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Existence and Boundary Stabilization of the Semilinear Mindlin-Timoshenko System

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Abstract

We consider dynamics of the one-dimensional Mindlin-Timoshenko model for beams with a nonlinear external forces and a boundary damping mechanism. We investigate existence and uniqueness of strong and weak solution. We also study the boundary stabilization of the solution, i.e., we prove that the energy of every solution decays exponentially as $t \to \infty$.

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1 Introduction

A widely accepted dynamical model describing the transverse vibrations of beams is the Mindlin-Timoshenko system of equations. This system is chosen because it is a more accurate model than the Euler-Bernoulli beam one and because it also takes into account transverse shear effects. The Mindlin-Timoshenko system is used, for example, to model aircraft wings. For a beam of length $L > 0$ this one-dimensional system reads as

$$\left| \begin{array}{l} \dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k\left(u + v_x\right) + f\left(u\right) = 0 \quad \text{in} \quad Q, \\ \rho h v_{tt} - k\left(u + v_x\right)_x + g\left(v\right) = 0 \quad \text{in} \quad Q, \end{array} \right. \tag{1.1}$$

where $Q = (0, L) \times (0, T)$ and $T > 0$ is a given time. In (1.1) subscripts mean partial derivatives. Here the function $u = u\left(x, t\right)$ is the angle of deflection of a filament (it is measure of transverse shear effects) and $v = v\left(x, t\right)$ is the transverse displacement of the beam at time t. The constant $h > 0$ represents the thickness of the beam that, for this

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model, is considered to be small and uniform, independent of x. The constant ρ is the mass density per unit volume of the beam and the parameter k is the so called modulus of elasticity in shear. It is given by the formula $k = \widehat{k}Eh/2\,(1+\mu)$, where $\widehat{k}$ is a shear correction coefficient, E is the Young's modulus and μ is the Poisson's ratio, $0 < \mu < 1/2$. The functions f and g represent nonlinear external forces. For details concerning the Mindlin-Timoshenko hypotheses and governing equations see, for example, Lagnese [7] and Lagnese-Lions [8].

We impose the following boundary conditions:

$$\left| \begin{array}{l} u\,(0,\cdot) = v\,(0,\cdot) = 0 \quad \text{on} \quad (0,T)\,, \\ u_x\,(L,\cdot) + u_t\,(L,\cdot) = 0 \quad \text{on} \quad (0,T)\,, \\ u\,(L,\cdot) + v_x\,(L,\cdot) + v_t\,(L,\cdot) = 0 \quad \text{on} \quad (0,T)\,. \end{array} \right. \tag{1.2}$$

The conditions (1.2) assure that the beam stays clamped in the end $x = 0$ and in the end $x = L$ it is supported and suffering action of a dissipative force.

To complete the system, let us include the initial conditions:

$$u\,(\cdot,0) = u^0,\ u_t\,(\cdot,0) = u^1,\ v\,(\cdot,0) = v^0,\ v_t\,(\cdot,0) = v^1 \quad \text{in} \quad (0,L)\,. \tag{1.3}$$

Several authors analyzed different aspects of the Mindlin-Timosheko system. In the linear case ($f \equiv g \equiv 0$) we can cite Lagnese-Lions [8], Medeiros [12], which studied the exact controllability property using the Hilbert Uniqueness Method (HUM) introduced by Lions (see [11]) and Lagnese [7] which analyzed the asymptotic behavior (as $t \to \infty$) of the system. In Araruna-Zuazua [2] was made a spectral analysis of the system allowing to obtain a controllability using HUM combined with arguments of non-harmonic analysis. In the semilinear case, we can mention Parente et. al. [16], which treat about existence and uniqueness for the problem $(1.1)-(1.3)$, with the functions f and g being Lipschitz continuous, applying the same method used in Milla Miranda-Medeiros [15]. The existence of a compact global attractor, in the 2-dimensional case, was studied in Chueshov-Lasiecka [4] with the nonlinearities f and g being locally Lipschitz. All the mentioned papers are treated with different boundary conditions involving several situations that appear in the engineering.

In this work we state a result of existence of solutions for the system $(1.1)-(1.3)$, when the nonlinearities f and g satisfy the following conditions:

$$f, g \text{ are continuous function, such that } f\,(s)\,s \geq 0 \text{ and } g(s)s \geq 0,\ \forall s \in \mathbb{R}. \tag{1.4}$$

Furthermore, we analyze the asymptotic behavior (as $t \to \infty$) of the solutions with the nonlinearities satisfy the additional growth condition:

$$\exists \delta_1 > 0 \text{ such that } f(s)s \geq (2+\delta_1)F(s),\ \forall s \in \mathbb{R},\ \text{where } F\,(s) = \int_0^s f\,(t)\,dt \tag{1.5}$$

and

$$\exists \delta_2 > 0 \text{ such that } g(s)s \geq (2+\delta_2)G(s),\ \forall s \in \mathbb{R},\ \text{where } G\,(s) = \int_0^s g\,(t)\,dt. \tag{1.6}$$

Precisely, we show the existence of positive constants $C > 0$ and $\kappa > 0$ such that the energy of the system (1.1) defined by

$$\begin{aligned}E(t) = \; & \frac{1}{2}\left[\frac{\rho h^3}{12}\int_0^L |u_t(x,t)|^2\,dx + \rho h\int_0^L |v_t(x,t)|^2\,dx + k\int_0^L |(u+v_x)(x,t)|^2\,dx\right.\\ & \left. + \int_0^L |u_x(x,t)|^2\,dx + 2\int_0^L F(u(x,t))dx + 2\int_0^L G(v(x,t))dx\right]\end{aligned} \tag{1.7}$$

verifies the estimate

$$E(t) \le CE(0)\,e^{-\kappa t},\ \forall t \ge 0. \tag{1.8}$$

The uniqueness for the semilinear Mindlin-Timoshenko system $(1.1)-(1.3)$ with the general nonlinearities considered here is a open problem.

To obtain existence of solution of the semilinear Mindlin-Timoshenko problem $(1.1)-(1.3)$, we found difficulties to show that the solution verifies the boundary conditions (1.2) and to overcome them, we use the same techniques applied in [1], that consists essentially in to combine results involving non-homogeneous boundary value problem with hidden regularity arguments. Boundary stability is also analyzed, that is, we show that the energy (1.7) associated to weak solution of the problem $(1.1)-(1.3)$ tends to zero exponentially as $t \to \infty$. In order, the exponential decay was obtained by constructing perturbed energy functional for which differential inequality leads to this rate decay. We apply this method motivated by work of Komornik-Zuazua [6], whose authors treated this issue for semilinear wave equation.

The paper is organized as follows. Section 2 contains some notations and essential results which we apply in this work. In Section 3 we prove existence and uniqueness of strong solution for $(1.1)-(1.3)$ employing the Faedo-Galerkin's method with a special basis like in [15] with f and g being Lipschitz continuous functions satisfying a sign condition. Section 4 is devoted to get existence of weak solution of $(1.1)-(1.3)$, with f and g satisfying (1.4). For this, we approached the functions f and g by Lipschitz functions, as in Strauss [17], and we obtain the weak solution as limit of sequence of strong solutions acquired in the Section 3. We still analyze the uniqueness only for some particular cases of f and g which permit the application of the energy method as in Lions [9]. Finally, in Section 5 we prove the exponential decay for the energy associated to weak solution of the problem $(1.1)-(1.3)$ making use of the perturbed energy method as in [6].

2 Some Notations and Results

Let us represent by $\mathcal{D}(0,T)$ the space of the test functions defined in $(0,T)$ and $H^1(0,L)$ the usual Sobolev space. We define the Hilbert space

$$V = \left\{v \in H^1(0,L)\,;\ v(0) = 0\right\}$$

equipped with the inner product and norm given by

$$((u,v)) = (u_x, v_x)\,, \qquad \|u\|^2 = |u_x|^2\,,$$

where $(\cdot,\cdot)$ and $|\cdot|$ are, respectively, the inner product and norm in $L^2(0,L)$. By V' we denote the dual of V.

Let us consider the operator $-\frac{d^2}{dx^2}$ defined by tripled $\{V,\ L^2(0,L);\ ((\cdot,\cdot))\}$ with domain

$$D=\left\{u\in V\cap H^2(0,L);\ u_x(L)=0\right\}.$$

Let us represent by E the Banach space

$$E=\left\{v\in L^2(\Omega);\ v_{xx}\in L^1(\Omega)\right\}$$

with the norm

$$\|v\|_E=|v|+\|v_{xx}\|_{L^1(\Omega)}.$$

The trace application $\gamma: E\to\mathbb{R}^4$ defined by $\gamma v=(v(0),v(L),v_x(0),v_x(L))$ is linear and continuous, see Milla Miranda-Medeiros [14, Proposition 3.2].

In what follows, we will use C to denote a generic positive constant which may vary from line to line (unless otherwise stated).

We will now establish some results of elliptic regularity essential for the development of this work.

Proposition 2.1 *Let us consider $f\in L^2(0,L)$ and $\beta\in\mathbb{R}$. Then the solution u of the boundary value problem*

$$\left|\begin{array}{l}-u_{xx}=f \ \text{ in }\ (0,L),\\ u(0)=0,\\ u_x(L)=\beta,\end{array}\right. \tag{2.1}$$

belongs to $V\cap H^2(0,L)$. Furthermore, there exists a constant $C>0$ such that

$$\|u\|_{H^2(0,L)}\le C\left[|f|+|\beta|\right]. \tag{2.2}$$

Proof. We consider the function $h:[0,L]\to\mathbb{R}$, defined by $h(x)=\beta x$. Thus

$$\|h\|_{H^2(0,L)}=C\,|\beta|, \tag{2.3}$$

where $C=\sqrt{(L^3/3)+L}$.

Let w be the unique solution of the following boundary value problem:

$$\left|\begin{array}{l}-w_{xx}=f \ \text{ in }\ (0,L),\\ w(0)=0,\\ w_x(L)=0.\end{array}\right.$$

Since $f\in L^2(0,L)$, we have by classical elliptic result (see for instance [3]) that $w\in D$ and the existence of a constant $C>0$ such that

$$\|w\|_{H^2(0,L)}\le C\,|f|. \tag{2.4}$$

In this way, $u = w + h \in V \cap H^2(0, L)$ solves (2.1) satisfying (2.2). ■

We would like to prove existence and uniqueness of solution for the problem

$$\left| \begin{array}{l} -u_{xx} = f \text{ in } (0, L), \text{ with } f \in L^1(0, L), \\ u(0) = 0, \\ u_x(L) = 0. \end{array} \right. \tag{2.5}$$

Formally, we obtain from (2.5) that

$$\int_0^L u(-v_{xx})dx + u_x(0)\, v(0) + u(L)\, v_x(L) = \int_0^L fv dx. \tag{2.6}$$

Taking in (2.6) $v \in D$, we obtain

$$\int_0^L u(-v_{xx})dx = \int_0^L fv dx, \ \forall v \in D. \tag{2.7}$$

We adopt (2.7) as definition of solution of (2.5) in the sense of transposition (see [10]). To guarantee the existence and uniqueness of (2.5) we consider the follow result:

Proposition 2.2 *If $f \in L^1(0, L)$, then there exists a unique function $u \in E$ satisfying (2.7). The application $T : L^1(0, L) \to L^2(0, L)$ such that $Tf = u$ is linear, continuous and $-u_{xx} = f$.*

Proof. Let $g \in L^2(0, L)$ and v be a solution of the problem

$$\left| \begin{array}{l} -v_{xx} = g \text{ in } (0, L), \\ v(0) = 0, \\ v_x(L) = 0. \end{array} \right. \tag{2.8}$$

We have $v \in D$.

Let us consider the application $S : L^2(0, L) \to C^0([0, L])$ such that $Sg = v$, where v is the solution of (2.8). Then S is linear and continuous. Let S^* be the transpose of S, that is,

$$S^* : \left[C^0([0, L])\right]' \to L^2(0, L); \quad \langle S^*\theta, \phi \rangle = \langle \theta, S\phi \rangle, \ \forall \phi \in L^2(0, L),$$

where $\langle \cdot, \cdot \rangle$ represents different pairs of duality. Let us prove that the function $u = S^* f$ satisfies (2.7). In fact, we have $\langle S^* f, g \rangle = \langle f, Sg \rangle$, which means

$$\int_0^L u(-v_{xx})dx = \int_0^L fv dx.$$

For the uniqueness, we consider u_1, $u_2 \in L^2(0, L)$ satisfying (2.7). Then

$$\int_0^L (u_1 - u_2)(-v_{xx})\, dx = 0, \ \forall v \in D. \tag{2.9}$$

Considering $g \in L^2(0,L)$ and v be a solution of (2.8), we get

$$\int_0^L (u_1 - u_2)\, g dx = 0, \ \forall g \in L^2(0,L).$$

Therefore $u_1 = u_2$ and the uniqueness is proved. Since $T = S^*$ and S^* is linear and continuous, it follows that T has the same properties. ■

For the non-homogeneous boundary value problem

$$\left| \begin{array}{l} -u_{xx} = f \text{ in } (0,L), \\ u(0) = 0, \\ u_x(L) = \beta, \end{array} \right. \tag{2.10}$$

we consider the following result:

Proposition 2.3 *Let $f \in L^1(0,L)$ and $\beta \in \mathbb{R}$. Then there exists a unique solution $u \in E$ for the problem* (2.10).

Proof. Let us consider the function $\xi : [0,L] \to \mathbb{R}$, defined by $\xi(x) = \beta x$. Let w be the solution of the problem

$$\left| \begin{array}{l} -w_{xx} = f \text{ in } (0,L), \\ w(0) = 0, \\ w_x(L) = 0. \end{array} \right.$$

Since $f \in L^1(0,L)$, by Proposition 2.2, it follows that $w \in E$. Taking $u = w + \xi$, we have $u \in E$ is a solution of (2.10).

For the uniqueness, let u_1 and u_2 two solutions of (2.10). Then $v = u_1 - u_2$ is solution of

$$\left| \begin{array}{l} -v_{xx} = 0 \text{ in } (0,L), \\ v(0) = 0, \\ v_x(L) = 0. \end{array} \right.$$

Hence, by Proposition 2.2, we have $v = 0$, which implies $u_1 = u_2$. ■

Proposition 2.4 *In $V \cap H^2(0,L)$ the norms $H^2(0,L)$ and the norm*

$$u \mapsto \left(|-u_{xx}|^2 + |u_x(L)|^2 \right)^{\frac{1}{2}}, \tag{2.11}$$

are equivalents.

Proof. Let $u \in V \cap H^2(0,L)$. Then, according to Proposition 2.1, we can guarantee that

$$\|u\|_{H^2(0,L)} \leq C \left(|-u_{xx}|^2 + |u_x(L)|^2 \right)^{\frac{1}{2}}.$$

On the other hand, since the embedding of $H^2(0,L)$ in $C^1([0,L])$ is continuous, we have

$$|u_x(L)| \leq \|u\|_{C^1([0,L])} \leq C \|u\|_{H^2(0,L)}.$$

We also have $|-u_{xx}|^2 \leq C \|u\|_{H^2(0,L)}$. In this way we obtain the result. ■

We consider $V \cap H^2(0,L)$ equipped with the norm (2.11).

Proposition 2.5 *Let us suppose* $u^0, v^0 \in V \cap H^2(0,L)$ *and* $u^1, v^1 \in V$ *such that* $u_x^0(L) + u^1(L) = 0$ *and* $u^0(L) + v_x^0(L) + v^1(L) = 0$. *Then, for each* $\epsilon > 0$, *there exist* $w^{(1)}$, $z^{(1)}$, $w^{(2)}$ *and* $z^{(2)}$ *in* $V \cap H^2(0,L)$ *such that*

$$\left\| w^{(1)} - u^0 \right\|_{V \cap H^2(0,L)} < \epsilon, \quad \left\| z^{(1)} - u^1 \right\|_V < \epsilon,$$

$$\left\| w^{(2)} - v^0 \right\|_{V \cap H^2(0,L)} < \epsilon \quad \textit{and} \quad \left\| z^{(2)} - v^1 \right\|_V < \epsilon,$$

with

$$w_x^{(1)}(L) + z^{(1)}(L) = 0 \quad \textit{and} \quad u^0(L) + w_x^{(2)}(L) + z^{(2)}(L) = 0.$$

Proof. Since $V \cap H^2(0,L)$ is dense in V, for each $\epsilon > 0$, there exist $z^{(1)}, z^{(2)} \in V \cap H^2(0,L)$ such that $\left\| z^{(1)} - u^1 \right\|_V < \epsilon$ and $\left\| z^{(2)} - v^1 \right\|_V < \epsilon$.

Let us consider $w^{(1)}$ to be a solution of the problem

$$\left| \begin{array}{l} -w_{xx}^{(1)} = -u_{xx}^0 \text{ in } (0,L), \\ w^{(1)}(0) = 0, \\ w_x^{(1)}(L) = -z^{(1)}(L). \end{array} \right.$$

According to Proposition 2.1, it follows that $w^{(1)} \in V \cap H^2(0,L)$ and

$$\begin{array}{l} \left\| w^{(1)} - u^0 \right\|_{V \cap H^2(0,L)}^2 = \left| -w_{xx}^{(1)} + u_{xx}^0 \right|^2 + \left| w_x^{(1)}(L) - u_x^0(L) \right|^2 = \left| -z^{(1)}(L) + u^1(L) \right|^2 \\ \leq C \left\| z^{(1)} - u^1 \right\|_V^2 < C\epsilon^2. \end{array}$$

Analogously, let us consider $w^{(2)}$ to be a solution of the problem

$$\left| \begin{array}{l} -w_{xx}^{(2)} = -v_{xx}^0 \text{ in } (0,L), \\ w^{(2)}(0) = 0, \\ w_x^{(2)}(L) = -u^0(L) - z^{(2)}(L). \end{array} \right.$$

By Proposition 2.1, we have that $w^{(2)} \in V \cap H^2(0,L)$ and

$$\begin{array}{l} \left\| w^{(2)} - v^0 \right\|_{V \cap H^2(0,L)}^2 = \left| -w_{xx}^{(2)} + v_{xx}^0 \right|^2 + \left| w_x^{(2)}(L) - v_x^0(L) \right|^2 \\ = \left| -u^0(L) - z^{(2)}(L) - (-u^0(L) - v^1(L)) \right|^2 = \left| -z^{(2)}(L) + v^1(L) \right|^2 \\ \leq C \left\| z^{(2)} - v^1 \right\|_V^2 = C\epsilon^2, \end{array}$$

concluding the result. ∎

3 Strong Solution

Our goal in this section is to prove existence and uniqueness of solutions for the problem $(1.1) - (1.3)$, when u^0, v^0, u^1 and v^1 are smooth.

Let be f, g functions defined in $\mathbb{R}$ and u^0, v^0, u^1, v^1 functions defined in $(0, L)$ satisfying

$$\begin{aligned}&f, g : \mathbb{R} \to \mathbb{R} \text{ are Lipschitz function with constant } c_f,\ c_g, \text{ respectively,}\\ &\text{and } sf(s) \geq 0,\ sg(s) \geq 0,\ \forall s \in \mathbb{R},\end{aligned} \tag{3.1}$$

$$\left(u^0, u^1\right) \in \left[V \cap H^2(0, L)\right] \times V, \tag{3.2}$$

$$\left(v^0, v^1\right) \in \left[V \cap H^2(0, L)\right] \times V, \tag{3.3}$$

$$u_x^0(L) + u^1(L) = 0, \tag{3.4}$$

$$u^0(L) + v_x^0(L) + v^1(L) = 0. \tag{3.5}$$

Theorem 3.1 *Let f, g, u^0, v^0, u^1 and v^1 satisfying the hypotheses $(3.1) - (3.5)$. Then there exist unique functions $u, v : Q \to \mathbb{R}$, such that*

$$u, v \in L^\infty(0, T, V) \cap L^2\left(0, T, H^2(0, L)\right), \tag{3.6}$$

$$u_t, v_t \in L^\infty(0, T, V), \tag{3.7}$$

$$u_{tt}, v_{tt} \in L^2(Q), \tag{3.8}$$

$$\frac{\rho h^3}{12} u_{tt} - u_{xx} + k(u + v_x) + f(u) = 0 \ \ in \ L^2(Q), \tag{3.9}$$

$$\rho h v_{tt} - k(u + v_x)_x + g(v) = 0 \ \ in \ L^2(Q), \tag{3.10}$$

$$u_x(L, \cdot) + u_t(L, \cdot) = 0 \ \ in \ \ (0, T), \tag{3.11}$$

$$u(L, \cdot) + v_x(L, \cdot) + v_t(L, \cdot) = 0 \ \ in \ \ (0, T), \tag{3.12}$$

$$u(0) = u^0,\ u_t(0) = u^1,\ v(0) = v^0,\ v_t(0) = v^1 \ \ in \ \ \ (0, L). \tag{3.13}$$

Proof. We employ the Faedo-Galerkin's method with the special basis in $V \cap H^2(0, L)$. Since the data u^0, v^0, u^1 and v^1 verify $(3.2) - (3.5)$, it follows by Proposition 2.5 the existence of four sequences $(u^{0\nu})_{\nu \in \mathbb{N}}$, $(u^{1\nu})_{\nu \in \mathbb{N}}$, $(v^{0\nu})_{\nu \in \mathbb{N}}$ and $(v^{1\nu})_{\nu \in \mathbb{N}}$ of vectors in $V \cap H^2(0, L)$ such that

$$u^{0\nu} \to u^0 \text{ strongly in } V \cap H^2(0, L), \tag{3.14}$$

$$v^{0\nu} \to v^0 \text{ strongly in } V \cap H^2(0, L), \tag{3.15}$$

$$u^{1\nu} \to u^1 \text{ strongly in } V, \tag{3.16}$$

$$v^{1\nu} \to v^1 \text{ strongly in } V, \tag{3.17}$$

$$u_x^{0\nu}(L) + u^{1\nu}(L) = 0,\ \forall \nu \in \mathbb{N}, \tag{3.18}$$

$$u^{0\nu}(L) + v_x^{0\nu}(L) + v^{1\nu}(L) = 0,\ \forall\ \nu \in \mathbb{N}. \tag{3.19}$$

We fix $\nu \in \mathbb{N}$. If $A = \{u^{0\nu}, v^{0\nu}, u^{1\nu}, v^{1\nu}\}$ is a linearly independent set, we take

$$w_1^\nu = \frac{u^{0\nu}}{\|u^{0\nu}\|_{V \cap H^2(0,L)}},\ w_2^\nu = \frac{u^{1\nu}}{\|u^{1\nu}\|_{V \cap H^2(0,L)}},\ w_3^\nu = \frac{v^{1\nu}}{\|v^{1\nu}\|_{V \cap H^2(0,L)}} \text{ and } w_4^\nu = \frac{v^{0\nu}}{\|v^{0\nu}\|_{V \cap H^2(0,L)}}$$

as being the first four vectors of the basis. By Gram-Schmidt's orthonormalization process, we construct, for each $\nu \in \mathbb{N}$, a basis in $V \cap H^2(\Omega)$ represented by $\{w_1^\nu, w_2^\nu, w_3^\nu, w_4^\nu, \ldots, w_n^\nu, \ldots\}$. Otherwise, if A is a linearly dependent set, we can extract a linearly independent subset of A and continue the above process. For each $m \in \mathbb{N}$, we consider $V_m^\nu = [w_1^\nu, w_2^\nu, w_3^\nu, w_4^\nu, \ldots, w_m^\nu]$ the subspace of $V \cap H^2(\Omega)$ generated by the first m vectors of basis. Let us find an "approximate solution" $(u^{\nu m}, v^{\nu m}) \in V_m^\nu \times V_m^\nu$ of the type

$$u^{\nu m}(x,t) = \sum_{j=1}^{m} \mu^{j\nu m}(t)\, w_j^\nu(x), \quad v^{\nu m}(x,t) = \sum_{j=1}^{m} h^{j\nu m}(t)\, w_j^\nu(x),$$

where $\mu^{j\nu m}(t)$ and $h^{j\nu m}(t)$ are solutions of the initial value problem

$$\left| \begin{array}{l} \dfrac{\rho h^3}{12}\left(u_{tt}^{\nu m}(t), \psi\right) - \left(u_{xx}^{\nu m}(t), \psi\right) + k\left(\left(u^{\nu m} + v_x^{\nu m}\right)(t), \psi\right) + \left(f(u^{\nu m}(t)), \psi\right) = 0, \\ \rho h\left(v_{tt}^{\nu m}(t), \varphi\right) - k\left(\left(\left(u^{\nu m} + v_x^{\nu m}\right)(t)\right)_x, \varphi\right) + \left(g(v^{\nu m}(t)), \varphi\right) = 0, \\ u^{\nu m}(0) = u^{0\nu m},\ u_t^{\nu m}(0) = u^{1\nu m},\ v^{\nu m}(0) = v^{0\nu m},\ v_t^{\nu m}(0) = v^{1\nu m} \ \text{ in } (0, L), \end{array} \right. \tag{3.20}$$

for all $\psi, \varphi \in V_m^\nu$, where

$$\left(u^{0\nu m}, u^{1\nu m}, v^{0\nu m}, v^{1\nu m}\right) \rightarrow \left(u^0, u^1, v^0, v^1\right) \text{ strongly in } \left[V \cap H^2(0,L) \times V\right]^2. \tag{3.21}$$

The system (3.20) has solution on an interval $[0, t_{\nu m}]$, with $t_{\nu m} < T$. This solution can be extended to the whole interval $[0, T]$ as a consequence of *a priori* estimates that shall be proved in the next step.

Adding the equations in (3.20) results

$$\begin{array}{l} \dfrac{\rho h^3}{12}\left(u_{tt}^{\nu m}(t), \psi\right) + \rho h\left(v_{tt}^{\nu m}(t), \varphi\right) + k\left(\left(u^{\nu m} + v_x^{\nu m}\right)(t), \psi + \varphi_x\right) + \left(\left(u^{\nu m}(t), \psi\right)\right) \\ + u_t^{\nu m}(L,t)\, \psi(L,t) + v_t^{\nu m}(L,t)\, \varphi(L,t) + \left(f(u^{\nu m}(t)), \psi\right) + \left(g(v^{\nu m}(t)), \varphi\right) = 0, \end{array} \tag{3.22}$$

for all $\psi, \varphi \in V_m^\nu$.

Estimates I. Making $\psi = 2u_t^{\nu m}(t)$, $\varphi = 2v_t^{\nu m}(t)$ in (3.22), integrating from 0 to $t \leq t_{\nu m}$ and using (3.21), we get

$$\begin{array}{l} \dfrac{\rho h^3}{12}\left|u_t^{\nu m}(t)\right|^2 + \rho h\left|v_t^{\nu m}(t)\right|^2 + k\left|\left(u^{\nu m} + v_x^{\nu m}\right)(t)\right|^2 + \left\|u^{\nu m}(t)\right\|^2 \\ + 2\displaystyle\int_0^t \left|u_t^{\nu m}(L,t)\right|^2 dt + 2\int_0^t \left|v_t^{\nu m}(L,t)\right|^2 dt + 2\int_0^L F(u^{\nu m}(x,t))dx \\ + 2\displaystyle\int_0^L G(v^{\nu m}(x,t))dx \leq C + 2\int_0^L F(u^{0\nu m})dx + 2\int_0^L G(v^{0\nu m})dx, \end{array} \tag{3.23}$$

where $F(t) = \int_0^t f(s)ds$, $G(t) = \int_0^t g(s)ds$ and the constant $C > 0$ is independent of m, ν and t. We must obtain estimates for the terms $2\int_0^L F(u^{0\nu m})dx$ and $2\int_0^L G(v^{0\nu m})dx$. Since

$f(s)s \geq 0$ and $g(s)s \geq 0$, it follows that $F(t) \geq 0$ and $G(t) \geq 0$, for all $t \in [0,T]$ and $f(0) = g(0) = 0$. So, by (3.1), we have

$$\int_0^L F(u^{0\nu m})dx \leq c_f \left|u^{0\nu m}\right|^2 \text{ and } \int_0^L G(v^{0\nu m})dx \leq c_g \left|v^{0\nu m}\right|^2. \tag{3.24}$$

From (3.21) and (3.24), the inequality (3.23) becomes

$$\begin{aligned}&\frac{\rho h^3}{12}\left|u_t^{\nu m}(t)\right|^2 + \rho h \left|v_t^{\nu m}(t)\right|^2 + k\left|(u^{\nu m} + v_x^{\nu m})(t)\right|^2 + \left\|u^{\nu m}(t)\right\|^2 + 2\int_0^t \left|u_t^{\nu m}(L,s)\right|^2 ds \\ &+2\int_0^t \left|v_t^{\nu m}(L,s)\right|^2 ds + 2\int_0^L F(u^{\nu m}(x,t))dx + 2\int_0^L G(v^{\nu m}(x,t))dx \leq C,\end{aligned} \tag{3.25}$$

where $C > 0$ is a constant which is independent of m, ν and t. In this way, we can prolong the solution to the whole interval $[0,T]$.

Estimates II. Considering the temporal derivative of the approximate equation (3.22), setting $\psi = u_{tt}^{\nu m}(t)$ and $\varphi = v_{tt}^{\nu m}(t)$ in the resulting equation and integrating from 0 to $t \leq T$ we get

$$\begin{aligned}&\frac{\rho h^3}{12}\left|u_{tt}^{\nu m}(t)\right|^2 + \rho h\left|v_{tt}^{\nu m}(t)\right|^2 + k\left|(u^{\nu m} + v_x^{\nu m})_t(t)\right|^2 + \left\|u_t^{\nu m}(t)\right\|^2 + 2\int_0^t \left|u_{tt}^{\nu m}(L,s)\right|^2 ds \\ &+2\int_0^t \left|v_{tt}^{\nu m}(L,s)\right|^2 dt \leq \frac{\rho h^3}{12}\left|u_{tt}^{\nu m}(0)\right|^2 + \rho h\left|v_{tt}^{\nu m}(0)\right|^2 + k\left|u^{1\nu m} + v_x^{1\nu m}\right|^2 + \left\|u^{1\nu m}\right\|^2 \\ &+2\int_0^t \left|(f_t(u^{\nu m}(s)\, u_t^{\nu m}(s), u_{tt}^{\nu m}(s))\right| ds + 2\int_0^t \left|(g_t(v^{\nu m}(s)\, v_t^{\nu m}(s), v_{tt}^{\nu m}(s))\right| ds.\end{aligned} \tag{3.26}$$

We need estimates for the terms involving $u_{tt}^{\nu m}(0)$, $v_{tt}^{\nu m}(0)$ and for last two integrals in (3.26). For this, we consider in (3.22) $t = 0$, $\psi = u_{tt}^{\nu m}(0)$ and $\varphi = v_{tt}^{\nu m}(0)$. So, using (3.1) and (3.21) we obtain

$$\frac{\rho h^3}{12}\left|u_{tt}^{\nu m}(0)\right|^2 + \rho h\left|v_{tt}^{\nu m}(0)\right|^2 \leq C, \tag{3.27}$$

where $C > 0$ is a constant independent of m, ν and t. We also have by (3.1) that $|f_t(s)| \leq c_f$ and $|g_t(s)| \leq c_g$, a. e. in $\mathbb{R}$. Then

$$\begin{aligned}&2\int_0^t \left|(f_t(u^{\nu m}(s)\, u_t^{\nu m}(s), u_{tt}^{\nu m}(s))\right| ds + 2\int_0^t \left|(g_t(v^{\nu m}(s)\, v_t^{\nu m}(s), v_{tt}^{\nu m}(s))\right| ds \\ &\leq \frac{c_f}{2}\int_0^t \left|u_t^{\nu m}(t)\right|^2 dt + \frac{c_f}{2}\int_0^t \left|u_{tt}^{\nu m}(t)\right|^2 dt + \frac{c_g}{2}\int_0^t \left|v_t^{\nu m}(t)\right|^2 dt + \frac{c_g}{2}\int_0^t \left|v_{tt}^{\nu m}(t)\right|^2 dt.\end{aligned} \tag{3.28}$$

Thus, using (3.21) and the estimates (3.27), (3.28) in (3.26) we get

$$\begin{aligned}&\left|u_{tt}^{\nu m}(t)\right|^2 + \left|v_{tt}^{\nu m}(t)\right|^2 + \left|(u^{\nu m} + v_x^{\nu m})_t(t)\right|^2 + \left\|u_t^{\nu m}(t)\right\|^2 + 2\int_0^t \left|u_{tt}^{\nu m}(L,t)\right|^2 dt \\ &+2\int_0^t \left|v_{tt}^{\nu m}(L,t)\right|^2 dt \leq C,\end{aligned} \tag{3.29}$$

where $C = C(c_f, c_g) > 0$ is a constant independent of t, ν and m.

According to (3.25) and (3.29), we have

$$(u^{\nu m}) \text{ is bounded in } L^\infty(0,T,V), \tag{3.30}$$

$$(u_t^{\nu m}) \text{ is bounded in } L^\infty(0,T,V), \tag{3.31}$$

$$(u_{tt}^{\nu m}) \text{ is bounded in } L^2(Q), \tag{3.32}$$

$$(v^{\nu m}) \text{ is bounded in } L^\infty(0,T,V), \tag{3.33}$$

$$(v_t^{\nu m}) \text{ is bounded in } L^\infty(0,T,V), \tag{3.34}$$

$$(v_{tt}^{\nu m}) \text{ is bounded in } L^2(Q), \tag{3.35}$$

From (3.30) $-$ (3.35), we can obtain subsequences of $(u^{\nu m})$ and $(v^{\nu m})$, which will be also denoted by $(u^{\nu m})$ and $(v^{\nu m})$, such that

$$u^{\nu m} \to u^\nu \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.36}$$

$$u_t^{\nu m} \to u_t^\nu \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.37}$$

$$u_{tt}^{\nu m} \to u_{tt}^\nu \text{ weakly in } L^2(Q), \tag{3.38}$$

$$v^{\nu m} \to v^\nu \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.39}$$

$$v_t^{\nu m} \to v_t^\nu \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.40}$$

$$v_{tt}^{\nu m} \to v_{tt}^\nu \text{ weakly in } L^2(Q). \tag{3.41}$$

According to (3.1), (3.30), (3.33) and the compact injection of $H^1(Q)$ in $L^2(Q)$, there exists a subsequence of $(u^{\nu m})$ and $(v^{\nu m})$, which will be also denoted by $(u^{\nu m})$ and $(v^{\nu m})$, such that

$$f(u^{\nu m}) \to f(u^\nu) \text{ strongly in } L^2(Q), \tag{3.42}$$

$$g(v^{\nu m}) \to g(v^\nu) \text{ strongly in } L^2(Q). \tag{3.43}$$

We can see that the estimates (3.25) and (3.29) are also independent of ν. So, using the same arguments to obtain u^ν and v^ν, we can pass to the limit, as $\nu \to \infty$, to obtain functions u and v such that

$$u^\nu \to u \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.44}$$

$$u_t^\nu \to u_t \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.45}$$

$$u_{tt}^\nu \to u_{tt} \text{ weakly in } L^2(Q), \tag{3.46}$$

$$v^\nu \to v \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.47}$$

$$v_t^\nu \to v_t \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{3.48}$$

$$v_{tt}^\nu \to v_{tt} \text{ weakly in } L^2(Q), \tag{3.49}$$

$$f(u^\nu) \to f(u) \text{ strongly in } L^2(Q), \tag{3.50}$$

$$g(v^{\nu}) \to g(v) \text{ strongly in } L^2(Q). \tag{3.51}$$

Making $m \to \infty$ and $\nu \to \infty$ in the equations in (3.20) and using the convergences (3.36) – (3.51) we have

$$\begin{aligned}
&\frac{\rho h^3}{12}\int_0^T (u_{tt}(t), \psi)\,\theta dt + \int_0^T ((u(t), \psi))\,\theta dt + \int_0^T u_t(L,t)\psi(L,t)\theta dt \\
&+k\int_0^T ((u+v_x)(t), \psi)\,\theta dt + \int_0^T (f(u(t)), \psi)\,\theta dt = 0,\ \forall \psi \in V,\ \forall \theta \in \mathcal{D}(0,T)
\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}
&\rho h\int_0^T (v_{tt}(t), \varphi)\,\theta dt + k\int_0^T ((u+v_x)(t), \varphi_x)\theta dt + k\int_0^T v_t(L,t)\varphi(L,t)\theta dt \\
&+\int_0^T (g(v(t)), \varphi)\,\theta dt = 0,\ \forall \varphi \in V,\ \forall \theta \in \mathcal{D}(0,T).
\end{aligned} \tag{3.53}$$

Taking $\varphi, \psi \in \mathcal{D}(0,L)$, it follows that

$$\frac{\rho h^3}{12} u_{tt} - u_{xx} + k(u+v_x) + f(u) = 0 \quad \text{in} \quad L^2(Q) \tag{3.54}$$

and

$$\rho h v_{tt} - k(u+v_x)_x + g(v) = 0 \quad \text{in} \quad L^2(Q). \tag{3.55}$$

Multiplying (3.54) by $\psi\theta$, $\psi \in V$ and $\theta \in \mathcal{D}(0,T)$, integrating in Q and comparing with (3.52), we get

$$\int_0^T [u_t(L,t) + u_x(L,t)]\,\psi(L,t)\theta dt = 0, \forall \theta \in \mathcal{D}(0,T), \forall \psi \in V.$$

Consequently

$$u_t(L) + u_x(L) = 0 \text{ on } (0,T). \tag{3.56}$$

Now, multiplying (3.55) by $\varphi\theta$, $\varphi \in V$ and $\theta \in \mathcal{D}(0,T)$, integrating in Q and comparing with (3.53), we obtain

$$k\int_0^T (v_t(L,t) + u(L,t) + v_x(L,t))\varphi(L,t)\theta dt = 0, \forall \theta \in \mathcal{D}(0,T),\ \forall \varphi \in V,$$

which implies

$$v_t(L) + u(L) + v_x(L) = 0 \text{ on } (0,T). \tag{3.57}$$

To complete the proof of the theorem, we need to show that $u, v \in L^2(0,T,H^2(0,L))$. For this, we consider the following boundary value problem:

$$\left|\begin{array}{l}
-u_{xx}(t) = -\dfrac{\rho h^3}{12}u_{tt}(t) - k(u+v_x)(t) - f(u(t)) \quad \text{in} \quad (0,L), \\
u(0,t) = 0, \\
u_x(L,t) = -u_t(L,t)
\end{array}\right. \tag{3.58}$$

and

$$
\left| \begin{array}{l} -v_{xx}(t) = -\dfrac{\rho h}{k} v_{tt}(t) + u_x(t) - \dfrac{1}{k} g(v(t)) \quad \text{in} \quad (0,L), \\ v(0,t) = 0, \\ v_x(L,t) = -(u(L,t) + v_t(L,t)). \end{array} \right. \tag{3.59}
$$

Since $-\frac{\rho h^3}{12} u_{tt} - k(u+v_x) - f(u)$, $\frac{\rho h}{k} v_{tt} + u_x - \frac{1}{k} g(v) \in L^2(Q)$, it follows by Proposition 2.1 that $u, v \in L^2(0,T,H^2(0,L))$. Using a standard argument, we can verify the initial conditions. The uniqueness of solution is proved by energy method. ■

4 Weak Solution

The purpose of this section is to obtain existence of solutions for the problem (1.1) – (1.3), with less regularity on the initial data and now f, g being continuous functions and $sf(s) \geq 0$, $sg(s) \geq 0$, $\forall s \in \mathbb{R}$. Owing to few regularity of the initial data, the corresponding solutions shall be called weak.

Theorem 4.1 *Let us consider*

$$
f, g : \mathbb{R} \to \mathbb{R} \textit{ are continuous functions such that } f(s)s \geq 0 \textit{ and } g(s)s \geq 0, \ \forall s \in \mathbb{R}, \tag{4.1}
$$

$$
(u^0, u^1, v^0, v^1) \in \left[V \times L^2(0,L)\right]^2, \tag{4.2}
$$

$$
F(u^0), G(v^0) \in L^1(0,L). \tag{4.3}
$$

Then there exist at least two functions $u, v : Q \to \mathbb{R}$ such that

$$
u, v \in L^\infty(0,T,V), \tag{4.4}
$$

$$
u_t, v_t \in L^\infty(0,T,L^2(0,L)), \tag{4.5}
$$

$$
\frac{\rho h^3}{12} u_{tt} - u_{xx} + k(u + v_x) + f(u) = 0 \ \textit{in} \ L^1(0,T,V' + L^1(0,L)), \tag{4.6}
$$

$$
\rho h v_{tt} - k(u + v_x)_x + g(v) = 0 \ \textit{in} \ L^1(0,T,V' + L^1(0,L)), \tag{4.7}
$$

$$
u_x(L,\cdot) + u_t(L,\cdot) = 0 \ \textit{in} \ L^2(0,T), \tag{4.8}
$$

$$
u(L,\cdot) + v_x(L,\cdot) + v_t(L,\cdot) = 0 \ \textit{in} \ L^2(0,T), \tag{4.9}
$$

$$
u(0) = u^0, \ u_t(0) = u^1, \ v(0) = v^0, \ v_t(0) = v^1 \ \textit{in} \ (0,L). \tag{4.10}
$$

Proof. There exist two sequences of functions $(f_\nu)_{\nu \in \mathbb{N}}$ and $(g_\nu)_{\nu \in \mathbb{N}}$, such that, for each $\nu \in \mathbb{N}$, $f_\nu, g_\nu : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants c_{f_ν} and c_{g_ν}, respectively, satisfying $sf_\nu(s) \geq 0$ and $sg_\nu(s) \geq 0$, $\forall s \in \mathbb{R}$ and $(f_\nu)_{\nu \in \mathbb{N}}$, $(g_\nu)_{\nu \in \mathbb{N}}$ approximate f and g, respectively, uniformly on bounded sets of $\mathbb{R}$. The construction of these sequences can be seen in Strauss [17].

Since the initial data u^0 and v^0 are not necessarily bounded, we approximate u^0 and v^0 by bounded functions of V. We consider the functions $\xi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_j(s) = \begin{cases} -j, & \text{if} \quad s < -j, \\ s, & \text{if} \quad |s| \le j, \\ j, & \text{if} \quad s > j. \end{cases}$$

Considering $\xi_j(u^0) = u^{0j}$ and $\xi_j(v^0) = v^{0j}$, we have by Kinderlehrer-Stampacchia [5] that the sequences $(u^{0j})_{j\in\mathbb{N}}$ and $(v^{0j})_{j\in\mathbb{N}}$ in V are bounded in $[0, L]$ and

$$u^{0j} \to u^0 \text{ strongly in } V, \tag{4.11}$$

$$v^{0j} \to v^0 \text{ strongly in } V. \tag{4.12}$$

Let us take the sequences $(u^{0jp})_{p\in\mathbb{N}}$, $(v^{0jp})_{p\in\mathbb{N}}$ in $V \cap H^2(0, L)$ and $(u^{1p})_{p\in\mathbb{N}}$, $(v^{1p})_{p\in\mathbb{N}}$ in V such that

$$u^{0jp} \to u^{0j} \text{ strongly in } V, \tag{4.13}$$

$$v^{0jp} \to v^{0j} \text{ strongly in } V, \tag{4.14}$$

$$u^{1p} \to u^1 \text{ strongly in } L^2(0, L), \tag{4.15}$$

$$v^{1p} \to v^1 \text{ strongly in } L^2(0, L), \tag{4.16}$$

$$u_x^{0jp}(L, \cdot) + u^{1p}(L, \cdot) = 0 \text{ in } (0, T)\,, \tag{4.17}$$

$$u_x^{0jp}(L, \cdot) + v_x^{0jp}(L, \cdot) + v_t^{0jp}(L, \cdot) = 0 \text{ in } (0, T)\,. \tag{4.18}$$

We fix $(j, p, \nu) \in \mathbb{N}$. For the initial data $(u^{0jp}, u^{1p}, v^{0jp}, v^{1p}) \in \{[V \cap H^2(0, L)] \times V\}^2$, there exist unique functions $u_{jp\nu}, v_{jp\nu} : Q \to \mathbb{R}$ in the conditions of the Theorem 3.1. By the same argument employed in the Estimates I (see (3.23)), we obtain

$$\begin{aligned} &\frac{\rho h^3}{12} \left|u_t^{jp\nu}(t)\right|^2 + \rho h \left|v_t^{jp\nu}(t)\right|^2 + k \left|(u^{jp\nu} + v_x^{jp\nu})(t)\right|^2 + \left\|u^{jp\nu}(t)\right\|^2 + 2\int_0^t \left|u_t^{jp\nu}(L, s)\right|^2 ds \\ &+2\int_0^t \left|v_t^{jp\nu}(L, s)\right|^2 ds + 2\int_0^L F_\nu\left(u^{jp\nu}(x, t)\right) dx + 2\int_0^L G_\nu\left(v^{jp\nu}(x, t)\right) dx \le \frac{\rho h^3}{12} \left|u^{1p}\right|^2 \\ &+\rho h \left|v^{1p}\right|^2 + k \left|u^{0jp} + v_x^{0p\nu}\right|^2 + \left\|u^{0jp}\right\|^2 + 2\int_0^L F_\nu(u^{0jp}) dx + 2\int_0^L G_\nu(v^{0jp}) dx, \end{aligned} \tag{4.19}$$

where $F_\nu(t) = \int_0^t f_\nu(s)ds$ and $G_\nu(t) = \int_0^t g_\nu(s)ds$.

We need estimates for the terms $\int_0^L F_\nu\left(u^{0jp}\right) dx$ and $\int_0^L G_\nu\left(v^{0jp}\right) dx$. Since u^{0j} and v^{0j} are bounded a. e. in $[0, L]$, $\forall j \in \mathbb{N}$, it follows that

$$f_\nu(u^{0j}) \to f\left(u^{0j}\right) \text{ uniformly in } (0, L)\,,$$

$$g_\nu(v^{0j}) \to g\left(v^{0j}\right) \text{ uniformly in } (0, L)\,.$$

So

$$\int_0^L F_\nu\left(u^{0j}(x)\right)dx \to \int_0^L F\left(u^{0j}(x)\right)dx \text{ uniformly in } \mathbb{R}, \tag{4.20}$$

$$\int_0^L G_\nu\left(v^{0j}(x)\right)dx \to \int_0^L G\left(v^{0j}(x,t)\right)dx \text{ uniformly in } \mathbb{R}. \tag{4.21}$$

From (4.11) and (4.12), there exist subsequences of $(u^{0j})_{j\in\mathbb{N}}$ and $(v^{0j})_{j\in\mathbb{N}}$, which still be also denoted by $(u^{0j})_{j\in\mathbb{N}}$ and $(v^{0j})_{j\in\mathbb{N}}$, such that

$$u^{0j} \to u^0 \text{ a. e. in } (0,L)\,,$$

$$v^{0j} \to v^0 \text{ a. e. in } (0,L)\,.$$

By continuity of F and G, it follows that $F(u^{0j}) \to F(u^0)$ and $G(v^{0j}) \to G(v^0)$ a. e. in $[0,L]$. We also have $F(u^{0j}) \leq F(u^0)$ and $G(v^{0j}) \leq G(v^0)$. Thus, by (4.3) and the Lebesgue's dominated convergence theorem, we get

$$F(u^{0j}) \to F(u^0) \text{ strongly in } L^1(0,L), \tag{4.22}$$

$$G(v^{0j}) \to G(v^0) \text{ strongly in } L^1(0,L). \tag{4.23}$$

Making the same arguments for F_ν and G_ν, it follows that

$$F_\nu(u^{0jp}) \to F_\nu(u^{0j}) \text{ strongly in } L^1(0,L), \tag{4.24}$$

$$G_\nu(v^{0jp}) \to G_\nu(v^{0j}) \text{ strongly in } L^1(0,L). \tag{4.25}$$

By (4.20) – (4.25), we obtain

$$\int_0^L F_\nu(u^{0jp}(x))dx \to \int_0^L F\left(u^0(x)\right)dx \text{ in } \mathbb{R}, \tag{4.26}$$

$$\int_0^L G_\nu(v^{0jp}(x))dx \to \int_0^L G(v^0(x))dx \text{ in } \mathbb{R}. \tag{4.27}$$

Then

$$\int_0^L F_\nu(u^{0jp}(x)dx \leq C \quad \text{and} \quad \int_0^L G_\nu(v^{0jp}(x)dx \leq C, \tag{4.28}$$

where the constant $C > 0$ is independent of j, p and ν.

Using (4.11) – (4.16) and (4.28) in (4.19), we have

$$\frac{\rho h^3}{12}\left|u_t^{jp\nu}(t)\right|^2 + \rho h\left|v_t^{jp\nu}(t)\right|^2 + k\left|(u^{jp\nu} + v_x^{jp\nu})(t)\right|^2 + \left\|u^{jp\nu}(t)\right\|^2 + 2\int_0^t \left|u_t^{jp\nu}(L,s)\right|^2 ds$$
$$+2\int_0^t \left|v_t^{jp\nu}(L,s)\right|^2 ds \leq C, \tag{4.29}$$

where $C > 0$ is independent of j, p, ν and t.

From (4.29), we get

$$(u^{jp\nu}) \text{ is bounded in } L^\infty(0,T,V), \tag{4.30}$$

$$(v^{jp\nu}) \text{ is bounded in } L^\infty(0,T,V), \tag{4.31}$$

$$(u_t^{jp\nu}) \text{ is bounded in } L^2(Q), \tag{4.32}$$

$$(v_t^{jp\nu}) \text{ is bounded in } L^2(Q), \tag{4.33}$$

$$(u_t^{jp\nu}(L,\cdot)) \text{ is bounded in } L^2(0,T), \tag{4.34}$$

$$(v_t^{jp\nu}(L,\cdot)) \text{ is bounded in } L^2(0,T). \tag{4.35}$$

According to (3.11), (3.12), (4.30), (4.34) and (4.35), we have

$$u_x^{jp\nu}(L,\cdot) \text{ is bounded in } L^2(0,T), \tag{4.36}$$

$$v_x^{jp\nu}(L,\cdot) \text{ is bounded in } L^2(0,T). \tag{4.37}$$

As the estimates above are hold for all $(j,p,\nu) \in \mathbb{N}^3$ and, in particular for $(\nu,\nu,\nu) \in \mathbb{N}^3$, we can take subsequences $(u^{\nu\nu\nu})_{\nu\in\mathbb{N}}$ and $(v^{\nu\nu\nu})_{\nu\in\mathbb{N}}$, which we denote by $(u^\nu)_{\nu\in\mathbb{N}}$ and $(v^\nu)_{\nu\in\mathbb{N}}$, such that

$$u^\nu \to u \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{4.38}$$

$$v^\nu \to v \text{ weak} * \text{ in } L^\infty(0,T,V), \tag{4.39}$$

$$u_t^\nu \to u_t \text{ weakly in } L^2(Q), \tag{4.40}$$

$$v_t^\nu \to v_t \text{ weakly in } L^2(Q), \tag{4.41}$$

$$u_x^\nu(L,\cdot) \to \chi \text{ weakly in } L^2(0,T), \tag{4.42}$$

$$v_x^\nu(L,\cdot) \to \Sigma \text{ weakly in } L^2(0,T), \tag{4.43}$$

$$u_t^\nu(L,\cdot) \to u_t(L,\cdot) \text{ weakly in } L^2(0,T), \tag{4.44}$$

$$v_t^\nu(L,\cdot) \to v_t(L,\cdot) \text{ weakly in } L^2(0,T), \tag{4.45}$$

We note that the Theorem 3.1 gives us

$$\frac{\rho h^3}{12} u_{tt}^\nu - u_{xx}^\nu + k(u^\nu + v_x^\nu) + f_\nu(u^\nu) = 0 \text{ in } L^2(Q), \tag{4.46}$$

$$\rho h v_{tt}^\nu - k\left(u^\nu + v_x^\nu\right)_x + g_\nu(v^\nu) = 0 \text{ in } L^2(Q) \tag{4.47}$$

$$u_x^\nu(L,\cdot) + u_t^\nu(L,\cdot) = 0 \text{ in } (0,T), \tag{4.48}$$

$$u^\nu(L,\cdot) + v_x^\nu(L,\cdot) + v_t^\nu(L,\cdot) = 0 \text{ in } (0,T). \tag{4.49}$$

From (4.38) – (4.41) and the compact embedding of $H^1(Q)$ in $L^2(Q)$, we can guarantee the existence of subsequences of (u^ν) and (v^ν), which we still denote with the index ν, such that

$$u^\nu \to u \text{ a. e. in } Q, \tag{4.50}$$

$$v^{\nu} \rightarrow v \text{ a. e. in } Q. \tag{4.51}$$

As f, g are continuous, it follows

$$f(u^{\nu}) \rightarrow f(u) \text{ a. e. in } Q,$$

$$g(v^{\nu}) \rightarrow g(v) \text{ a. e. in } Q.$$

We also have

$$f_{\nu}(u^{\nu}) \rightarrow f(u^{\nu}) \text{ a. e. in } Q,$$

$$g_{\nu}(v^{\nu}) \rightarrow g(v^{\nu}) \text{ a. e. in } Q,$$

because $u^{\nu}(x,t)$ and $v^{\nu}(x,t)$ are bounded in $\mathbb{R}$. Therefore

$$f_{\nu}(u^{\nu}) \rightarrow f(u) \text{ a. e. in } Q, \tag{4.52}$$

$$g_{\nu}(v^{\nu}) \rightarrow g(v) \text{ a. e. in } Q. \tag{4.53}$$

Making the inner product in $L^2(Q)$ of (4.46) with $u^{\nu}(t)$, we obtain

$$\begin{aligned}
&\int_0^T (f_{\nu}(u^{\nu}(t)), u^{\nu}(t)) = \frac{\rho h^3}{12}\int_0^T |u_t^{\nu}(t)|^2 \, dt - \frac{\rho h^3}{12}(u_t^{\nu}(T), u^{\nu}(T)) \\
&+\frac{\rho h^3}{12}(u_t^{\nu}(0), u^{\nu}(0)) - \int_0^T u_t^{\nu}(L,T)u^{\nu}(L,T)dt - \int_0^T ((u^{\nu}(t), u^{\nu}(t)))dt \\
&-k\int_0^T ((u^{\nu} + v_x^{\nu})(t), u^{\nu}(t))dt.
\end{aligned} \tag{4.54}$$

Observing $(4.13), (4.15), (4.30) - (4.32)$ and (4.34), we have by (4.54) that

$$\int_0^T (f_{\nu}(u^{\nu}(t)), u^{\nu}(t)) \leq C, \tag{4.55}$$

where $C > 0$ is independent of ν.

From (4.52), (4.55) and Strauss' Theorem (see [17]), it follows

$$f_{\nu}(u^{\nu}) \rightarrow f(u) \text{ strongly in } L^1(Q). \tag{4.56}$$

Analogously, taking the inner product in $L^2(Q)$ of (4.47) with $v^{\nu}(t)$ and after using $(4.14), (4.16), (4.30), (4.31), (4.33)$ and (4.35) we get

$$\int_0^T (g_{\nu}(v^{\nu}(t)), v^{\nu}(t))dt \leq C, \tag{4.57}$$

where $C > 0$ is independent of ν.

From (4.53), (4.57) and Strauss' Theorem (see [17]), it follows

$$g_{\nu}(v^{\nu}) \rightarrow g(v) \text{ strongly in } L^1(Q). \tag{4.58}$$

Applying the convergences (4.38) − (4.45), (4.56) and (4.58) in (4.46) − (4.49), we conclude

$$\frac{\rho h^3}{12}u_{tt} - u_{xx} + k(u+v_x) + f(u) = 0 \text{ in } L^1(0,T,V' + L^1(0,L)), \tag{4.59}$$

$$\rho h v_{tt} - k(u+v_x)_x + g(v) = 0 \text{ in } L^1(0,T,V' + L^1(0,L)), \tag{4.60}$$

$$\chi + u_t\,(L,\cdot) = 0 \text{ in } L^2(0,T), \tag{4.61}$$

$$u\,(L,\cdot) + \Sigma + v_t\,(L,\cdot) = 0 \text{ in } L^2(0,T). \tag{4.62}$$

Let us prove that $u_x\,(L,\cdot) = \chi$ and $v_x\,(L,\cdot) = \Sigma$.

- $u_x(L,\cdot) = \chi$

According to (4.59), we deduce

$$-u_{xx} = -\frac{\rho h^3}{12}u_{tt} - k(u+v_x) - f(u). \tag{4.63}$$

Since $u_t, (u+v_x) \in L^2(Q)$ and $f(u) \in L^1(Q)$, by Propositions 2.1 and 2.3, there exist functions $z, w \in L^2(0,T,V \cap H^2(0,L))$ and $\eta \in L^1(0,T,E)$ such that $-z_{xx} = u_t$, $-w_{xx} = u + v_x$ and $-\eta_{xx} = f(u)$. Hence

$$-u_{xx} = \frac{\rho h^3}{12}(z_{xx})_t + kw_{xx} + \eta_{xx}. \tag{4.64}$$

Multiplying (4.64) by $\theta \in \mathcal{D}(0,T)$ and integrating from 0 to T, we obtain

$$-\left[\int_0^T u\theta dt - \frac{\rho h^3}{12}\int_0^T z\theta' dt + k\int_0^T w\theta dt\right]_{xx} = -\left[\int_0^T -\eta\theta dt\right]_{xx}.$$

From the uniqueness given by Proposition 2.3, we get

$$\int_0^T \left(u + \frac{\rho h^3}{12}z_t + kw + \eta\right)\theta dt = 0,\ \forall\theta \in \mathcal{D}(0,T),$$

that is,

$$u = -\frac{\rho h^3}{12}z_t - kw - \eta. \tag{4.65}$$

Since $z_{xt}(L,\cdot) = (z_x(L,\cdot))_t$ (see [14, Lemma 3.2]), we can apply the trace theorem in (4.65) to obtain

$$u_x\,(L,\cdot) = -\frac{\rho h^3}{12}\left(z_x\,(L,\cdot)\right)_t - kw_x\,(L,\cdot) - \eta_x\,(L,\cdot) \in H^{-1}\,(0,T) + L^1\,(0,T)\,.$$

For other side, by (4.46) we have

$$-u^\nu_{xx} = -\frac{\rho h^3}{12}u^\nu_{tt} - k(u^\nu + v^\nu_x) - f_\nu(u^\nu)$$

with $u_t^\nu, (u^\nu + v_x^\nu), f_\nu(u^\nu) \in L^2(Q)$. By Propositions 2.1, there exist functions $z^\nu, w^\nu, \eta^\nu \in L^2(0,T,V\cap H^2(0,L))$ such that $-z_{xx}^\nu = u_t^\nu$, $-w_{xx}^\nu = u^\nu + v_x^\nu$ and $-\eta_{xx}^\nu = f_\nu(u^\nu)$. Thus, as it was done before, we have

$$-u_{xx}^\nu = \frac{\rho h^3}{12}(z_{xx}^\nu)_t + k w_{xx}^\nu + \eta_{xx}^\nu$$

and

$$u^\nu = -\frac{\rho h^3}{12} z_t^\nu - k w^\nu - \eta^\nu. \tag{4.66}$$

By (4.38) – (4.40) and (4.56), we get

$$z^\nu \to z \text{ weakly in } L^2(0,T,V\cap H^2(0,L)), \tag{4.67}$$

$$w^\nu \to w \text{ weakly in } L^2(0,T,V\cap H^2(0,L)), \tag{4.68}$$

$$\eta^\nu \to \eta \text{ strongly in } L^1(0,T,E). \tag{4.69}$$

According to [13] and (4.67), we have

$$z_t^\nu \to z_t \text{ weakly in } H^{-1}(0,T,V\cap H^2(0,L)). \tag{4.70}$$

From (4.67) – (4.70) and by continuity of the trace, we obtain

$$\eta_x^\nu(L,\cdot) \to \eta_x(L,\cdot) \text{ strongly in } L^1(0,T), \tag{4.71}$$

$$z_{xt}^\nu(L,\cdot) \to z_{xt}(L,\cdot) \text{ weakly in } H^{-1}(0,T), \tag{4.72}$$

$$w_x^\nu(L,\cdot) \to w_x(L,\cdot) \text{ weakly in } L^2(0,T). \tag{4.73}$$

Taking into account the convergences (4.71) – (4.73), it follows by (4.65) and (4.66) that

$$u_x^\nu(L,\cdot) \to u_x(L,\cdot) \text{ weakly in } \left[H^1(0,T)\cap L^\infty(0,T)\right]'. \tag{4.74}$$

In this way, comparing (4.42) and (4.74), we can conclude

$$\chi = u_x(L,\cdot) \quad \text{in} \quad L^2(0,T). \tag{4.75}$$

- $v_x(L,\cdot) = \Sigma$.

Making the same procedure as before, from (4.60), it follows that

$$-v_{xx} = u_x - \frac{\rho h}{k} v_{tt} - \frac{1}{k} g(v), \tag{4.76}$$

with $u_x,\ v_t \in L^2(Q)$ and $g(v) \in L^1(Q)$. By Propositions 2.1 and 2.3, there exist functions $\beta,\ \phi \in L^2(0,T,V\cap H^2(0,L))$ and $\zeta \in L^1(0,T,E)$ such that $-\beta_{xx} = u_x,\ \ -\phi_{xx} = \frac{\rho h}{k} v_t$ and $-\zeta_{xx} = \frac{1}{k} g(v)$. So, we can find

$$-v_{xx} = -\beta_{xx} + \frac{\rho h}{k}(\phi_{xx})_t + \frac{1}{k}\zeta_{xx} \tag{4.77}$$

and

$$v=\beta-\frac{\rho h}{k}\phi_t-\frac{1}{k}\zeta. \tag{4.78}$$

Applying the trace theorem in (4.78), we obtain

$$v_x(L,\cdot)=\beta_x(L,\cdot)-\frac{\rho h}{k}\left(\phi_x(L,\cdot)\right)_t-\frac{1}{k}\zeta_x(L,\cdot)\in H^{-1}(0,T)+L^1(0,T).$$

We know by (4.47) that

$$-v^\nu_{xx}=u^\nu_x-\frac{\rho h}{k}v^\nu_{tt}-\frac{1}{k}g_\nu(v^\nu).$$

Since $u^\nu_x,\ v^\nu_t,\ g_\nu(v^\nu)\in L^2(Q)$, it follows the existence of functions $\beta^\nu,\ \phi^\nu,\ \zeta^\nu\in L^2(0,T,V\cap H^2(0,L))$ such that $-\beta^\nu_{xx}=u^\nu_x$, $-\phi^\nu_{xx}=v^\nu_t$ and $-\zeta^\nu_{xx}=g_\nu(v^\nu)$. Thus, for analogy to the that we did before, we get

$$-kv^\nu_{xx}=-k\beta^\nu_{xx}+\rho h(\phi^\nu_{xx})_t+\zeta^\nu_{xx}$$

and

$$kv^\nu=k\beta^\nu-\rho h\phi^\nu_t-\zeta^\nu \tag{4.79}$$

By (4.38), (4.41) and (4.58), we have

$$\beta^\nu\to\beta \text{ weakly in } L^2(0,T,V\cap H^2(0,L)), \tag{4.80}$$

$$\phi^\nu\to\phi \text{ weakly in } L^2(0,T,V\cap H^2(0,L)), \tag{4.81}$$

$$\zeta^\nu\to\zeta \text{ strongly in } L^1(0,T,E), \tag{4.82}$$

$$\phi^\nu_t\to\phi_t \text{ weakly in } H^{-1}(0,T,V\cap H^2(0,L)). \tag{4.83}$$

According to convergences (4.80) – (4.83) and the continuity of trace, it follows that

$$\beta^\nu_x(L,\cdot)\to\beta(L,\cdot) \text{ weakly in } L^2(0,T), \tag{4.84}$$

$$\zeta^\nu_x(L,\cdot)\to\zeta(L,\cdot) \text{ strongly in } L^1(0,T), \tag{4.85}$$

$$\phi^\nu_x(L,\cdot)\to\phi(L,\cdot) \text{ weakly in } H^{-1}(0,T). \tag{4.86}$$

Using the convergences (4.84) – (4.86), we can conclude from (4.78) and (4.79) that

$$v^\nu_x(L,\cdot)\to v_x(L,\cdot) \text{ weakly in } \left[H^1(0,T)\cap L^\infty(0,T)\right]', \tag{4.87}$$

which comparing with (4.43), we deduce

$$\Sigma=v_x(L,\cdot)\quad\text{in}\quad L^2(0,T).$$

To verify the initial conditions (4.10), we use the standard method. ■

Remark 4.1 *The uniqueness of solution in the conditions of the Theorem 4.1 is a open question. But, for some particular cases of the nonlinearities, for example* $f(s)=|s|^{p-1}s$ *and* $g(s)=|s|^{q-1}s$ *with* $p,q\in[1,\infty)$, *we can use the energy method as in Lions* [9, p. 15] *to obtain the uniqueness of solution.*

5 Asymptotic Behavior

The aim of this section is to study the asymptotic behavior of the energy $E(t)$ associated to weak solution of the problem $(1.1)-(1.3)$. As it was mentioned in the introduction, this energy is defined by

$$\begin{aligned} E(t) = \; & \frac{1}{2}\left[\frac{\rho h^3}{12}\left|u_t\left(t\right)\right|^2 + \rho h\left|v_t\left(t\right)\right|^2 + k\left|\left(u+v_x\right)\left(t\right)\right|^2 + \left\|u\left(t\right)\right\|^2\right. \\ & \left. +2\int_0^L F(u(x,t))dx + 2\int_0^L G(v(x,t))dx\right]. \end{aligned} \tag{5.1}$$

Let us consider the following additional hypotheses:

$$\exists\ \delta_1 > 0 \text{ such that } f(s)s \geq (2+\delta_1)F(s),\ \forall s \in \mathbb{R}, \tag{5.2}$$

and

$$\exists\ \delta_2 > 0 \text{ such that } g(s)s \geq (2+\delta_2)G(s),\ \forall s \in \mathbb{R}. \tag{5.3}$$

The functions f and g given in the Remark 4.1 satisfy the conditions (4.1), (4.3), (5.2) and (5.3).

The mean result of this section is:

Theorem 5.1 *Let $L < \min\{2, 2/k\}$ and f, g, u^0, v^0, u^1, v^1 in the conditions of the Theorem 4.1 plus the hypotheses (5.2) and (5.3). Then there exists a positive constant $\kappa > 0$ such that the energy $E(t)$ satisfy*

$$E(t) \leq 4E(0)e^{-\kappa t},\ \forall t \geq 0. \tag{5.4}$$

Proof. Taking the inner product in $L^2(0,L)$ of (4.46) and (4.47) with $u_t^\nu(t)$ and $v_t^\nu(t)$, respectively, we obtain

$$E_\nu'(t) = -\left|u^\nu(L,\cdot)\right|^2 - \left|v^\nu(L,\cdot)\right|^2, \tag{5.5}$$

where $E_\nu(t)$ is the energy associated to strong solution (u^ν, v^ν), obtained in Section 3, when f and g are replaced by f_ν and g_ν, respectively. Thus this energy is non-increasing.

It is important to emphasize that, for each $\nu \in \mathbb{N}$, the functions f_ν and g_ν of the approximating sequences also satisfy the conditions (5.2) and (5.3), respectively (cf. [17]).

For an arbitrary $\epsilon > 0$, let us define the perturbed energy

$$E_{\nu_\epsilon}(t) = E_\nu(t) + \epsilon\Psi(t), \tag{5.6}$$

with

$$\Psi(t) = \alpha(\frac{\rho h^3}{12}u_t^\nu(t), xu_x^\nu(t)) + \alpha(\rho h v_t^\nu(t), xv_x^\nu(t)) + \beta(\frac{\rho h^3}{12}u_t^\nu(t), u^\nu(t)) + \beta(\rho h v_t^\nu(t), v^\nu(t)), \tag{5.7}$$

where $\alpha > 0$ and $\beta > 0$ are constants such that

$$\alpha + 2\beta > \max\{\alpha k L, \alpha L, 4\beta\}, \tag{5.8}$$

$$\exists\ \gamma_1 > 0 \text{ such that } \beta f(s)s \geq (\alpha+\gamma_1)F(s),\ \forall s \in \mathbb{R}, \tag{5.9}$$

and

$$\exists\ \gamma_2 > 0 \text{ such that } \beta g(s)s \geq (\alpha+\gamma_2)G(s),\ \forall s \in \mathbb{R}. \tag{5.10}$$

The choice of β is possible because

$$\frac{(\alpha+\gamma_1)}{(2+\delta_1)} f(s)s \geq (\alpha+\gamma_1)F(s), \quad \frac{(\alpha+\gamma_2)}{(2+\delta_2)} g(s)s \geq (\alpha+\gamma_2)G(s)$$

and

$$0 < \frac{(\alpha+\gamma_i)}{(2+\delta i)} < \frac{\alpha}{2}, \text{ for } 0 < \gamma_i < \frac{\alpha\delta_i}{2}\ \ (i=1,2).$$

After some calculations, we find

$$\begin{aligned}
|\Psi(t)| &\leq \left(\frac{\alpha L+\beta L}{2}\right)\frac{\rho h^3}{12}\left|u_t^\nu(t)\right|^2 + (\alpha L+\beta L)\rho h\left|v_t^\nu(t)\right|^2 \\
&+ \left(\frac{\alpha\rho hL+\beta\rho hL}{2k}\right)k\left|(u^\nu+v_x^\nu)(t)\right|^2 + \left[\frac{\rho h^3 L(\alpha+\beta)+12\rho hL^3(\alpha+\beta)}{24}\right]\left\|u^\nu(t)\right\|^2,
\end{aligned}$$

which implies

$$|\Psi(t)| \leq C_1 E_\nu(t), \tag{5.11}$$

where $C_1 = \left[3+(\rho h/k)+(\rho h^3/12)+\rho hL^2\right]L(\alpha+\beta)$. It follows by (5.6) and (5.11) that

$$\left|E_{\nu_\epsilon}(t) - E_\nu(t)\right| = \epsilon\left|\Psi(t)\right| \leq \epsilon C_1 E_\nu(t),$$

that is,

$$(1-\epsilon C_1)E_\nu(t) \leq E_{\nu_\epsilon}(t) \leq (1+\epsilon C_1)E_\nu(t).$$

Taking $0 \leq \epsilon \leq \epsilon_0 = 1/2C_1$, we have

$$\frac{E_\nu(t)}{2} \leq E_{\nu_\epsilon}(t) \leq 2E_\nu(t). \tag{5.12}$$

Deriving the function (5.7) and using the equations (4.46) and (4.47), it follows that

$$\begin{aligned}
&\Psi'(t) = \alpha(u_{xx}^\nu(t), xu_x^\nu(t)) + \alpha k((u^\nu+v_x^\nu)_x(t), x(u^\nu+v_x^\nu)(t)) \\
&+\alpha k\left((u^\nu+v_x^\nu)(t), u^\nu(t)\right) + \alpha kLu^\nu(L,t)v_t^\nu(L,t) - \alpha(f_\nu(u^\nu(t)), xu_x^\nu(t)) \\
&+\alpha(\frac{\rho h^3}{12}u_t^\nu(t), xu_{xt}^\nu(t)) - \alpha(g_\nu(v^\nu(t)), xv_x^\nu(t)) + \alpha(\rho hv_t^\nu(t), xv_{xt}^\nu(t)) - \beta\left\|u^\nu(t)\right\|^2 \\
&-\beta u^\nu(L,t)u_t^\nu(L,t) - \beta k\left|(u^\nu+v_x^\nu)(t)\right|^2 - \beta(f_\nu(u^\nu(t)), u^\nu(t)) \\
&+\frac{\beta\rho h^3}{12}\left|u_t^\nu(t)\right|^2 - \beta kv^\nu(L,t)v_t^\nu(L,t) - \beta(g_\nu(v^\nu(t)), v^\nu(t)) + \beta\rho h\left|v_t^\nu(t)\right|^2.
\end{aligned} \tag{5.13}$$

Now, we will analyze some terms that appear on the right side of (5.13).

- Analysis of $\alpha(u^{\nu}_{xx}(t), xu^{\nu}_x(t))$.

$$\alpha(u^{\nu}_{xx}(t), xu^{\nu}_x(t)) = \frac{\alpha}{2}\int_0^L x\frac{d}{dx}\left|u^{\nu}_x(t)\right|^2 dx = \frac{\alpha L}{2}\left|u^{\nu}_t(L,t)\right|^2 - \frac{\alpha}{2}\int_0^L \left|u^{\nu}_x(t)\right|^2 dx. \quad (5.14)$$

- Analysis of $\frac{\alpha\rho h^3}{12}(u^{\nu}_t(t), xu^{\nu}_{tx}(t))$

$$\begin{aligned}&\frac{\alpha\rho h^3}{12}(u^{\nu}_t(t), xu^{\nu}_{tx}(t)) = \frac{\alpha\rho h^3}{24}\int_0^L x\frac{d}{dx}\left|u^{\nu}_t(t)\right|^2 dx = \frac{\alpha\rho h^3 L}{24}\left|u^{\nu}_t(L,t)\right|^2 \\ &-\frac{\alpha\rho h^3}{24}\int_0^L \left|u^{\nu}_t(t)\right|^2 dx.\end{aligned} \quad (5.15)$$

- Analysis of $\alpha\rho h(v^{\nu}_t(t), xv^{\nu}_{tx}(t))$

$$\alpha\rho h(v^{\nu}_t(t), xv^{\nu}_{tx}(t)) = \frac{\alpha\rho h}{2}\int_0^L x\frac{d}{dx}\left|v^{\nu}_t(t)\right|^2 dx = \frac{\alpha\rho h L}{2}\left|v^{\nu}_t(L,t)\right|^2 - \frac{\alpha\rho h}{2}\int_0^L \left|v^{\nu}_t(t)\right|^2 dx. \quad (5.16)$$

- Analysis of $-\beta u^{\nu}(L,t)u^{\nu}_t(L,t)$

$$-\beta u^{\nu}(L,t)u^{\nu}_t(L,t) \leq \beta c_0\left|u^{\nu}_t(L,t)\right|\left\|u^{\nu}(t)\right\| \leq C_2\left|u^{\nu}_t(L,t)\right|^2 + \xi\left\|u^{\nu}(t)\right\|^2, \quad (5.17)$$

where $C_2 = \beta^2 c_0^2/4\xi$, $c_0 > 0$ such that $|u^{\nu}(L,t)| \leq c_0\|u^{\nu}(t)\|$ and $\xi > 0$ a constant to be chosen.

- Analysis of $-\beta k v^{\nu}(L,t)v^{\nu}_t(L,t)$

$$\begin{aligned}&-\beta k v^{\nu}(L,t)v^{\nu}_t(L,t) \leq \beta k c_0\left|v^{\nu}_t(L,t)\right|\left(\left|(u^{\nu}+v^{\nu}_x)(t)\right| + L\left\|u^{\nu}(t)\right\|\right) \\ &\leq \frac{\beta^2 k c_0^2}{4\xi}\left|v^{\nu}_t(L,t)\right|^2 + \xi k\left|(u^{\nu}+v^{\nu}_x)(t)\right|^2 + \frac{\beta^2 k^2 c_0^2 L^2}{4\xi}\left|v^{\nu}_t(L,t)\right|^2 + \xi\left\|u^{\nu}(t)\right\|^2 \\ &\leq C_3\left|v^{\nu}_t(L,t)\right|^2 + \xi k\left|(u^{\nu}+v^{\nu}_x)(t)\right|^2 + \xi\left\|u^{\nu}(t)\right\|^2,\end{aligned} \quad (5.18)$$

where $C_3 = (1+kL^2)\,\beta^2 k c_0^2/4\xi$.

- Analysis of $\alpha(f_\nu(u^{\nu}(t)), xu^{\nu}_x(t))$

Observing that F is of class $C^1(\mathbb{R})$ and $F(0) = 0$, so

$$\begin{aligned}&\alpha(f_\nu(u^{\nu}(t)), xu^{\nu}_x(t)) = \alpha\int_0^L x f_\nu(u^{\nu}(t))u^{\nu}_x(t)\,dx = \alpha\int_0^L x\frac{d}{dx}F_\nu(u^{\nu}(x,t))dx \\ &= \alpha L F(u^{\nu}(L,t)) - \alpha\int_0^L F_\nu(u^{\nu}(x,t))dx.\end{aligned} \quad (5.19)$$

- Analysis of $\alpha(g_\nu(v^\nu(t)), xv_x^\nu(t))$

Noting that G belongs to class $C^1(\mathbb{R})$ and $G(0)=0$, we obtain, in a manner analogous to the last analysis, that

$$\alpha(g_\nu(v^\nu(t)), xv_x^\nu(t)) = \alpha LG(v^\nu(L,t)) - \alpha\int_0^L G_\nu(v^\nu(x,t))dx. \tag{5.20}$$

- Analysis of $\alpha((u^\nu + v_x^\nu)_x(t), x(u^\nu + v_x^\nu)(t))$

$$\begin{aligned}((u^\nu + v_x^\nu)_x(t), x(u^\nu + v_x^\nu)(t)) &= \frac{\alpha}{2}\int_0^L x\frac{d}{dx}\left|(u^\nu + v_x^\nu)(t)\right|^2 dx\\ = \frac{\alpha L}{2}\left|v_t(L,t)\right| - \frac{\alpha}{2}\int_0^L \left|(u^\nu + v_x^\nu)(t)\right|^2 dx.\end{aligned} \tag{5.21}$$

- Analysis of $\alpha k\left((u^\nu + v_x^\nu)(t), u^\nu(t)\right)$

$$\alpha k((u^\nu + v_x^\nu)(t), u^\nu(t)) \le \frac{\alpha L}{2}k\left|(u^\nu + v_x^\nu)(t)\right|^2 + \frac{\alpha Lk}{2}\left\|u^\nu(t)\right\|^2. \tag{5.22}$$

- Analysis of $\alpha kLu^\nu(L,t)v_t^\nu(L,t)$

$$\alpha kLv_t^\nu(L,t)u^\nu(L,t) \le c_0\alpha kL\left|v_t^\nu(L,t)\right|\left\|u^\nu(t)\right\| \le C_4\left|v_t^\nu(L,t)\right|^2 + \xi\left\|u^\nu(t)\right\|^2, \tag{5.23}$$

where $C_4 = c_0^2\alpha^2k^2L^2/4\xi$.

Substituting (5.14) – (5.23) in (5.13) and using (5.9), (5.10), we get

$$\begin{aligned}\Psi'(t) \le{}& \frac{\alpha L}{2}\left|u_t^\nu(L,t)\right|^2 - \frac{\alpha}{2}\left\|u^\nu(t)\right\|^2 + \frac{\alpha kL}{2}\left|v_t(L,t)\right| - \frac{\alpha}{2}k\left|(u^\nu + v_x^\nu)(t)\right|^2\\ &+\frac{\alpha L}{2}k\left|(u^\nu + v_x^\nu)(t)\right|^2 + \frac{\alpha kL}{2}\left\|u^\nu(t)\right\|^2 + C_4\left|v_t(L,t)\right|^2 + \xi\left\|u^\nu(t)\right\|^2\\ &-\alpha LF_\nu(u^\nu(L,t)) + \frac{\alpha\rho h^3L}{24}\left|u_t^\nu(L,t)\right|^2 - \frac{\alpha\rho h^3}{24}\left|u_t^\nu(t)\right|^2 - \alpha LG_\nu(v^\nu(L,t))\\ &+\frac{\alpha\rho hL}{2}\left|v_t^\nu(L,t)\right|^2 - \frac{\alpha\rho h}{2}\left|v_t^\nu(t)\right|^2 - \beta\left\|u^\nu(t)\right\|^2 + C_2\left|u_t^\nu(L,t)\right|^2 + \xi\left\|u^\nu(t)\right\|^2\\ &-\beta k\left|(u^\nu + v_x^\nu)(t)\right|^2 - \gamma_1\int_0^L F_\nu(u^\nu(x,t))dx + \frac{\beta\rho h^3}{12}\left|u_t^\nu(t)\right|^2 + C_3\left|v_t^\nu(L,t)\right|^2\\ &+\xi k\left|(u^\nu + v_x^\nu)(t)\right|^2 + \xi\left\|u^\nu(t)\right\|^2 - \gamma_2\int_0^L G_\nu(v^\nu(x,t))dx + \beta\rho h\left|v_t^\nu(t)\right|^2,\end{aligned} \tag{5.24}$$

which implies

$$\Psi'(t) \le -\left(C_5 - 2\xi\right)E_\nu(t) + C_6\left|u_t^\nu(L,t)\right|^2 + C_7\left|v_t^\nu(L,t)\right|^2, \tag{5.25}$$

where C_5, C_6 and C_7 are the following positive constants:

$$C_5 = \min\{\alpha + 2\beta - \alpha k L, \alpha + 2\beta - \alpha L, \alpha - 2\beta, \gamma_1, \gamma_2\},$$

$$C_6 = \frac{\alpha L}{2} + \frac{\alpha \rho h^3}{24} + C_2 \quad \text{and} \quad C_7 = \frac{\alpha k L}{2} + \frac{\alpha \rho h L}{2} + C_3 + C_4.$$

The positivity of C_5 is guaranteed by (5.8) – (5.10).

Now, deriving (5.6) and, soon after, substituting the expressions (5.5) and (5.25), we obtain

$$E'_{\nu_\epsilon}(t) \leq -\epsilon\,(C_5 - 2\xi)\,E_\nu(t) - (1 - \epsilon C_6)\,|u_t^\nu(L,t)|^2 - (1 - \epsilon C_7)\,|v_t^\nu(L,t)|^2.$$

Taking $0 < \epsilon \leq \epsilon_1 = \min\{1/C_6, 1/C_7\}$ and $0 < \xi < C_5/2$, we have

$$E'_{\nu_\epsilon}(t) + \kappa E_{\nu_\epsilon} \leq 0,$$

with $\kappa = \min\{\epsilon_0, \epsilon_1\}\,(C_5 - 2\xi) > 0$. In this way

$$E_{\nu_\epsilon}(t) \leq E_{\nu_\epsilon}(0)e^{-\kappa t},\ \forall t \geq 0.$$

Combining the last inequality with (5.12), we deduce

$$E_\nu(t) \leq 4E_\nu(0)e^{-\kappa t},\ \ \forall t \geq 0. \tag{5.26}$$

Since F_ν and G_ν are continuous, it follows by (4.50) and (4.51) the convergences

$$F_\nu(u^\nu(\cdot,t)) \to F_\nu(u(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0, \tag{5.27}$$

$$G_\nu(v^\nu(\cdot,t)) \to G_\nu(v(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0. \tag{5.28}$$

For other hand, since $f_\nu \to f$ and $g_\nu \to g$ uniformly on bounded sets of $\mathbb{R}$, then

$$F_\nu(u(\cdot,t)) \to F(u(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0, \tag{5.29}$$

$$G_\nu(v(\cdot,t)) \to G(v(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0. \tag{5.30}$$

According to (5.27) – (5.30), we can conclude that

$$F_\nu(u^\nu(\cdot,t)) \to F(u(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0, \tag{5.31}$$

$$G_\nu(v^\nu(\cdot,t)) \to G(v(\cdot,t)) \text{ a. e. in } (0,L),\ \forall t \geq 0. \tag{5.32}$$

From (5.26) we have the following estimates

$$\int_0^L F_\nu(u^\nu(x,t))dx \leq 4E_\nu(0), \quad \int_0^L G_\nu(v^\nu(x,t))dx \leq 4E_\nu(0). \tag{5.33}$$

Using the convergences (4.11) – (4.16), (4.26) and (4.27) in (5.33), it follows that

$$\liminf_{\nu\to\infty}\int_0^L F_\nu(u^\nu(x,t))dx \le 4E(0), \quad \liminf_{\nu\to\infty}\int_0^L G_\nu(v^\nu(x,t))dx \le 4E(0). \tag{5.34}$$

By (5.31), (5.32), (5.34) and Fatou's lemma, we have

$$\int_0^L F(u(x,t))dx \le \liminf_{\nu\to\infty}\int_0^L F_\nu(u^\nu(x,t))dx,$$

and

$$\int_0^L G(v(x,t))dx \le \liminf_{\nu\to\infty}\int_0^L G_\nu(v^\nu(x,t))dx.$$

In this way, taking the $\liminf_{\nu\to\infty}$ in (5.26), we can deduce the inequality (5.4). ∎

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