Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **33**, 1-24; **http://www.math.u-szeged.hu/ejqtde/**

Multivalued Evolution Equations with Infinite Delay in Fréchet Spaces

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Abstract

In this paper, sufficient conditions are given to investigate the existence of mild solutions on a semi-infinite interval for two classes of first order semilinear functional and neutral functional differential evolution inclusions with infinite delay using a recent nonlinear alternative for contractive multivalued maps in Fréchet spaces due to Frigon, combined with semigroup theory.

Key words : Functional and neutral evolution inclusions, mild solution, fixed-point, semigroup theory, contractive multivalued maps, Fréchet spaces, infinite delay.
AMS Subject Classification : 34A60, 34G25, 34K40, 49K24.

1 Introduction

In this paper, we consider the existence of mild solutions defined on a semi-infinite positive real interval $J := [0, +\infty)$, for two classes of first order semilinear functional and neutral functional differential evolution inclusions with infinite delay in a real Banach space $(E, |\cdot|)$. Firstly, in Section 3, we study the following evolution inclusion of the form

$$y'(t) \in A(t)y(t) + F(t, y_t), \quad \text{a.e.} \quad t \in J \tag{1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all subsets of E, $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $0 \leq s \leq t < +\infty$.

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belongs to some abstract *phase space* $\mathcal{B}$, to be specified later.

In Section 4, we consider the following neutral evolution inclusion of the form

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in A(t)y(t) + F(t, y_t), \quad \text{a.e.} \quad t \in J \tag{3}$$

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$$y_0 = \phi \in \mathcal{B}, \tag{4}$$

where $A(\cdot)$, F and ϕ are as in problem $(1)-(2)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally in Section 5, two examples are provided illustrating the abstract theory.

Functional differential and partial differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for functional differential equations is the books by Hale [25] and Hale and Verduyn Lunel [27], Kolmanovskii and Myshkis [36], Kuang [37] and Wu [41] and the references therein.

During the last decades, several authors considered the problem of existence of mild solutions for semilinear evolution equations with finite delay. Some results can be found in the books by Ahmed [1, 2], Heikkila and Lakshmikantham [28] and Pazy [38] and Wu [41] and the references therein. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [26], see also Corduneanu and Lakshmikantham [17], Hino *et al.* [31], Kappel and Schappacher [33], and Schumacher [39].

An extensive theory is developed for inclusion (1) with $A(t) = A$. We refer the reader to the books by Heikkila and Lakshmikantham [28], Kamenski *et al* [34] and the pioneer Hino and Murakami paper [30]. By means of fixed point arguments, Benchohra and his collaborators have studied many classes of first and second order functional differential inclusions with local and nonlocal conditions in [7, 8, 10, 11, 13, 14, 15, 23] on a bounded interval. Extension to the semiinfinite interval is given by Benchohra and Ntouyas in [9, 12] and by Henderson and Ouahab [29] with finite delay.

When A is depending on the time, Arara *et al* [3] considered control multivalued problem on a bounded interval $[0, b]$ and very recently Baghli and Benchohra [4, 5] provided uniqueness results for some classes of partial and neutral functional differential evolution equations on the interval $J = [0, +\infty)$ when the delay is finite. The perturbed problem with infinite delay is studied in [6]. Our main purpose in this paper is to look for the multivalued version of these problems.

Sufficient conditions are established to get existence results of mild solutions which are fixed points of the appropriate operators of the semilinear functional and the neutral functional differential evolution problems by applying the nonlinear alternative of Leray-Schauder type due to Frigon [21] for contractive multivalued maps in Fréchet spaces, combined with the semigroup theory [1, 2, 38].

2 Preliminaries

We introduce here notations, definitions and preliminary facts from multivalued analysis which are used throughout this paper.

Let $C([0, +\infty); E)$ be the space of continuous functions from $[0, +\infty)$ into E and

$B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : [0, +\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see Yosida [42] for instance).

Let $L^1([0, +\infty), E)$ denotes the Banach space of measurable functions $y : [0, +\infty) \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \ dt.$$

Consider the following space

$$B_{+\infty} = \{y : (-\infty, +\infty) \to E : y|_J \in C(J, E), \quad y_0 \in \mathcal{B}\},$$

where $y|_J$ is the restriction of y to $J = [0, +\infty)$.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [26] and follow the terminology used in [31]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $y(t)$ with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{ \ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(A_2) For the function $y(.)$ in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1

1. *(ii) is equivalent to* $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ *for every* $\phi \in \mathcal{B}$.
2. *Since* $\|\cdot\|_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can verify* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leq 0$.
3. *From the equivalence of (ii), we can see that for all* $\phi, \psi \in \mathcal{B}$ *such that* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *: This implies necessarily that* $\phi(0) = \psi(0)$.

Hereafter are some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [31].

Example 2.2 *The spaces BC, BUC, C^∞ and C^0. Let :*

BC *the space of bounded continuous functions defined from $(-\infty, 0]$ to E;*

BUC *the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;*

$$C^\infty := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E \right\};$$

$$C^0 := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \le 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 2.3 *The spaces C_g, UC_g, C_g^∞ and C_g^0. Let g be a positive continuous function on $(-\infty, 0]$. We define :*

$$C_g := \left\{ \phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0] \right\};$$

$$C_g^0 := \left\{ \phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{ \frac{|\phi(\theta)|}{g(\theta)} : \ \theta \le 0 \right\}.$$

We consider the following condition on the function g.

(g_1) *For all $a > 0$,* $\displaystyle \sup_{0 \le t \le a} \sup \left\{ \frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t \right\} < \infty.$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 2.4 *The space C_γ. For any real constant γ, we define the functional space C_γ by*

$$C_\gamma := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta \to -\infty} e^{\gamma\theta} \phi(\theta) \text{ exist in } E \right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta} |\phi(\theta)| : \ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

In what follows, we assume that $\{A(t),\ t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.5 *We say that a family $\{A(t)\}_{t\geq 0}$ generates a unique linear evolution system $\{U(t,s)\}_{(t,s)\in\Delta}$ for $\Delta := \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$ satisfying the following properties :*

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)y$ is continuous.*

More details on evolution systems and their properties could be found on the books of Ahmed [1], Engel and Nagel [19] and Pazy [38].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that F is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \text{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \quad \text{for every } x \in X.$$

Let (X,d) be a metric space. We use the following notations :

$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \ closed\}, \quad \mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \ bounded\},$$

$$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \ convexe\}, \quad \mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \ compact\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B}) = \max\left\{ \sup_{a\in\mathcal{A}} \ d(a,\mathcal{B}) \ , \ \sup_{b\in\mathcal{B}} \ d(\mathcal{A},b) \right\},$$

where $d(\mathcal{A},b) = \inf_{a\in\mathcal{A}} \ d(a,b)$, $d(a,\mathcal{B}) = \inf_{b\in\mathcal{B}} \ d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [35]).

Definition 2.6 *A multivalued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if for each $x \in E$, the function $Y : J \to X$ defined by*

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable where d is the metric induced by the normed Banach space X.

Definition 2.7 *A function $F : J \times \mathcal{B} \longrightarrow \mathcal{P}(X)$ is said to be an L^1_{loc}-Carathéodory multivalued map if it satisfies :*

(i) $x \mapsto F(t, y)$ is continuous (with respect to the metric H_d) for almost all $t \in J$;

(ii) $t \mapsto F(t, y)$ is measurable for each $y \in \mathcal{B}$;

(iii) for every positive constant k there exists $h_k \in L^1_{loc}(J; \mathbb{R}^+)$ such that

$$\|F(t, y)\| \le h_k(t) \quad \textit{for all } \|y\|_{\mathcal{B}} \le k \textit{ and for almost all } t \in J.$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x \in B} \{\sup\{ \ \|y\| \ : \ y \in G(x)\}\} < \infty.$$

Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

For each $y \in B_{+\infty}$ let the set $S_{F,y}$ known as *the set of selectors* from F defined by

$$S_{F,y} = \{v \in L^1(J; E) : v(t) \in F(t, y_t) \ , \ a.e. \ t \in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [18], Górniewicz [24], Hu and Papageorgiou [32] and Tolstonogov [40].

Definition 2.8 *A multivalued map $F : X \to \mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n \in \mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that*

i) $H_d(F(x), F(y)) \le k_n \ \|x - y\|_n$ for all $x, y \in X$.

ii) For every $x \in X$ and every $\epsilon \in (0, \infty)^n$, there exists $y \in F(x)$ such that

$$\|x - y\|_n \le \|x - F(x)\|_n + \epsilon_n \textit{ for every } n \in \mathbb{N}$$

Theorem 2.9 *(Nonlinear Alternative of Frigon, [21, 22]). Let X be a Fréchet space and U an open neighborhood of the origin in X and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds :*

$(C1)$ *N has a fixed point;*

$(C2)$ *There exists $\lambda \in [0, 1)$ and $x \in \partial U$ such that $x \in \lambda \ N \ (x)$.*

3 Semilinear Evolution Inclusions

The main result of this section concerns the semilinear evolution problem $(1)-(2)$. Before stating and proving this one, we give first the definition of the mild solution.

Definition 3.1 *We say that the function $y(\cdot):(-\infty,+\infty)\to E$ is a mild solution of the evolution system $(1)-(2)$ if $y(t)=\phi(t)$ for all $t\in(-\infty,0]$ and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot)\in L^1(J;E)$: $f(t)\in F(t,y_t)$ a.e. in J such that y satisfies the following integral equation :*

$$y(t)=U(t,0)\ \phi(0)+\int_0^t U(t,s)\ f(s)\ ds \quad \text{for each } t\in[0,+\infty). \tag{5}$$

We will need to introduce the following hypothesis which are assumed hereafter :

$(H1)$ There exists a constant $\widehat{M}\geq 1$ such that :

$$\|U(t,s)\|_{B(E)}\leq \widehat{M} \quad \text{for every } (t,s)\in\Delta.$$

$(H2)$ The multifunction $F:J\times\mathcal{B}\longrightarrow\mathcal{P}(E)$ is L^1_{loc}-Carathéodory with compact and convex values for each $u\in\mathcal{B}$ and there exist a function $p\in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:J\to(0,\infty)$ such that:

$$\|F(t,u)\|_{\mathcal{P}(E)}\leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

$(H3)$ For all $R>0$, there exists $l_R\in L^1_{loc}(J;\mathbb{R}_+)$ such that :

$$H_d(F(t,u)-F(t,v))\leq l_R(t)\ \|u-v\|_{\mathcal{B}}$$

for each $t\in J$ and for all $u,v\in\mathcal{B}$ with $\|u\|_{\mathcal{B}}\leq R$ and $\|v\|_{\mathcal{B}}\leq R$ and

$$d(0,F(t,0))\leq l_R(t) \quad \text{a.e.} \quad t\in J.$$

For every $n\in\mathbb{N}$, we define in $B_{+\infty}$ the family of semi-norms by :

$$\|y\|_n:=\sup\ \{\ e^{-\tau\ L_n^*(t)}\ |y(t)|:t\in[0,n]\ \}$$

where : $L_n^*(t)=\displaystyle\int_0^t \overline{l}_n(s)\ ds$, $\overline{l}_n(t)=\widehat{M}K_n l_n(t)$ and l_n is the function from $(H3)$. Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$. In what follows we will choose $\tau>1$.

Theorem 3.2 *Suppose that hypotheses $(H1)-(H3)$ are satisfied and moreover*

$$\int_{c_n}^{+\infty}\frac{ds}{\psi(s)}>K_n\widehat{M}\int_0^n p(s)\ ds \quad \text{for each } n\in\mathbb{N} \tag{6}$$

with $c_n=(K_n\widehat{M}H+M_n)\|\phi\|_{\mathcal{B}}$. Then evolution problem $(1)-(2)$ has a mild solution.

Proof. Transform the problem (1) – (2) into a fixed-point problem. Consider the multivalued operator $N : B_{+\infty} \to \mathcal{P}(B_{+\infty})$ defined by :

$$N(y) = \left\{ h \in B_{+\infty} : h(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0) + \displaystyle\int_0^t U(t,s)\ f(s)\ ds, & \text{if } t \geq 0, \end{cases} \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e.}\, t \in J\}$.

Clearly, the fixed points of the operator N are mild solutions of the problem (1)–(2). We remark also that, for each $y \in B_{+\infty}$, the set $S_{F,y}$ is nonempty since, by $(H2)$, F has a measurable selection (see [16], Theorem III.6).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty,+\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{7}$$

It is obvious that y satisfies (5) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s)\ f(s)\ ds \quad \text{ for } t \in J.$$

where $f(t) \in F(t, z_t + x_t)$ *a.e.* $t \in J$.

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

Define in $B^0_{+\infty}$, the multivalued operator $\mathcal{F} : B^0_{+\infty} \to \mathcal{P}(B^0_{+\infty})$ by :

$$\mathcal{F}(z) = \left\{ h \in B^0_{+\infty} : h(t) = \int_0^t U(t,s)\ f(s)\ ds, \quad t \in J \right\},$$

where $f \in S_{F,z} = \{v \in L^1(J,E) : v(t) \in F(t, z_t + x_t) \text{ for a.e.}\, t \in J\}$.

Obviously the operator inclusion N has a fixed point is equivalent to the operator inclusion $\mathcal{F}$ has one, so it turns to prove that $\mathcal{F}$ has a fixed point.

Let $z \in B^0_{+\infty}$ be a possible fixed point of the operator $\mathcal{F}$. Given $n \in \mathbb{N}$, then z should be solution of the inclusion $z \in \lambda\ \mathcal{F}(z)$ for some $\lambda \in (0,1)$ and there exists

$f \in S_{F,z} \Leftrightarrow f(t) \in F(t, z_t + x_t)$ such that, for each $t \in [0, n]$, we have

$$\begin{aligned} |z(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \ |f(s)| \ ds \\ &\leq \widehat{M} \int_0^t p(s) \ \psi \left(\|z_s + x_s\|_{\mathcal{B}}\right) \ ds. \end{aligned}$$

Assumption (A_1) gives

$$\begin{aligned} \|z_s + x_s\|_{\mathcal{B}} &\leq \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\ &\leq K(s)|z(s)| + M(s)\|z_0\|_{\mathcal{B}} + K(s)|x(s)| + M(s)\|x_0\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + M_n\|\phi\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + K_n\widehat{M}|\phi(0)| + M_n\|\phi\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + K_n\widehat{M}H\|\phi\|_{\mathcal{B}} + M_n\|\phi\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + (K_n\widehat{M}H + M_n)\|\phi\|_{\mathcal{B}}. \end{aligned}$$

Set $c_n := (K_n\widehat{M}H + M_n)\|\phi\|_{\mathcal{B}}$, then we have

$$\|z_s + x_s\|_{\mathcal{B}} \leq K_n|z(s)| + c_n \tag{8}$$

Using the nondecreasing character of ψ, we get

$$|z(t)| \leq \widehat{M} \int_0^t p(s) \ \psi \left(K_n|z(s)| + c_n\right) \ ds.$$

Then

$$K_n|z(t)| + c_n \leq K_n\widehat{M} \int_0^t p(s)\psi(K_n|z(s)| + c_n)ds + c_n.$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ K_n|z(s)| + c_n \ : \ 0 \leq s \leq t \ \}, \quad 0 \leq t < +\infty.$$

Let $t^\star \in [0, t]$ be such that

$$\mu(t) = K_n|z(t^\star)| + c_n.$$

By the previous inequality, we have

$$\mu(t) \leq K_n\widehat{M} \int_0^t p(s) \ \psi(\mu(s)) \ ds + c_n \quad \text{for} \quad t \in [0, n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \leq v(t) \ \text{ for all } t \in [0, n].$$

From the definition of v, we have

$$v(0) = c_n \quad \text{and} \quad v'(t) = K_n \widehat{M} p(t)\ \psi(\mu(t)) \quad \text{a.e.} \;\; t \in [0, n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq K_n \widehat{M}\ p(t)\ \psi(v(t)) \;\; \text{a.e. } t \in [0, n].$$

This implies that for each $t \in [0, n]$ and using the condition (6), we get

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq K_n \widehat{M} \int_0^t p(s)\ ds \leq K_n \widehat{M} \int_0^n p(s)\ ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus, for every $t \in [0, n]$, there exists a constant Λ_n such that $v(t) \leq N_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$. Set

$$\mathcal{U} = \{\ z \in B^0_{+\infty} :\ \sup\{\ |z(t)|\ :\ 0 \leq t \leq n\} < \Lambda_n + 1 \quad \text{for all} \;\; n \in \mathbb{N}\}.$$

Clearly, $\mathcal{U}$ is an open subset of $B^0_{+\infty}$.

We shall show that $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{P}(B^0_{+\infty})$ is a contraction and an admissible operator.

First, we prove that $\mathcal{F}$ is a contraction ; Let $z, \overline{z} \in B^0_{+\infty}$ and $h \in \mathcal{F}(z)$. Then there exists $f(t) \in F(t, z_t + x_t)$ such that for each $t \in [0, n]$

$$h(t) = \int_0^t U(t, s)\ f(s)\ ds.$$

From $(H3)$ it follows that

$$H_d(F(t, z_t + x_t), F(t, \overline{z}_t + x_t)) \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}}.$$

Hence, there is $\rho \in F(t, \overline{z}_t + x_t)$ such that

$$|f(t) - \rho| \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}} \;\; t \in [0, n].$$

Consider $\mathcal{U}_\star : [0, n] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\rho \in E : |f(t) - \rho| \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}}\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{z}_t + x_t)$ is measurable (in [16], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{z}_t + x_t)$ and using (A_1), we obtain for each $t \in [0, n]$

$$\begin{array}{rcl} |f(t) - \overline{f}(t)| & \leq & l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}} \\ & \leq & l_n(t)\ [K(t)\ |z(t) - \overline{z}(t)| + M(t)\ \|z_0 - \overline{z}_0\|_{\mathcal{B}}] \\ & \leq & l_n(t)\ K_n\ |z(t) - \overline{z}(t)| \end{array}$$

Let us define, for each $t \in [0, n]$

$$\overline{h}(t) = \int_0^t U(t,s)\ \overline{f}(s)\ ds.$$

Then we have

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \left|f(s) - \overline{f}(s)\right|\ ds \\
&\leq \int_0^t \widehat{M}\ K_n\ l_n(s)\ |z(s) - \overline{z}(s)|\ ds \\
&\leq \int_0^t [\bar{l}_n(s)\ e^{\tau\ L_n^*(s)}\ ]\ [e^{-\tau\ L_n^*(s)}\ |z(s) - \overline{z}(s)|]\ ds \\
&\leq \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]'\ ds\ \|z - \overline{z}\|_n \\
&\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|h - \overline{h}\|_n \leq \frac{1}{\tau}\|z - \overline{z}\|_n.$$

By an analogous relation, obtained by interchanging the roles of z and $\overline{z}$, it follows that

$$H_d(\mathcal{F}(z), \mathcal{F}(\overline{z})) \leq \frac{1}{\tau}\|z - \overline{z}\|_{\overline{B}}.$$

So, $\mathcal{F}$ is a contraction for all $n \in \mathbb{N}$.

Now we shall show that $\mathcal{F}$ is an admissible operator. Let $z \in B^0_{+\infty}$. Set, for every $n \in \mathbb{N}$, the space

$$B_n^0 := \left\{y : (-\infty, n] \to E : y|_{[0,n]} \in C([0,n], E), \quad y_0 \in \mathcal{B}\right\},$$

and let us consider the multivalued operator $\mathcal{F} : B_n^0 \to \mathcal{P}_{cl}(B_n^0)$ defined by :

$$\mathcal{F}(z) = \left\{h \in B_n^0 : h(t) = \int_0^t\ U(t,s)\ f(s)\ ds, \quad t \in [0,n]\right\}$$

where $f \in S^n_{F,y} = \{v \in L^1([0,n], E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in [0,n]\}$.

From $(H1) - (H3)$ and since F is a multivalued map with compact values, we can prove that for every $z \in B_n^0$, $\mathcal{F}(z) \in \mathcal{P}_{cl}(B_n^0)$ and there exists $z_\star \in B_n^0$ such that $z_\star \in \mathcal{F}(z_\star)$. Let $h \in B_n^0$, $\overline{y} \in \overline{\mathcal{U}}$ and $\epsilon > 0$. Assume that $z_\star \in \mathcal{F}(\overline{z})$, then we have

$$\begin{aligned}
|\overline{z}(t) - z_\star(t)| &\leq |\overline{z}(t) - h(t)| + |z_\star(t) - h(t)| \\
&\leq e^{\tau\ L_n^*(t)}\ \|\overline{z} - \mathcal{F}(\overline{z})\|_n + \|z_\star - h\|.
\end{aligned}$$

Since h is arbitrary, we may suppose that $h \in B(z_\star, \epsilon) = \{h \in B_n^0 : \|h - z_\star\|_n \leq \epsilon\}$. Therefore,

$$\|\overline{z} - z_\star\|_n \leq \|\overline{z} - \mathcal{F}(\overline{z})\|_n + \epsilon.$$

If z is not in $\mathcal{F}(\overline{z})$, then $\|z_\star - \mathcal{F}(\overline{z})\| \neq 0$. Since $\mathcal{F}(\overline{z})$ is compact, there exists $x \in \mathcal{F}(\overline{z})$ such that $\|z_\star - \mathcal{F}(\overline{z})\| = \|z_\star - x\|$. Then we have

$$\begin{aligned} |\overline{z}(t) - z_\star(t)| &\leq |\overline{z}(t) - h(t)| + |x(t) - h(t)| \\ &\leq e^{\tau\ L_n^*(t)}\ \|\overline{z} - \mathcal{F}(\overline{z})\|_n + |x(t) - h(t)|. \end{aligned}$$

Thus,

$$\|\overline{z} - x\|_n \leq \|\overline{z} - \mathcal{F}(\overline{z})\|_n + \epsilon.$$

So, $\mathcal{F}$ is an admissible operator contraction. From the choice of $\mathcal{U}$ there is no $z \in \partial\mathcal{U}$ such that $z = \lambda\ \mathcal{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.9 does not hold. We deduce that the operator $\mathcal{F}$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is a mild solution of the evolution inclusion problem $(1) - (2)$.

4 Semilinear Neutral Evolution Inclusions

In this section, we give existence results for the neutral functional differential evolution problem with infinite delay $(3) - (4)$. Firstly we define the mild solution.

Definition 4.1 *We say that the function $y(\cdot) : (-\infty, +\infty) \to E$ is a mild solution of the neutral evolution system $(3) - (4)$ if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_t)$ a.e. in J such that y satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_t) + \int_0^t U(t,s)A(s)g(s,y_s)ds \\ &+ \int_0^t U(t,s)f(s)\ ds \qquad for\ each\ t \in [0, +\infty). \end{aligned} \tag{9}$$

We consider the hypotheses $(H1) - (H3)$ and we will need the following assumptions :

$(H4)$ There exists a constant $\overline{M}_0 > 0$ such that :

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \qquad for\ all\ t \in J.$$

$(H5)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_n}$ such that :

$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H6)$ There exists a constant $L_\star > 0$ such that :

$$|A(s)\ g(s,\phi) - A(\overline{s})\ g(\overline{s},\overline{\phi})| \leq L_\star\ (|s-\overline{s}| + \|\phi - \overline{\phi}\|_{\mathcal{B}})$$

for all $s, \overline{s} \in J$ and $\phi, \overline{\phi} \in \mathcal{B}$.

For every $n \in \mathbb{N}$, let us take here $\overline{l}_n(t) = \widehat{M} K_n [L_\star + l_n(t)]$ for the family of semi-norm $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ defined in Section 3. In what follows we fix $\tau > 1$ and assume $\left[\overline{M}_0 L_\star K_n + \dfrac{1}{\tau}\right] < 1$.

Theorem 4.2 *Suppose that hypotheses* $(H1)-(H6)$ *are satisfied and moreover*

$$\int_{\delta_n}^{+\infty} \frac{ds}{s+\psi(s)} > \frac{\widehat{M} K_n}{1-\overline{M}_0 L K_n} \int_0^n \max(L, p(s))ds \quad for\ each \quad n \in \mathbb{N} \tag{10}$$

with

$$\begin{aligned}\delta_n \ :=&\ (K_n \widehat{M} H + M_n)\|\phi\|_{\mathcal{B}} \ + \frac{K_n}{1-\overline{M}_0 L K_n}\Big[(\widehat{M}+1)\overline{M}_0 L + \widehat{M} L n \\ &+ \overline{M}_0 L \left[\widehat{M}(K_n H + 1) + M_n\right] \|\phi\|_{\mathcal{B}}\Big].\end{aligned}$$

Then the neutral evolution problem $(3)-(4)$ *has a mild solution.*

Proof. Transform the neutral evolution problem $(3)-(4)$ into a fixed-point problem. Consider the multivalued operator $\widetilde{N} : B_{+\infty} \to \mathcal{P}(B_{+\infty})$ defined by :

$$\widetilde{N}(y) = \left\{ h \in B_{+\infty} : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t,y_t) & \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_s)ds & \\ + \displaystyle\int_0^t U(t,s)f(s)ds, & \text{if } t \in J, \end{array} \right. \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in J\}$.

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(3)-(4)$. We remark also that, for each $y \in B_{+\infty}$, the set $S_{F,y}$ is nonempty since, by $(H2)$, F has a measurable selection (see [16], Theorem III.6).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty,+\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{11}$$

It is obvious that y satisfies (9) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= g(t, z_t + x_t) - U(t,0)g(0,\phi) \\ &+ \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds + \int_0^t U(t,s)f(s)ds. \end{aligned}$$

where $f(t) \in F(t, z_t + x_t)$ *a.e.* $t \in J$.

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

Define in $B^0_{+\infty}$, the multivalued operator $\widetilde{\mathcal{F}} : B^0_{+\infty} \to \mathcal{P}(B^0_{+\infty})$ by :

$$\begin{aligned} \widetilde{\mathcal{F}}(z) &= \Big\{h \in B^0_{+\infty} : h(t) = g(t, z_t + x_t) - U(t,0)g(0,\phi) \\ &\quad + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds \\ &\quad + \int_0^t U(t,s)f(s)ds, \quad t \in J\Big\} \end{aligned}$$

where $f \in S_{F,z} = \{v \in L^1(J,E) : v(t) \in F(t, z_t + x_t) \text{ for a.e.}\, t \in J\}$.

Obviously the operator inclusion $\widetilde{N}$ has a fixed point is equivalent to the operator inclusion $\widetilde{\mathcal{F}}$ has one, so it turns to prove that $\widetilde{\mathcal{F}}$ has a fixed point.

Let $z \in B^0_{+\infty}$ be a possible fixed point of the operator $\widetilde{\mathcal{F}}$. Given $n \in \mathbb{N}$, then z should be solution of the inclusion $z \in \lambda\, \widetilde{\mathcal{F}}(z)$ for some $\lambda \in (0,1)$ and there exists $f \in S_{F,z} \Leftrightarrow f(t) \in F(t, z_t + x_t)$ such that, for each $t \in [0,n]$, we have

$$\begin{aligned} |z(t)| &\leq \|A^{-1}(t)\|_{B(E)}|A(t)g(t, z_t + x_t)| + \|U(t,0)\|_{B(E)}\|A^{-1}(0)\|_{B(E)}|A(0)\; g(0,\phi)| \\ &+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s, z_s + x_s)|ds + \int_0^t \|U(t,s)\|_{B(E)}|f(s)|ds \\ &\leq \overline{M}_0 L(\|z_t + x_t\|_{\mathcal{B}} + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) \\ &+ \widehat{M}\int_0^t L(\|z_s + x_s\|_{\mathcal{B}} + 1)ds + \widehat{M}\int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds \\ &\leq \overline{M}_0 L\|z_t + x_t\|_{\mathcal{B}} + \overline{M}_0 L(1 + \widehat{M}) + \widehat{M}Ln + \widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}} \\ &+ \widehat{M}\int_0^t L\|z_s + x_s\|_{\mathcal{B}}ds + \widehat{M}\int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds. \end{aligned}$$

Using the inequality (8) and the nondecreasing character of ψ, we obtain

$$\begin{aligned}
|z(t)| &\leq \overline{M}_0 L(K_n|z(t)|+c_n)+\overline{M}_0 L(1+\widehat{M})+\widehat{M}Ln+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}\\
&+ \widehat{M}\int_0^t L(K_n|z(s)|+c_n)ds+\widehat{M}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\\
&\leq \overline{M}_0 LK_n|z(t)|+\overline{M}_0 L(1+\widehat{M})+\widehat{M}Ln+\overline{M}_0 Lc_n+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}\\
&+ \widehat{M}\left[\int_0^t L(K_n|z(s)|+c_n)ds+\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\right].
\end{aligned}$$

Then

$$\begin{aligned}
(1-\overline{M}_0 LK_n)|z(t)| &\leq (\widehat{M}+1)\overline{M}_0 L+\widehat{M}Ln+\overline{M}_0 Lc_n+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}\\
&+ \widehat{M}\left[\int_0^t L(K_n|z(s)|+c_n)ds+\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\right].
\end{aligned}$$

Set $\delta_n := c_n+\dfrac{K_n}{1-\overline{M}_0 LK_n}\left[(\widehat{M}+1)\overline{M}_0 L+\widehat{M}Ln+\overline{M}_0 Lc_n+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}\right]$. Thus

$$\begin{aligned}
K_n|z(t)|+c_n &\leq \delta_n\\
&+ \frac{\widehat{M}K_n}{1-\overline{M}_0 LK_n}\left[\int_0^t L(K_n|z(s)|+c_n)ds+\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\right].
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\{\ K_n|z(s)|+c_n\ :\ 0\leq s\leq t\ \},\quad 0\leq t<+\infty.$$

Let $t^\star\in[0,t]$ be such that $\mu(t)=K_n|z(t^\star)|+c_n$. By the previous inequality, we have

$$\mu(t)\leq\delta_n+\frac{\widehat{M}K_n}{1-\overline{M}_0 LK_n}\left[\int_0^t L\mu(s)ds+\int_0^t p(s)\psi(\mu(s))ds\right]\quad\text{for}\quad t\in[0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t)\leq v(t)\ \text{ for all } t\in[0,n].$$

From the definition of v, we have $v(0)=\delta_n$ and

$$v'(t)=\frac{\widehat{M}K_n}{1-\overline{M}_0 LK_n}\left[L\mu(t)+p(t)\psi(\mu(t))\right]\quad\text{a.e. } t\in[0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t)\leq\frac{\widehat{M}K_n}{1-\overline{M}_0 LK_n}\left[Lv(t)+p(t)\psi(v(t))\right]\quad\text{a.e. } t\in[0,n].$$

This implies that for each $t \in [0,n]$ and using the condition (10), we get

$$\begin{aligned}
\int_{\delta_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq \frac{\widehat{M}K_n}{1-\overline{M}_0 L K_n} \int_0^t \max(L, p(s))ds \\
&\leq \frac{\widehat{M}K_n}{1-\overline{M}_0 L K_n} \int_0^n \max(L, p(s))ds \\
&< \int_{\delta_n}^{+\infty} \frac{ds}{s+\psi(s)}.
\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$.

We can show as in Section 3 that $\widetilde{\mathcal{F}}$ is an admissible operator and we shall prove now that $\widetilde{\mathcal{F}} : \overline{\mathcal{U}} \to \mathcal{P}(B^0_{+\infty})$ is a contraction.

Let $z, \overline{z} \in B^0_{+\infty}$ and $h \in \widetilde{\mathcal{F}}(z)$. Then there exists $f(t) \in F(t, z_t + x_t)$ such that for each $t \in [0,n]$

$$h(t) = g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds + \int_0^t U(t,s)f(s)ds$$

From $(H3)$ it follows that

$$H_d(F(t, z_t + x_t), F(t, \overline{z}_t + x_t)) \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}}.$$

Hence, there is $\rho \in F(t, \overline{z}_t + x_t)$ such that

$$|f(t) - \rho| \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}} \ \ t \in [0,n].$$

Consider $\mathcal{U}_\star : [0,n] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\rho \in E : |f(t) - \rho| \leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}}\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{z}_t + x_t)$ is measurable (in [16], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{z}_t + x_t)$ and using (A_1), we obtain for each $t \in [0,n]$

$$\begin{aligned}
|f(t) - \overline{f}(t)| &\leq l_n(t)\ \|z_t - \overline{z}_t\|_{\mathcal{B}} \\
&\leq l_n(t)\ [K(t)\ |z(t) - \overline{z}(t)| + M(t)\ \|z_0 - \overline{z}_0\|_{\mathcal{B}}] \\
&\leq l_n(t)\ K_n\ |z(t) - \overline{z}(t)|.
\end{aligned} \tag{12}$$

Let us define, for each $t \in [0,n]$

$$\overline{h}(t) = g(t, \overline{z}_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, \overline{z}_s + x_s)ds + \int_0^t U(t,s)\overline{f}(s)ds$$

Then, for each $t \in [0,n]$ and $n \in \mathbb{N}$ and using $(H1)$ and $(H3)$ to $(H6)$, we get

$$\begin{aligned}
|h(t)-\overline{h}(t)| &\leq |g(t,z_t+x_t)-g(t,\overline{z}_t+x_t)| \\
&+ \int_0^t |U(t,s)A(s)[g(s,z_s+x_s)-g(s,\overline{z}_s+x_s)]|\,ds \\
&+ \int_0^t \left|U(t,s)[f(s)-\overline{f}(s)]\right|\,ds \\
&\leq \|A^{-1}(t)\|_{B(E)}\,|A(t)g(t,z_t+x_t)-A(t)g(t,\overline{z}_t+x_t)| \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s,z_s+x_s)-A(s)g(s,\overline{z}_s+x_s)|\,ds \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|f(s)-\overline{f}(s)|\,ds \\
&\leq \overline{M}_0L_\star\|z_t-\overline{z}_t\|_{\mathcal{B}}+\int_0^t \widehat{M}L_\star\|z_s-\overline{z}_s\|_{\mathcal{B}}\,ds+\int_0^t \widehat{M}|f(s)-\overline{f}(s)|\,ds.
\end{aligned}$$

Using (A_1) and (12), we obtain

$$\begin{aligned}
|h(t)-\overline{h}(t)| &\leq \overline{M}_0L_\star K(t)|z(t)-\overline{z}(t)|+\int_0^t \widehat{M}L_\star K(s)|z(s)-\overline{z}(s)|\,ds \\
&+ \int_0^t \widehat{M}l_n(s)K_n|z(s)-\overline{z}(s)|\,ds \\
&\leq \overline{M}_0L_\star K_n|z(t)-\overline{z}(t)|+\int_0^t \widehat{M}K_n[L_\star+l_n(s)]|z(s)-\overline{z}(s)|\,ds \\
&\leq \overline{M}_0L_\star K_n|z(t)-\overline{z}(t)|+\int_0^t \overline{l}_n(s)|z(s)-\overline{z}(s)|\,ds \\
&\leq \overline{M}_0L_\star K_n[e^{\tau\ L_n^*(t)}]\ [e^{-\tau\ L_t^*(t)}\ |z(t)-\overline{z}(t)|] \\
&+ \int_0^t \left[\overline{l}_n(s)\ e^{\tau\ L_n^*(s)}\right]\ \left[e^{-\tau\ L_n^*(s)}\ |z(s)-\overline{z}(s)|\right]\ ds \\
&\leq \overline{M}_0L_\star K_n\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n+\int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]'\ ds\ \|z-\overline{z}\|_n \\
&\leq \overline{M}_0L_\star K_n\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n+\frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n \\
&\leq \left[\overline{M}_0L_\star K_n+\frac{1}{\tau}\right]\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|h-\overline{h}\|_n \leq \left[\overline{M}_0L_\star K_n+\frac{1}{\tau}\right]\|z-\overline{z}\|_n.$$

By an analogous relation, obtained by interchanging the roles of z and $\overline{z}$, it follows that

$$H_d(\widetilde{\mathcal{F}}(z), \widetilde{\mathcal{F}}(\overline{z})) \leq \left[\overline{M}_0 L_\star K_n + \frac{1}{\tau}\right] \|z - \overline{z}\|_{\overline{B}}.$$

So the operator $\widetilde{\mathcal{F}}$ is a contraction for all $n \in \mathbb{N}$ and an admissible operator. From the choice of $\mathcal{U}$ there is no $z \in \partial\mathcal{U}$ such that $z = \lambda\, \widetilde{\mathcal{F}}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.9 does not hold. This implies that the operator $\widetilde{\mathcal{F}}$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem $(3) - (4)$.

5 Applications

To illustrate the previous results, we give in this section two applications:

Example 5.1 *Consider the following model*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) & \in & a(t,\xi)\, \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) \\ & & + \displaystyle\int_{-\infty}^{0} P(\theta) R(t, v(t+\theta,\xi))d\theta \qquad \xi \in [0,\pi] \\ v(t,0) & = & v(t,\pi) \; = 0 \qquad\qquad t \in [0,+\infty) \\ v(\theta,\xi) & = & v_0(\theta,\xi) \qquad\qquad -\infty < \theta \leq 0, \quad \xi \in [0,\pi], \end{cases} \tag{13}$$

where $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t ; $P : (-\infty,0] \to \mathbb{R}$ and $v_0 : (-\infty,0] \times [0,\pi] \to \mathbb{R}$ are continuous functions and $R : [0,+\infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain

$$D(A) = \{\ w \in E \ :\ w,\ w' \text{ are absolutely continuous},\ w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption $(H1)$ (see [20]).

For the phase space $\mathcal{B}$, we choose the well known space $BUC(\mathbb{R}^-,E)$: the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| \quad \text{for} \quad \varphi \in \mathcal{B}.$$

If we put for $\varphi \in BUC(\mathbb{R}^-,E)$ and $\xi \in [0,\pi]$

$$y(t)(\xi) = v(t,\xi),\ t \in [0,+\infty),\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi)=v_0(\theta,\xi),\ -\infty<\theta\leq 0,\ \xi\in[0,\pi],$$

and

$$F(t,\varphi)(\xi)=\int_{-\infty}^{0}P(\theta)R(t,\varphi(\theta)(\xi))d\theta,\ -\infty<\theta\leq 0,\ \xi\in[0,\pi].$$

Then, the problem (13) takes the abstract semilinear functional evolution inclusion form $(1)-(2)$. *In order to prove the existence of mild solutions of problem (13), we suppose the following assumptions :*

- *There exist* $p\in L^1(J,\mathbb{R}^+)$ *and a nondecreasing continuous function* $\psi:[0,+\infty)\to(0,+\infty)$ *such that*

$$|R(t,u)|\leq p(t)\psi(|u|),\ \textit{for}\ \in J,\ \textit{and}\ u\in\mathbb{R}.$$

- P *is integrable on* $(-\infty,0]$.

By the dominated convergence theorem, one can show that $f\in S_{F,y}$ *is a continuous function from* $\mathcal{B}$ *to* E. *On the other hand, we have for* $\varphi\in\mathcal{B}$ *and* $\xi\in[0,\pi]$

$$|F(t,\varphi)(\xi)|\leq\int_{-\infty}^{0}|p(t)P(\theta)|\,\psi(|(\varphi(\theta))(\xi)|)d\theta.$$

Since the function ψ *is nondecreasing, it follows that*

$$\|F(t,\varphi)\|_{\mathcal{P}(E)}\leq p(t)\int_{-\infty}^{0}|P(\theta)|\,d\theta\psi(|\varphi|),\quad \textit{for}\ \varphi\in\mathcal{B}.$$

Proposition 5.2 *Under the above assumptions, if we assume that condition (6) in Theorem 3.2 is true,* $\varphi\in\mathcal{B}$, *then the problem (13) has a mild solution which is defined in* $(-\infty,+\infty)$.

Example 5.3 *Consider the following model*

$$\begin{cases} \dfrac{\partial}{\partial t}\ \left[v(t,\xi)-\displaystyle\int_{-\infty}^{0}T(\theta)u(t,v(t+\theta,\xi))d\theta\right] & \\ \qquad\in\ a(t,\xi)\ \dfrac{\partial^2 v}{\partial\xi^2}(t,\xi) & \\ \qquad+\displaystyle\int_{-\infty}^{0}P(\theta)R(t,v(t+\theta,\xi))d\theta & t\in[0,+\infty),\quad \xi\in[0,\pi] \\ v(t,0)\ =\ v(t,\pi)\ =0 & t\in[0,+\infty) \\ v(\theta,\xi)\ =\ v_0(\theta,\xi) & -\infty<\theta\leq 0,\quad \xi\in[0,\pi], \end{cases}\tag{14}$$

where $a(t,\xi)$ *is a continuous function and is uniformly Hölder continuous in* t *;* $T, P : (-\infty, 0] \to \mathbb{R}$ *;* $u : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ *and* $v_0 : (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ *are continuous functions and* $R : [0,+\infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map with compact convex values.*

Consider $E = L^2([0,\pi],\mathbb{R})$ *and define* $A(t)$ *by* $A(t)w = a(t,\xi)w''$ *with domain*

$$D(A) = \{\ w \in E\ :\ w,\ w' \text{ are absolutely continuous},\ w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

Then $A(t)$ *generates an evolution system* $U(t,s)$ *satisfying assumption* $(H1)$ *(see [20]).*

For the phase space $\mathcal{B}$*, we choose the well known space* $BUC(\mathbb{R}^-, E)$ *: the space of uniformly bounded continuous functions endowed with the following norm*

$$\|\varphi\| = \sup_{\theta\le 0} |\varphi(\theta)| \quad \text{for} \quad \varphi \in \mathcal{B}.$$

If we put for $\varphi \in BUC(\mathbb{R}^-, E)$ *and* $\xi \in [0,\pi]$

$$y(t)(\xi) = v(t,\xi),\ t \in [0,+\infty),\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = v_0(\theta,\xi),\ -\infty < \theta \le 0,\ \xi \in [0,\pi],$$

$$g(t,\varphi)(\xi) = \int_{-\infty}^{0} T(\theta)u(t,\varphi(\theta)(\xi))d\theta,\ -\infty < \theta \le 0,\ \xi \in [0,\pi],$$

and

$$F(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta)R(t,\varphi(\theta)(\xi))d\theta,\ -\infty < \theta \le 0,\ \xi \in [0,\pi].$$

Then, the problem (14) takes the abstract neutral functional evolution inclusion form $(3)-(4)$*. In order to prove the existence of mild solutions of problem (14), we suppose the following assumptions :*

- u *is Lipschitz with respect to its second argument. Let* $lip(u)$ *denotes the Lipschitz constant of* u*.*

- *There exist* $p \in L^1(J,\mathbb{R}^+)$ *and a nondecreasing continuous function* $\psi : [0,+\infty) \to (0,+\infty)$ *such that*

$$|R(t,x)| \le p(t)\psi(|x|),\ \text{for}\ \in J,\ \text{and}\ x \in \mathbb{R}.$$

- T, P *are integrable on* $(-\infty, 0]$*.*

By the dominated convergence theorem, one can show that $f \in S_{F,y}$ is a continuous function from $\mathcal{B}$ to E. Moreover the mapping g is Lipschitz continuous in its second argument, in fact, we have

$$|g(t,\varphi_1) - g(t,\varphi_2)| \leq \overline{M}_0 L_* lip(u) \int_{-\infty}^{0} |T(\theta)|\, d\theta\, |\varphi_1 - \varphi_2|\,, \;\; \textit{for}\ \varphi_1, \varphi_2 \in \mathcal{B}.$$

On the other hand, we have for $\varphi \in \mathcal{B}$ and $\xi \in [0,\pi]$

$$|F(t,\varphi)(\xi)| \leq \int_{-\infty}^{0} |p(t)P(\theta)|\, \psi(|(\varphi(\theta))(\xi)|)d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$\|F(t,\varphi)\|_{\mathcal{P}(E)} \leq p(t) \int_{-\infty}^{0} |P(\theta)|\, d\theta \psi(|\varphi|), \quad \textit{for}\ \varphi \in \mathcal{B}.$$

Proposition 5.4 *Under the above assumptions, if we assume that condition (10) in Theorem 4.2 is true, $\varphi \in \mathcal{B}$, then the problem (14) has a mild solution which is defined in $(-\infty, +\infty)$.*

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(Received July 5, 2008)