the authors made extensive use of the almost periodicity to study the existence and uniqueness of a quadratic-mean almost periodic solution to the class of nonautonomous semilinear stochastic evolution equations

$$dX(t) = A(t)X(t)\, dt + F(t, X(t))\, dt + G(t, X(t))\, dW(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operators satisfying the so-called Acquistapace and Terreni conditions [1], $F : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \to L^2(\mathbf{P}; \mathbb{H})$, $G : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \to L^2(\mathbf{P}; L_2^0)$ are jointly continuous satisfying some additional conditions, and $W(t)$ is a Wiener process.

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The present paper is definitely inspired by [2, 3] and [7, 8] and consists of studying the existence of Stepanov almost periodic (respectively, quadratic-mean almost periodic) solutions to the Eq. (1.2) when the forcing terms F and G are both S^2-almost periodic. It is worth mentioning that the existence results of this paper generalize those obtained in Bezandry and Diagana [3], as S^2-almost periodicity is weaker than the concept of quadratic-mean almost periodicity.

The existence of almost periodic (respectively, periodic) solutions to autonomous stochastic differential equations has been studied by many authors, see, e.g., [1], [2], [9], and [17] and the references therein. In particular, Da Prato and Tudor [5], have studied the existence of almost periodic solutions to Eq. (1.2) in the case when $A(t)$ is periodic. In this paper, it goes back to studying the existence and uniqueness of a S^2-almost periodic (respectively, quadratic-mean almost periodic) solution to Eq. (1.2) when the operators $A(t)$ satisfy the so-called Acquistapace and Terreni conditions and the forcing terms F, G are S^2-almost periodic. Next, we make extensive use of our abstract results to establish the existence of Stepanov (quadratic mean) almost periodic solutions to an n-dimensional system of stochastic parabolic partial differential equations.

The organization of this work is as follows: in Section 2, we recall some preliminary results that we will use in the sequel. In Section 3, we introduce and study the notion of Stepanov almost periodicity for stochastic processes. In Section 4, we give some sufficient conditions for the existence and uniqueness of a Stepanov almost periodic (respectively, quadratic-mean almost periodic) solution to Eq. (1.2). Finally, an example is given to illustrate our main results.

2. Preliminaries

For details of this section, we refer the reader to [2, 4] and the references therein. Throughout the rest of this paper, we assume that $(\mathbb{K}, \|\cdot\|_K)$ and $(\mathbb{H}, \|\cdot\|)$ are separable real Hilbert spaces and that $(\Omega, \mathcal{F}, \mathbf{P})$ stands for a probability space. The space $L_2(\mathbb{K}, \mathbb{H})$ denotes the collection of all Hilbert-Schmidt operators acting from $\mathbb{K}$ into $\mathbb{H}$, equipped with the classical Hilbert-Schmidt norm, which we denote $\|\cdot\|_2$. For a symmetric nonnegative operator $Q \in L_2(\mathbb{K}, \mathbb{H})$ with finite trace we assume that $\{W(t) : t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathbb{K}$. It is worth mentioning that the Wiener process W can obtained as follows: let $\{W_i(t) : t \in \mathbb{R}\}$, $i = 1, 2$, be independent $\mathbb{K}$-valued Q-Wiener processes, then

$$W(t) = \begin{cases} W_1(t) & \text{if } t \geq 0, \\ \\ W_2(-t) & \text{if } t \leq 0, \end{cases}$$

is Q-Wiener process with the real number line as time parameter. We then let $\mathcal{F}_t = \sigma\{W(s),\ s \leq t\}$.

The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L^2(\mathbf{P}; \mathbb{H})$, is a Banach space when it is equipped with norm

$$\|X\|_{L^2(\mathbf{P};\mathbb{H})} = \sqrt{\mathbf{E}\|X\|^2}$$

where the expectation $\mathbf{E}$ is defined by

$$\mathbf{E}[X] = \int_\Omega X(\omega) d\mathbf{P}(\omega).$$

Let $\mathbb{K}_0 = Q^{\frac{1}{2}}(\mathbb{K})$ and let $L_2^0 = L_2(\mathbb{K}_0; \mathbb{H})$ with respect to the norm

$$\|\Phi\|^2_{\mathbb{L}_2^0} = \|\Phi\, Q^{\frac{1}{2}}\|_2^2 = \mathrm{Trace}(\Phi\, Q\, \Phi^*) .$$

Throughout, we assume that $A(t) : D(A(t)) \subset L^2(\mathbf{P}; \mathbb{H}) \to L^2(\mathbf{P}; \mathbb{H})$ is a family of densely defined closed linear operators, and $F : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; \mathbb{H})$, $G : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; L_2^0)$ are jointly continuous functions.

In addition to the above-mentioned assumptions, we suppose that $A(t)$ for each $t \in \mathbb{R}$ satisfies the so-called Acquistapace and Terreni conditions given as follows: There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, L, $K \geq 0$, and α, $\beta \in (0, 1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \;\; \|R(\lambda,\ A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \tag{2.1}$$

and

$$\|(A(t) - \lambda_0)R(\lambda,\ A(t) - \lambda_0)[R(\lambda_0,\ A(t)) - R(\lambda_0,\ A(s))]\| \leq L|t - s|^\alpha |\lambda|^\beta$$

for all $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbf{C} - \{0\} : \ |\mathrm{arg}\lambda| \leq \theta\}$.

Note that the above-mentioned Acquistapace and Terreni conditions do guarantee the existence of an evolution family associated with $A(t)$. Throughout the rest of this paper, we denote by $\{U(t, s) : t \geq s \text{ with } \ t, s \in \mathbb{R}\}$, the evolution family of operators associated with the family of operators $A(t)$ for each $t \in \mathbb{R}$. For additional details on evolution families, we refer the reader to the landmark book by Lunardi [11].

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. This setting requires the following preliminary definitions.

Definition 2.1. A stochastic process $X : \mathbb{R} \to L^2(\mathbf{P}; \mathbb{B})$ is said to be continuous whenever

$$\lim_{t \to s} \mathbf{E}\|X(t) - X(s)\|^2 = 0.$$

Definition 2.2. A continuous stochastic process $X : \mathbb{R} \to L^2(\mathbf{P}; \mathbb{B})$ is said to be quadratic-mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} \mathbf{E}\|X(t + \tau) - X(t)\|^2 < \varepsilon.$$

The collection of all quadratic-mean almost periodic stochastic processes $X : \mathbb{R} \to L^2(\mathbf{P}; \mathbb{B})$ will be denoted by $AP(\mathbb{R}; L^2(\mathbf{P}; \mathbb{B}))$.

3. S^2-Almost Periodicity

Definition 3.1. The Bochner transform $X^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a stochastic process $X : \mathbb{R} \to L^2(\mathbf{P}; \mathbb{B})$ is defined by

$$X^b(t, s) := X(t + s).$$

Remark 3.2. A stochastic process $Z(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, is the Bochner transform of a certain stochastic process $X(t)$,

$$Z(t,s)=X^b(t,s)\,,$$

if and only if

$$Z(t+\tau,s-\tau)=Z(s,t)$$

for all $t\in\mathbb{R}$, $s\in[0,1]$ and $\tau\in[s-1,s]$.

Definition 3.3. The space $BS^2(L^2(\mathbf{P};\mathbb{B}))$ of all Stepanov bounded stochastic processes consists of all stochastic processes X on $\mathbb{R}$ with values in $L^2(\mathbf{P};\mathbb{B})$ such that $X^b\in L^\infty\big(\mathbb{R};L^2((0,1);L^2(\mathbf{P};\mathbb{B}))\big)$. This is a Banach space with the norm

$$\begin{aligned}\|X\|_S=\|X^b\|_{L^\infty(\mathbb{R};L^2)} &= \sup_{t\in\mathbb{R}}\left(\int_0^1 \mathbf{E}\|X(t+s)\|^2\,ds\right)^{1/2}\\ &= \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \mathbf{E}\|X(\tau)\|^2\,d\tau\right)^{1/2}.\end{aligned}$$

Definition 3.4. A stochastic process $X\in BS^2(L^2(\mathbf{P};\mathbb{B}))$ is called Stepanov almost periodic (or S^2-almost periodic) if $X^b\in AP\big(\mathbb{R};L^2((0,1);L^2(\mathbf{P};\mathbb{B}))\big)$, that is, for each $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\mathbf{E}\|X(s+\tau)-X(s)\|^2\,ds<\varepsilon.$$

The collection of such functions will be denoted by $S^2AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$.

The proof of the next theorem is straightforward and hence omitted.

Theorem 3.5. *If $X:\mathbb{R}\mapsto L^2(\mathbf{P};\mathbb{B})$ is a quadratic-mean almost periodic stochastic process, then X is S^2-almost periodic, that is, $AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))\subset S^2AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$.*

Lemma 3.6. *Let $(X_n(t))_{n\in\mathbb{N}}$ be a sequence of S^2-almost periodic stochastic processes such that*

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\mathbf{E}\|X_n(s)-X(s)\|^2ds\to 0,\quad \text{as}\ \ n\to\infty.$$

Then $X\in S^2AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$.

Proof. For each $\varepsilon>0$, there exists $N(\varepsilon)$ such that

$$\int_t^{t+1}\|X_n(s)-X(s)\|^2\,ds\le\frac{\varepsilon}{3},\quad \forall t\in\mathbb{R},\quad n\ge N(\varepsilon).$$

From the S^2-almost periodicity of $X_N(t)$, there exists $l(\varepsilon)>0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the following property

$$\int_t^{t+1}\mathbf{E}\|X_N(s+\tau)-X_N(s)\|^2\,ds<\frac{\varepsilon}{3},\quad \forall t\in\mathbb{R}.$$

Now

$$
\begin{aligned}
\mathbf{E}\|X(t+\tau)-X(t)\|^2 &= \mathbf{E}\|X(t+\tau)-X_N(t+\tau)+X_N(t+\tau)-X_N(t)+X_N(t)-X(t)\|^2\\
&\le \mathbf{E}\|X(t+\tau)-X_N(t+\tau)\|^2+\mathbf{E}\|X_N(t+\tau)-X_N(t)\|^2\\
&+ \mathbf{E}\|X_N(t)-X(t)\|^2
\end{aligned}
$$

and hence

$$
\sup_{t\in\mathbb{R}}\int_t^{t+1}\mathbf{E}\|X(s+\tau)-X(s)\|^2 ds < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,
$$

which completes the proof. □

Similarly,

Lemma 3.7. *Let $(X_n(t))_{n\in\mathbb{N}}$ be a sequence of quadratic-mean almost periodic stochastic processes such that*

$$
\sup_{s\in\mathbb{R}}\mathbf{E}\|X_n(s)-X(s)\|^2\to 0, \quad as \;\; n\to\infty
$$

Then $X\in AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$.

Using the inclusion $S^2AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))\subset BS^2(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$ and the fact that $(BS^2(\mathbb{R};L^2(\mathbf{P};\mathbb{B})),\|\cdot\|_S)$ is a Banach space, one can easily see that the next theorem is a straightforward consequence of Lemma 3.6.

Theorem 3.8. *The space $S^2AP(\mathbb{R};L^2(\mathbf{P};\mathbb{B}))$ equipped with the norm*

$$
\|X\|_{S^2}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\mathbf{E}\|X(s)\|^2\,ds\right)^{1/2}
$$

is a Banach space.

Let $(\mathbb{B}_1,\|\cdot\|_{\mathbb{B}_1})$ and $(\mathbb{B}_2,\|\cdot\|_{\mathbb{B}_2})$ be Banach spaces and let $L^2(\mathbf{P};\mathbb{B}_\mathbf{1})$ and $L^2(\mathbf{P};\mathbb{B}_\mathbf{2})$ be their corresponding L^2-spaces, respectively.

Definition 3.9. A function $F:\mathbb{R}\times L^2(\mathbf{P};\mathbb{B}_1)\to L^2(\mathbf{P};\mathbb{B}_2))$, $(t,Y)\mapsto F(t,Y)$ is said to be S^2-almost periodic in $t\in\mathbb{R}$ uniformly in $Y\in\tilde{\mathbb{K}}$ where $\tilde{\mathbb{K}}\subset L^2(\mathbf{P};\mathbb{B}_\mathbf{1})$ is a compact if for any $\varepsilon>0$, there exists $l(\varepsilon,\tilde{\mathbb{K}})>0$ such that any interval of length $l(\varepsilon,\tilde{\mathbb{K}})$ contains at least a number τ for which

$$
\sup_{t\in\mathbf{R}}\int_t^{t+1}\mathbf{E}\|F(s+\tau,Y)-F(s,Y)\|_{\mathbb{B}_2}^2\,ds<\varepsilon
$$

for each stochastic process $Y:\mathbb{R}\to\tilde{\mathbb{K}}$.

Theorem 3.10. *Let $F:\mathbb{R}\times L^2(\mathbf{P};\mathbb{B}_1)\to L^2(\mathbf{P};\mathbb{B}_\mathbf{2})$, $(t,Y)\mapsto F(t,Y)$ be a S^2-almost periodic process in $t\in\mathbb{R}$ uniformly in $Y\in\tilde{\mathbb{K}}$, where $\tilde{\mathbb{K}}\subset L^2(\mathbf{P};\mathbb{B}_\mathbf{1})$ is compact. Suppose that F is Lipschitz in the following sense:*

$$
\mathbf{E}\|F(t,Y)-F(t,Z)\|_{\mathbb{B}_2}^2\le M\;\mathbf{E}\|Y-Z\|_{\mathbb{B}_1}^2
$$

for all $Y,Z\in L^2(\mathbf{P};\mathbb{B}_\mathbf{1})$ and for each $t\in\mathbb{R}$, where $M>0$. Then for any S^2-almost periodic process $\Phi:\mathbb{R}\to L^2(\mathbf{P};\mathbb{B}_\mathbf{1})$, the stochastic process $t\mapsto F(t,\Phi(t))$ is S^2-almost periodic.

4. S^2-Almost Periodic Solutions

Let $C(\mathbb{R}, L^2(\mathbf{P};\mathbb{H}))$ (respectively, $C(\mathbb{R}, L^2(\mathbf{P};\mathbb{L}_2^0))$ denote the class of continuous stochastic processes from $\mathbb{R}$ into $L^2(\mathbf{P};\mathbb{H}))$ (respectively, the class of continuous stochastic processes from $\mathbb{R}$ into $L^2(\mathbf{P};\mathbb{L}_2^0)$).

To study the existence of S^2-almost periodic solutions to Eq. (1.2), we first study the existence of S^2-almost periodic solutions to the stochastic non-autonomous differential equations

$$dX(t) = A(t)X(t)dt + f(t)dt + g(t)dW(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where the linear operators $A(t)$ for $t \in \mathbb{R}$, satisfy the above-mentioned assumptions and the forcing terms $f \in S^2AP(\mathbb{R}, L^2(\mathbf{P};\mathbb{H})) \cap C(\mathbb{R}, L^2(\mathbf{P};\mathbb{H}))$ and $g \in S^2AP(\mathbb{R}, L^2(\mathbf{P};\mathbb{L}_2^0)) \cap C(\mathbb{R}, L^2(\mathbf{P};\mathbb{L}_2^0))$.

Our setting requires the following assumption:

(H.0) The operators $A(t)$, $U(r,s)$ commute and that the evolution family $U(t,s)$ is asymptotically stable. Namely, there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\delta(t-s)} \ \text{ for every } \ t \geq s.$$

In addition, $R(\lambda_0, A(\cdot)) \in S^2AP(\mathbb{R}; \mathcal{L}(L^2(\mathbf{P};\mathbb{H})))$ where λ_0 is as in Eq. (2.1).

Theorem 4.1. *Under previous assumptions, we assume that* (H.0) *holds. Then Eq. (4.1) has a unique solution $X \in S^2AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{H}))$.*

We need the following lemmas. For the proofs of Lemma 4.2 and Lemma 4.3, one can easily follow along the same lines as in the proof of Theorem 4.6.

Lemma 4.2. *Under assumptions of Theorem 4.1, then the integral defined by*

$$X_n(t) = \int_{n-1}^{n} U(t, t-\xi) f(t-\xi) d\xi$$

belongs to $S^2AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{H}))$ for each for $n = 1, 2, ...$

Lemma 4.3. *Under assumptions of Theorem 4.1, then the integral defined by*

$$Y_n(t) = \int_{n-1}^{n} U(t, t-\xi) g(t-\xi) dW(\xi).$$

belongs to $S^2AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{L}_2^0))$ for each for $n = 1, 2, ...$

Proof. (Theorem 4.1) By assumption there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\delta(t-s)} \ \text{ for every } \ t \geq s.$$

Let us first prove uniqueness. Assume that $X : \mathbb{R} \to L^2(\mathbf{P};\mathbb{H})$ is bounded stochastic process that satisfies the homogeneous equation

$$dX(t) = A(t)X(t)dt, \quad t \in \mathbb{R}. \tag{4.2}$$

Then $X(t) = U(t,s)X(s)$ for any $t \geq s$. Hence $\|X(t)\| \leq MDe^{-\delta(t-s)}$ with $\|X(s)\| \leq D$ for $s \in \mathbb{R}$ almost surely. Take a sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence

$(s_{n_k})_{k\in\mathbb{N}} \subset (s_n)_{n\in\mathbb{N}}$ such that $s_{n_k} < t$ for all $k = 1, 2, ...$ By letting $k \to \infty$, we get $X(t) = 0$ almost surely.

Now, if $X_1, X_2 : \mathbb{R} \to L^2(\mathbf{P}; \mathbb{H})$ are bounded solutions to Eq. (4.1), then $X = X_1 - X_2$ is a bounded solution to Eq. (4.2). In view of the above, $X = X_1 - X_2 = 0$ almost surely, that is, $X_1 = X_2$ almost surely.

Now let us investigate the existence. Consider for each $n = 1, 2, ...,$ the integrals

$$X_n(t) = \int_{n-1}^{n} U(t, t-\xi) f(t-\xi) d\xi$$

and

$$Y_n(t) = \int_{n-1}^{n} U(t, t-\xi) g(t-\xi) dW(\xi).$$

First of all, we know by Lemma 4.2 that the sequence X_n belongs to $S^2AP(\mathbb{R}; L^2(\mathbf{P}; \mathbb{H}))$. Moreover, note that

$$\begin{aligned}
\int_t^{t+1} \mathbf{E}\|X_n(s)\|^2\, ds &\leq \int_t^{t+1} \mathbf{E}\| \int_{n-1}^{n} U(s, s-\xi)\, f(s-\xi)\, d\xi\|^2\, ds \\
&\leq M^2 \int_{n-1}^{n} e^{-2\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\|f(s-\xi)\|^2\, ds \right\} d\xi \\
&\leq M^2 \|f\|_{S^2}^2 \left\{ \int_{n-1}^{n} e^{-2\delta\xi}\, d\xi \right\} \\
&\leq \frac{M^2}{2\delta}\, \|f\|_{S^2}^2\, e^{-2\delta n} (e^{2\delta} + 1)\, .
\end{aligned}$$

Since the series

$$\frac{M^2}{2\delta}\, (e^{2\delta} + 1) \sum_{n=2}^{\infty} e^{-2\delta n}$$

is convergent, it follows from the Weirstrass test that the sequence of partial sums defined by

$$L_n(t) := \sum_{k=1}^{n} X_k(t)$$

converges in the sense of the norm $\|\cdot\|_{S^2}$ uniformly on $\mathbb{R}$.

Now let

$$l(t) := \sum_{n=1}^{\infty} X_n(t)$$

for each $t \in \mathbb{R}$.

Observe that

$$l(t) = \int_{-\infty}^{t} U(t, \xi) f(\xi) d\xi, \quad t \in \mathbb{R},$$

and hence $l \in C(\mathbb{R}; L^2(\mathbf{P}; \mathbb{H}))$.

Similarly, the sequence Y_n belongs to $S^2AP(\mathbb{R}; L^2(\mathbf{P}; \mathbb{L}_2^0))$. Moreover, note that

$$\begin{aligned}
\int_t^{t+1} \mathbf{E}\|Y_n(s)\|^2 \, ds &= \mathrm{T}rQ \int_t^{t+1} \mathbf{E} \int_{n-1}^{n} \|U(s, s-\xi)\|^2 \, \|g(s-\xi)\|^2 \, d(\xi) \, ds \\
&\leq M^2 \, \mathrm{T}rQ \int_{n-1}^{n} e^{-2\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^2 \, ds \right\} d\xi \\
&\leq \frac{M^2}{2\delta} \, \mathrm{T}rQ \, \|g\|_{S^2}^2 \, e^{-2\delta n}(e^{2\delta}+1) \, .
\end{aligned}$$

Proceeding as before we can show easily that the sequence of partial sums defined by

$$M_n(t) := \sum_{k=1}^{n} Y_k(t)$$

converges in sense of the norm $\|\cdot\|_{S^2}$ uniformly on $\mathbb{R}$.

Now let

$$m(t) := \sum_{n=1}^{\infty} Y_n(t)$$

for each $t \in \mathbb{R}$.

Observe that

$$m(t) = \int_{-\infty}^{t} U(t, \xi) g(\xi) dW(\xi), \quad t \in \mathbb{R},$$

and hence $m \in C(\mathbb{R}, L^2(\mathbf{P}; \mathbb{L}_2^0))$.

Setting

$$X(t) = \int_{-\infty}^{t} U(t, \xi) f(\xi) \, d\xi + \int_{-\infty}^{t} U(t, \xi) g(\xi) \, dW(\xi),$$

one can easily see that X is a bounded solution to Eq. (4.1). Moreover,

$$\int_t^{t+1} \mathbf{E}\|X(s) - (L_n(s) + M_n(s))\,\|^2 ds \to 0 \quad \text{as} \quad n \to \infty$$

uniformly in $t \in \mathbb{R}$, and hence using Lemma 3.6, it follows that X is a S^2-almost periodic solution. In view of the above, it follows that X is the only bounded S^2-almost periodic solution to Eq. (4.1). □

Throughout the rest of this section, we require the following assumptions:

(H.1) The function $F : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \to L^2(\mathbf{P}; \mathbb{H})$, $(t, X) \mapsto F(t, X)$ is S^2-almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}$ ($\mathcal{O} \subset L^2(\mathbf{P}; \mathbb{H})$ being a compact). Moreover, F is Lipschitz in the following sense: there exists $K > 0$ for which

$$\mathbf{E}\|F(t, X) - F(t, Y)\|^2 \leq \, K \, \mathbf{E}\|X - Y\|^2$$

for all stochastic processes $X, Y \in L^2(\mathbf{P}; \mathbb{H})$ and $t \in \mathbb{R}$;

(H.2) The function $G : \mathbb{R} \times L^2(\mathbf{P};\mathbb{H}) \to L^2(\mathbf{P};\mathbb{L}_2^0)$, $(t,X) \mapsto G(t,X)$ be a S^2-almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}'$ ($\mathcal{O}' \subset L^2(\mathbf{P};\mathbb{H})$ being a compact). Moreover, G is Lipschitz in the following sense: there exists $K' > 0$ for which

$$\mathbf{E}\|G(t,X) - G(t,Y)\|^2_{\mathbb{L}_2^0} \le K'\,\mathbf{E}\|X - Y\|^2$$

for all stochastic processes $X, Y \in L^2(\mathbf{P};\mathbb{H})$ and $t \in \mathbb{R}$.

In order to study (1.2) we need the following lemma which can be seen as an immediate consequence of ([16], Proposition 4.4).

Lemma 4.4. *Suppose $A(t)$ satisfies the Acquistapace and Terreni conditions, $U(t,s)$ is exponentially stable and $R(\lambda_0, A(\cdot)) \in S^2AP(\mathbb{R};\mathcal{L}(L^2(\mathbf{P};\mathbb{H})))$. Let $h > 0$. Then, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length l contains at least a number τ with the property that*

$$\|U(t+\tau, s+\tau) - U(t,s)\| \le \varepsilon\, e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \ge h$.

Definition 4.5. A $\mathcal{F}_t$-progressively process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of (1.2) on $\mathbb{R}$ if

$$\begin{aligned} X(t) \;&=\; U(t,s)X(s) + \int_s^t U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma \\ &+\; \int_s^t U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma) \end{aligned} \tag{4.3}$$

for all $t \ge s$ for each $s \in \mathbb{R}$.

Now, we are ready to present our first main result.

Theorem 4.6. *Under assumptions* (H.0)-(H.1)-(H.2)*, then Eq. (1.2) has a unique S^2-almost period, which is also a mild solution and can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^t U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma + \int_{-\infty}^t U(t,\sigma)G(\sigma,X(\sigma))\,dW(\sigma) \quad \textit{for each } \; t \in \mathbb{R}$$

whenever

$$\Theta := M^2\left(2\frac{K}{\delta^2} + \frac{K'\cdot Tr(Q)}{\delta}\right) < 1.$$

Proof. Consider for each $n = 1, 2, \ldots$, the integral

$$R_n(t) = \int_{n-1}^n U(t,t-\xi)f(t-\xi)\,d\xi + \int_{n-1}^n U(t,t-\xi)g(t-\xi)\,dW(\xi)\,.$$

where $f(\sigma) = F(\sigma, X(\sigma))$ and $g(\sigma) = G(\sigma, X(\sigma))$.

Set

$$X_n(t) = \int_{n-1}^n U(t,t-\xi)f(t-\xi)\,d\xi \;\; \text{and,}$$

$$Y_n(t) = \int_{n-1}^{n} U(t, t-\xi) g(t-\xi)\, dW(\xi)\,.$$

Let us first show that $X_n(\cdot)$ is S^2-almost periodic whenever X is. Indeed, assuming that X is S^2-almost periodic and using (H.1), Theorem 3.10, and Lemma 4.4, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\int_t^{t+1} \mathbf{E}\|f(s+\tau) - f(s)\|^2\, ds < \eta(\varepsilon)$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For the S^2-almost periodicity of $X_n(\cdot)$, we need to consider two cases.
Case 1: $n \geq 2$.

$$\begin{aligned}
&\int_t^{t+1} \mathbf{E}\|X_n(s+\tau) - X_n(s)\|^2\, ds \\
&= \int_t^{t+1} \mathbf{E}\Big\| \int_{n-1}^{n} U(s+\tau, s+\tau-\xi) f(s+\tau-\xi)\, d\xi - \int_{n-1}^{n} U(s, s-\xi) f(s-\xi)\, d\xi \Big\|^2 ds \\
&\leq 2 \int_t^{t+1} \int_{n-1}^{n} \|U(s+\tau, s+\tau-\xi)\|^2 \mathbf{E}\|f(s+\tau-\xi) - f(s-\xi)\|^2\, d\xi\, ds \\
&+ 2 \int_t^{t+1} \int_{n-1}^{n} \|U(s+\tau, s+\tau-\xi) - U(s, s-\xi)\|^2 \mathbf{E}\|f(s-\xi)\|^2\, d\xi\, ds \\
&\leq 2\, M^2 \int_t^{t+1} \int_{n-1}^{n} e^{-2\delta\xi}\, \mathbf{E}\|f(s+\tau-\xi) - f(s-\xi)\|^2\, d\xi\, ds \\
&+ 2\, \varepsilon^2 \int_t^{t+1} \int_{n-1}^{n} e^{-\delta\xi} \mathbf{E}\|f(s-\xi)\|^2\, d\xi\, ds \\
&\leq 2\, M^2 \int_{n-1}^{n} e^{-2\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\|f(s+\tau-\xi) - f(s-\xi)\|^2\, ds \right\} d\xi \\
&+ 2\, \varepsilon^2 \int_{n-1}^{n} e^{-\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\|f(s-\xi)\|^2\, ds \right\} d\xi
\end{aligned}$$

Case 2: $n = 1$.
We have

$$\begin{aligned}
&\int_t^{t+1} \mathbf{E}\|X_1(s+\tau) - X_1(s)\|^2\, ds \\
&= \int_t^{t+1} \mathbf{E}\Big\| \int_0^1 U(s+\tau, s+\tau-\xi)\, f(s+\tau-\xi)\, d\xi - \int_0^1 U(s, s-\xi)\, f(s-\xi)\, d\xi \Big\|^2 ds \\
&\leq 3 \int_t^{t+1} \int_0^1 \|U(s+\tau, s+\tau-\xi)\|^2 \mathbf{E}\|f(s+\tau-\xi) - f(s-\xi)\|^2\, d\xi\, ds
\end{aligned}$$

$$+3\int_t^{t+1}\int_\varepsilon^1 \|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^2\mathbf{E}\|f(s-\xi)\|^2\,d\xi\,ds$$
$$+3\int_t^{t+1}\int_0^\varepsilon \|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^2\mathbf{E}\|f(s-\xi)\|^2\,d\xi\,ds$$
$$\leq 3\,M^2\int_t^{t+1}\int_0^1 e^{-2\delta\xi}\,\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^2\,d\xi\,ds$$
$$+3\varepsilon^2\int_t^{t+1}\int_\varepsilon^1 e^{-\delta\xi}\mathbf{E}\|f(s-\xi)\|^2\,d\xi\,ds+6\,M^2\int_t^{t+1}\int_0^\varepsilon e^{-2\delta\xi}\mathbf{E}\|f(s-\xi)\|^2\,d\xi\,ds$$
$$\leq 3\,M^2\int_0^1 e^{-2\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^2\,ds\right\}d\xi$$
$$+3\varepsilon^2\int_\varepsilon^1 e^{-\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s-\xi)\|^2\,ds\right\}d\xi+6M^2\int_0^\varepsilon e^{-\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s-\xi)\|^2\,ds\right\}d\xi$$

which implies that $X_n(\cdot)$ is S^2-almost periodic.

Similarly, assuming that X is S^2-almost periodic and using (H.2), Theorem 3.10, and Lemma 4.4, given $\varepsilon>0$, one can find $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\|U(t+\tau,s+\tau)-U(t,s)\|\leq\varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\geq\varepsilon$, and

$$\int_t^{t+1}\mathbf{E}\|g(s+\tau)-g(s)\|^2_{\mathbb{L}_2^0}\,ds<\eta(\varepsilon)$$

for each $t\in\mathbb{R}$, where $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

The next step consists in proving the S^2-almost periodicity of $Y_n(\cdot)$. Here again, we need to consider two cases.

Case 1: $n\geq 2$

$$\int_t^{t+1}\mathbf{E}\|Y_n(s+\tau)-Y_n(s)\|^2\,ds$$
$$=\int_t^{t+1}\mathbf{E}\|\int_{n-1}^n U(s+\tau,s+\tau-\xi)\,g(s+\tau-\xi)\,dW(\xi)$$
$$-\int_{n-1}^n U(s,s-\xi)\,g(s-\xi)\,dW(\xi)\|^2\,ds$$
$$\leq 2\,\mathrm{Tr}Q\int_t^{t+1}\int_{n-1}^n\|U(s+\tau,s+\tau-\xi)\|^2\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$+2\,\mathrm{Tr}Q\int_t^{t+1}\int_{n-1}^n\|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^2\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$\leq 2\,\mathrm{Tr}Q\,M^2\int_t^{t+1}\int_{n-1}^n e^{-2\delta\xi}\,\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$

$$+2\,\mathrm{Tr}Q\,\varepsilon^2\int_t^{t+1}\int_{n-1}^{n} e^{-\delta\xi}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$\leq 2\,\mathrm{Tr}Q\,M^2\int_{n-1}^{n} e^{-2\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,ds\right\}d\xi$$
$$+2\,\mathrm{Tr}Q\,\varepsilon^2\int_{n-1}^{n} e^{-\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,ds\right\}d\xi$$

Case 2: $n=1$

$$\int_t^{t+1}\mathbf{E}\|Y_1(s+\tau)-Y_1(s)\|^2\,ds$$
$$=\int_t^{t+1}\mathbf{E}\|\int_0^1 U(s+\tau,s+\tau-\xi)\,g(s+\tau-\xi)\,dW(\xi)$$
$$-\int_n^{n+1} U(s,s-\xi)\,g(s-\xi)\,dW(\xi)\|^2\,ds$$
$$\leq 3\,\mathrm{Tr}Q\int_t^{t+1}\int_0^1\|U(s+\tau,s+\tau-\xi)\|^2\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$+3\,\mathrm{Tr}Q\int_t^{t+1}\int_\varepsilon^1\|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^2\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$+3\,\mathrm{Tr}Q\int_t^{t+1}\int_0^\varepsilon\|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^2\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$\leq 3\,\mathrm{Tr}Q\,M^2\int_t^{t+1}\int_0^1 e^{-2\delta\xi}\,\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$+3\,\mathrm{Tr}Q\,\varepsilon^2\int_t^{t+1}\int_\varepsilon^1 e^{-\delta\xi}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$+6\,\mathrm{Tr}Q\,M^2\int_t^{t+1}\int_0^\varepsilon e^{-2\delta\xi}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi\,ds$$
$$\leq 3\,\mathrm{Tr}Q\,M^2\int_0^1 e^{-2\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\,ds\right\}d\xi$$
$$+3\,\mathrm{Tr}Q\,\varepsilon^2\int_\varepsilon^1 e^{-\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,ds\right\}d\xi$$
$$+6\,\mathrm{Tr}Q\,M^2\int_0^\varepsilon e^{-2\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\,ds\right\}d\xi,$$

which implies that $Y_n(\cdot)$ is S^2-almost periodic.

Setting

$$X(t):=\int_{-\infty}^t U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma+\int_{-\infty}^t U(t,\sigma)G(\sigma,X(\sigma))\,dW(\sigma)$$

and proceeding as in the proof of Theorem 4.1, one can easily see that

$$\int_t^{t+1} \mathbf{E}\| X(s) - (X_n(s) + Y_n(s)) \|^2 ds \to 0 \quad \text{as} \quad n \to \infty$$

uniformly in $t \in \mathbb{R}$, and hence using Lemma 3.6, it follows that X is a S^2-almost periodic solution.

Define the nonlinear operator Γ by

$$\Gamma X(t) := \int_{-\infty}^{t} U(t,\sigma) F(\sigma, X(\sigma))\, d\sigma + \int_{-\infty}^{t} U(t,\sigma) G(\sigma, X(\sigma))\, dW(\sigma)\,.$$

In view of the above, it is clear that Γ maps $S^2AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ into itself. Consequently, using the Banach fixed-point principle it follows that Γ has a unique fixed-point $\{X_0(t), \quad t \in \mathbb{R}\}$ whenever $\Theta < 1$, which in fact is the only S^2-almost periodic solution to Eq. (1.2). □

Our second main result is weaker than Theorem 4.6 although we require that G be bounded in some sense.

Theorem 4.7. *Under assumptions* (H.0)-(H.1)-(H.2)*, if we assume that there exists* $L > 0$ *such that* $\mathbf{E}\|G(t,Y)\|^2_{\mathbb{L}_2^0} \le L$ *for all* $t \in \mathbb{R}$ *and* $Y \in L^2(\mathbf{P};\mathbb{H})$*, then Eq. (1.2) has a unique quadratic-mean almost period mild solution, which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^{t} U(t,\sigma) F(\sigma, X(\sigma))\, d\sigma + \int_{-\infty}^{t} U(t,\sigma) G(\sigma, X(\sigma))\, dW(\sigma) \quad \textit{for each} \quad t \in \mathbb{R}$$

whenever

$$\Theta := M^2 \left(2\frac{K}{\delta^2} + \frac{K' \cdot Tr(Q)}{\delta} \right) < 1.$$

Proof. We use the same notations as in the proof of Theorem 4.6. Let us first show that $X_n(\cdot)$ is quadratic mean almost periodic upon the S^2-almost periodicity of $f = F(\cdot, X(\cdot))$. Indeed, assuming that X is S^2-almost periodic and using (H.1), Theorem 3.10, and Lemma 4.4, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \le \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \ge \varepsilon$, and

$$\int_t^{t+1} \mathbf{E}\|f(s+\tau) - f(s)\|^2\, ds < \eta(\varepsilon)$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The next step consists in proving the quadratic-mean almost periodicity of $X_n(\cdot)$. Here again, we need to consider two cases.

Case 1: $n \ge 2$.

$$\mathbf{E}\|X_n(t+\tau) - X_n(t)\|^2$$
$$= \mathbf{E}\| \int_{n-1}^{n} U(t+\tau, t+\tau-\xi)\, f(t+\tau-\xi)\, d\xi$$

$$
\begin{aligned}
& - \int_{n-1}^{n} U(t, t-\xi)\, f(s-\xi)\, d\xi\|^2 \\
& \leq 2 \int_{n-1}^{n} \|U(t+\tau, t+\tau-\xi)\|^2 \mathbf{E}\|f(t+\tau-\xi) - f(t-\xi)\|^2\, d\xi \\
& +2 \int_{n-1}^{n} \|U(t+\tau, t+\tau-\xi) - U(t, t-\xi)\|^2 \mathbf{E}\|f(t-\xi)\|^2\, d\xi \\
& \leq 2\, M^2 \int_{n-1}^{n} e^{-2\delta\xi}\, \mathbf{E}\|f(t+\tau-\xi) - f(t-\xi)\|^2\, d\xi \\
& +2\, \varepsilon^2 \int_{n-1}^{n} e^{-\delta\xi} \mathbf{E}\|f(t-\xi)\|^2\, d\xi \\
& \leq 2\, M^2 \int_{n-1}^{n} e^{-2\delta\xi}\, \mathbf{E}\|f(t+\tau-\xi) - f(t-\xi)\|^2\, d\xi \\
& +2\, \varepsilon^2 \int_{n-1}^{n} e^{-\delta\xi}\, \mathbf{E}\|f(t-\xi)\|^2\, d\xi \\
& \leq 2\, M^2 \int_{t-n+1}^{t-n} \mathbf{E}\|f(r+\tau) - f(r)\|^2\, dr + 2\, \varepsilon^2 \int_{t-n+1}^{t-n} \mathbf{E}\|f(r)\|^2\, dr
\end{aligned}
$$

Case 2: $n = 1$.

$$
\begin{aligned}
& \mathbf{E}\|X_1(t+\tau) - X_1(t)\|^2 \\
& = \mathbf{E}\| \int_0^1 U(t+\tau, t+\tau-\xi)\, f(t+\tau-\xi)\, d\xi - \int_0^1 U(t, t-\xi)\, f(t-\xi)\, d\xi\|^2 \\
& \leq 3\, \mathbf{E} \left[\int_0^1 \|U(t+\tau, t+\tau-\xi)\|\, \|f(t+\tau-\xi) - f(t-\xi)\|\, d\xi \right]^2 \\
& +3\, \mathbf{E} \left[\int_\varepsilon^1 \|U(t+\tau, t+\tau-\xi) - U(t, t-\xi)\|\, \|f(t-\xi)\|\, d\xi \right]^2 \\
& +3\, \mathbf{E} \left[\int_0^\varepsilon \|U(t+\tau, t+\tau-\xi) - U(t, t-\xi)\|\, \|f(t-\xi)\|\, d\xi \right]^2 \\
& \leq 3\, M^2\, \mathbf{E} \left[\int_0^1 e^{-\delta\xi}\, \|f(t+\tau-\xi) - f(t-\xi)\|\, d\xi \right]^2 \\
& +3\, \varepsilon^2\, \mathbf{E} \left[\int_\varepsilon^1 e^{-\frac{\delta}{2}\xi}\, \|f(t-\xi)\|\, d\xi \right]^2 + 12\, M^2\, \mathbf{E} \left[\int_0^\varepsilon e^{-\delta\xi}\, \|f(t-\xi)\|^2\, d\xi \right]^2
\end{aligned}
$$

Now, using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
& \leq 3\, M^2 \left(\int_0^1 e^{-\delta\xi}\, d\xi \right) \left(\int_0^1 e^{-\delta\xi} \mathbf{E}\|f(t+\tau-\xi) - f(t-\xi)\|^2\, d\xi \right) \\
& +3\, \varepsilon^2 \left(\int_\varepsilon^1 e^{-\frac{\delta}{2}\xi}\, d\xi \right) \left(\int_\varepsilon^1 e^{-\frac{\delta}{2}\xi} \mathbf{E}\|f(t-\xi)\|^2\, d\xi \right)
\end{aligned}
$$

$$+12\,M^2\left(\int_0^\varepsilon e^{-\delta\xi}\,d\xi\right)\left(\int_0^\varepsilon e^{-\delta\xi}\mathbf{E}\|f(t-\xi)\|^2\,d\xi\right)$$

$$\leq 3\,M^2\int_{t-1}^{t}\mathbf{E}\|f(r+\tau)-f(r)\|^2\,dr$$

$$+3\,\varepsilon^2\int_{t-1}^{t-\varepsilon}\mathbf{E}\|f(r)\|^2\,dr+12\,M^2\,\varepsilon\int_{t-\varepsilon}^{t}\mathbf{E}\|f(r)\|^2\,dr,$$

which implies that $X_n(\cdot)$ quadratic mean almost periodic.

Similarly, using (H.2), Theorem 3.10, and Lemma 4.4, given $\varepsilon>0$, one can find $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\|U(t+\tau,s+\tau)-U(t,s)\|\leq\varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\geq\varepsilon$, and

$$\int_t^{t+1}\mathbf{E}\|g(s+\tau)-g(s)\|^2_{\mathbb{L}_2^0}\,ds<\eta$$

for each $t\in\mathbb{R}$, where $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$. Moreover, there exists a positive constant $L>0$ such that

$$\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|g(\sigma)\|^2_{\mathbb{L}_2^0}\leq L.$$

The next step consists in proving the quadratic mean almost periodicity of $Y_n(\cdot)$.

Case 1: $n\geq 2$

$$\mathbf{E}\,\|Y_n(t+\tau)-Y_n(t)\|^2$$

$$=\mathbf{E}\left\|\int_{n-1}^{n}U(t+\tau,t+\tau-\xi)\,g(s+\tau-\xi)\,dW(\xi)-\int_{n-1}^{n}U(t,t-\xi)\,g(t-\xi)\,dW(\xi)\right\|^2$$

$$\leq 2\,\mathrm{Tr}Q\int_{n-1}^{n}\|U(t+\tau,t+\tau-\xi)\|^2\mathbf{E}\|g(t+\tau-\xi)-g(t-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi$$

$$+2\,\mathrm{Tr}Q\int_{n-1}^{n}\|U(t+\tau,t+\tau-\xi)-U(t,t-\xi)\|^2\mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi$$

$$\leq 2\,\mathrm{Tr}Q\,M^2\int_{n-1}^{n}e^{-2\delta\xi}\,\mathbf{E}\|g(t+\tau-\xi)-g(t-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi$$

$$+2\,\mathrm{Tr}Q\,\varepsilon^2\int_{n-1}^{n}e^{-\delta\xi}\mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\,d\xi$$

$$\leq 2\,\mathrm{Tr}Q\,M^2\int_{t-n}^{t-n+1}\mathbf{E}\|g(r+\tau)-g(r)\|^2_{\mathbb{L}_2^0}\,dr+2\,\mathrm{Tr}Q\,\varepsilon^2\int_{t-n}^{t-n+1}\mathbf{E}\|g(r)\|^2_{\mathbb{L}_2^0}\,dr.$$

Case 2: $n=1$

$$\mathbf{E}\|Y_1(t+\tau)-Y_1(t)\|^2$$

$$=\mathbf{E}\|\int_0^1 U(t+\tau,t+\tau-\xi)\,g(s+\tau-\xi)\,dW(\xi)$$

$$
\begin{aligned}
& - \int_0^1 U(t,t-\xi)\, g(t-\xi)\, dW(\xi)\|^2 \\
& \leq 3\, \mathrm{Tr}Q \int_0^1 \|U(t+\tau,t+\tau-\xi)\|^2 \mathbf{E}\|g(t+\tau-\xi)-g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& +3\, \mathrm{Tr}Q \int_t^{t+1} \left(\int_\varepsilon^1 + \int_0^\varepsilon \right) \|U(t+\tau,t+\tau-\xi)-U(t,t-\xi)\|^2 \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& \leq 3\, \mathrm{Tr}Q\, M^2 \int_0^1 e^{-2\delta\xi}\, \mathbf{E}\|g(t+\tau-\xi)-g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& +3\, \mathrm{Tr}Q\, \varepsilon^2 \int_\varepsilon^1 e^{-\delta\xi} \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi + 6\, \mathrm{Tr}Q\, M^2 \int_0^\varepsilon e^{-2\delta\xi} \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& \leq 3\, \mathrm{Tr}Q\, M^2 \int_0^1 \mathbf{E}\|g(t+\tau-\xi)-g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& +3\, \mathrm{Tr}Q\, \varepsilon^2 \int_\varepsilon^1 \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi + 6\, \mathrm{Tr}Q\, M^2 \int_0^\varepsilon \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& \leq 3\, \mathrm{Tr}Q\, M^2 \int_{t-1}^t \mathbf{E}\|g(r+\tau)-g(r)\|^2_{\mathbb{L}_2^0}\, dr \\
& +3\, \mathrm{Tr}Q\, \varepsilon^2 \int_{t-1}^t \mathbf{E}\|g(r)\|^2_{\mathbb{L}_2^0}\, dr + 6\, \mathrm{Tr}Q\, M^2 \int_0^\varepsilon \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi \\
& \leq 3\mathrm{Tr}QM^2 \int_{t-1}^t \mathbf{E}\|g(r+\tau)-g(r)\|^2_{\mathbb{L}_2^0} dr + 3\mathrm{Tr}Q\varepsilon^2 \int_{t-1}^t \mathbf{E}\|g(r)\|^2_{\mathbb{L}_2^0} dr + 6\varepsilon \mathrm{Tr}QM^2 L,
\end{aligned}
$$

which implies that $Y_n(\cdot)$ is quadratic-mean almost almost periodic. Moreover, setting

$$
X(t) = \int_{-\infty}^t U(t,\sigma)F(\sigma,X(\sigma))\, d\sigma + \int_{-\infty}^t U(t,\sigma)G(\sigma,X(\sigma))\, dW(\sigma)
$$

for each $t \in \mathbb{R}$ and proceeding as in the proofs of Theorem 4.1 and Theorem 4.6, one can easily see that

$$
\sup_{s\in\mathbb{R}} \mathbf{E}\|X(s) - (X_n(s)+Y_n(s))\,\|^2 \to 0 \quad \text{as} \quad n \to \infty
$$

and hence using Lemma 3.7, it follows that X is a quadratic mean almost periodic solution to Eq. (1.2).

In view of the above, the nonlinear operator Γ as in the proof of Theorem 4.6 maps $AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ into itself. Consequently, using the Banach fixed-point principle it follows that Γ has a unique fixed-point $\{X_1(t),\ t \in \mathbb{R}\}$ whenever $\Theta < 1$, which in fact is the only quadratic mean almost periodic solution to Eq. (1.2).

□

5. Example

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded subset whose boundary $\partial\mathcal{O}$ is both of class C^2 and locally on one side of $\mathcal{O}$. Of interest is the following stochastic parabolic partial differential equation

$$d_t X(t,x) = A(t,x)X(t,x)d_t + F(t, X(t,x))d_t + G(t, X(t,x))\, dW(t), \tag{5.1}$$

$$\sum_{i,j=1}^{n} n_i(x) a_{ij}(t,x) d_i X(t,x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial\mathcal{O}, \tag{5.2}$$

where $d_t = \dfrac{d}{dt}$, $d_i = \dfrac{d}{dx_i}$, $n(x) = (n_1(x), n_2(x), ..., n_n(x))$ is the outer unit normal vector, the family of operators $A(t,x)$ are formally given by

$$A(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}\right) + c(t,x), \quad t \in \mathbb{R}, \quad x \in \mathcal{O},$$

W is a real valued Brownian motion, and a_{ij}, c $(i,j = 1,2,...,n)$ satisfy the following conditions:

We require the following assumptions:

(H.3) The coefficients $(a_{ij})_{i,j=1,...,n}$ are symmetric, that is, $a_{ij} = a_{ji}$ for all $i,j = 1,...,n$. Moreover,

$$a_{ij} \in C_b^{\mu}(\mathbb{R}; L^2(\mathbf{P}; C(\overline{\mathcal{O}}))) \cap C_b(\mathbb{R}; L^2(\mathbf{P}; C^1(\overline{\mathcal{O}}))) \cap S^2AP(\mathbb{R}; L^2(\mathbf{P}; L^2(\mathcal{O})))$$

for all $i,j = 1,...n$, and

$$c \in C_b^{\mu}(\mathbb{R}; L^2(\mathbf{P}; L^2(\mathcal{O}))) \cap C_b(\mathbb{R}; L^2(\mathbf{P}; C(\overline{\mathcal{O}}))) \cap S^2AP(\mathbb{R}; L^2(\mathbf{P}; L^1(\mathcal{O})))$$

for some $\mu \in (1/2, 1]$.

(H.4) There exists $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\eta_i\, \eta_j \geq \delta_0 |\eta|^2,$$

for all $(t,x) \in \mathbb{R} \times \overline{\mathcal{O}}$ and $\eta \in \mathbb{R}^n$.

Under previous assumptions, the existence of an evolution family $U(t,s)$ satisfying (H.0) is guaranteed, see, eg., [16].

Now let $\mathbb{H} = L^2(\mathcal{O})$ and let $H^2(\mathcal{O})$ be the Sobolev space of order 2 on $\mathcal{O}$. For each $t \in \mathbb{R}$, define an operator $A(t)$ on $L^2(\mathbf{P}; \mathbb{H})$ by

$$\mathcal{D}(A(t)) = \{X \in L^2(\mathbf{P}, H^2(\mathcal{O})) : \sum_{i,j=1}^{n} n_i(\cdot) a_{ij}(t,\cdot) d_i X(t,\cdot) = 0 \text{ on } \partial\mathcal{O}\} \text{ and,}$$

$$A(t)X = A(t,x)X(x), \text{ for all } X \in \mathcal{D}(A(t))\,.$$

Let us mention that Corollary 5.1 and Corollary 5.2 are immediate consequences of Theorem 4.6 and Theorem 4.7, respectively.

Corollary 5.1. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)*, then Eqns.(5.1)-(5.2) has a unique mild solution, which obviously is S^2-almost periodic, whenever M is small enough.*

Similarly,

Corollary 5.2. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)*, if we suppose that there exists $L > 0$ such that $\mathbf{E}\|G(t,Y)\|^2_{\mathbb{L}_2^0} \leq L$ for all $t \in \mathbb{R}$ and $Y \in L^2(\mathbf{P}; L^2(\mathcal{O}))$. Then the system Eqns. (5.1)-(5.2) has a unique quadratic mean almost periodic, whenever M is small enough.*

References

1. P. Acquistapace and B. Terreni, A Unified Approach to Abstract Linear Parabolic Equations, *Tend. Sem. Mat. Univ. Padova* **78** (1987), pp. 47-107.
2. P. Bezandry and T. Diagana, Existence of Almost Periodic Solutions to Some Stochastic Differential Equations. *Applicable Anal.*. **86** (2007), no. 7, pp. 819 - 827.
3. P. Bezandry and T. Diagana, Square-mean almost periodic solutions nonautonomous stochastic differential equations. *Electron. J. Diff. Eqns.* Vol. **2007**(2007), no. 117, pp. 1-10.
4. C. Corduneanu, *Almost Periodic Functions*, 2nd Edition. Chelsea-New York, 1989.
5. G. Da Prato and C. Tudor, Periodic and Almost Periodic Solutions for Semilinear Stochastic Evolution Equations, *Stoch. Anal. Appl.* **13**(1) (1995), pp. 13-33.
6. T. Diagana, Pseudo Almost Periodic Functions in Banach Spaces. Nova Scieence Publishers, New York, 2007.
7. T. Diagana, Stepanov-like Pseudo Almost Periodicity and Its Applications to Some Nonautonmous Differential Equations. *Nonlinear Anal.* (in press).
8. T. Diagana, Stepanov-like Pseudo Almost Periodic Functions and Their Applications to Differential Equations, *Commun. Math. Anal.* **3**(2007), no. 1, pp. 9-18.
9. A. Ya. Dorogovtsev and O. A. Ortega, On the Existence of Periodic Solutions of a Stochastic Equation in a Hilbert Space. *Visnik Kiiv. Univ. Ser. Mat. Mekh.* **115** (1988), no. 30, pp. 21-30.
10. A. Ichikawa, Stability of Semilinear Stochastic Evolution Equations. *J. Math. Anal. Appl.* **90** (1982), no. 1, pp. 12-44.
11. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, PNLDE Vol. *16*, Birkhäauser Verlag, Basel, 1995.
12. D. Kannan and A.T. Bharucha-Reid, On a Stochastic Integro-differential Evolution of Volterra Type. *J. Integral Equations* **10** (1985), pp. 351-379.
13. T. Kawata, Almost Periodic Weakly Stationary Processes. *Statistics and probability: essays in honor of C. R. Rao*, pp. 383–396, North-Holland, Amsterdam-New York, 1982.
14. D. Keck and M. McKibben, Functional Integro-differential Stochastic Evolution Equations in Hilbert Space. *J. Appl. Math. Stochastic Anal.* **16**, no.2 (2003), pp. 141-161.
15. D. Keck and M. McKibben, Abstract Stochastic Integro-differential Delay Equations. *J. Appl. Math. Stoch. Anal.* 2005, no. 3, 275-305.
16. L. Maniar and R. Schnaubelt, Almost Periodicity of Inhomogeneous Parabolic Evolution Equations, Lecture Notes in Pure and Appl. Math. Vol. **234**, Dekker, New York, 2003, pp. 299-318.
17. C. Tudor, Almost Periodic Solutions of Affine Stochastic Evolutions Equations, *Stochastics and Stochastics Reports* **38** (1992), pp. 251-266.

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