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On the stability of a fractional-order differential equation with nonlocal initial condition

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Abstract

The topic of fractional calculus (integration and differentiation of fractional-order), which concerns singular integral and integro-differential operators, is enjoying interest among mathematicians, physicists and engineers (see [1]-[2] and [5]-[14] and the references therein). In this work, we investigate initial value problem of fractional-order differential equation with nonlocal condition. The stability (and some other properties concerning the existence and uniqueness) of the solution will be proved.

Key words: Fractional calculus; Banach contraction fixed point theorem; Nonlocal condition; Stability.

1 Introduction

Let $L_1[a,b]$ denote the space of all Lebesgue integrable functions on the interval $[a,b]$, $0 \leq a < b < \infty$, with the L_1-norm $||x||_{L_1} = \int_0^1 |x(t)|\ dt$.

Definition 1.1 *The fractional (arbitrary) order integral of the function $f \in L_1[a,b]$ of order $\beta \in R^+$ is defined by (see [11] - [14])*

$$I_a^\beta\ f(t) \ = \ \int_a^t \frac{(t\ -\ s)^{\beta\ -\ 1}}{\Gamma(\beta)}\ f(s)\ ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 *The (Caputo) fractional-order derivative D^α of order $\alpha \in (0,1]$ of the function $g(t)$ is defined as (see [12] - [14])*

$$D_a^\alpha\ g(t) \ = \ I_a^{1\ -\ \alpha}\ \frac{d}{dt}\ g(t), \quad t \ \in \ [a,b].$$

Now the following theorem (some properties of the fractional-order integration and the fractional-order differentiation) can be easily proved.

Theorem 1.1 *Let $\beta,\ \gamma\ \in R^+$ and $\alpha\ \in (0,1]$. Then we have:*

(i) $I_a^\beta\ :\ L_1 \rightarrow L_1$, *and if* $f(t) \in L_1$, *then* $I_a^\gamma\ I_a^\beta\ f(t)\ =\ I_a^{\gamma+\beta}\ f(t)$.

(ii) $\lim_{\beta \rightarrow n} I_a^\beta\ f(t)\ =\ I_a^n\ f(t)\ ,\ n\ =\ 1,\ 2,\ 3,\ \ldots$ *uniformly.*

If $f(t)$ is absolutely continuous on $[a,b]$, then

(iii) $\lim_{\alpha \rightarrow 1} D_a^\alpha\ f(t)\ =\ D\ f(t)$

(iv) If $f(t) = k \neq 0$, *k is a constant, then* $D_a^\alpha\ k\ =\ 0$.

In ([3]) the nonlocal initial value problem for first-order differential inclusions:

$$\begin{cases} x'(t)\ \in\ F(t,x(t)), \quad t\ \in\ (0,1], \\ x(0)\ +\ \sum_{k=1}^m\ a_k\ x(t_k)\ =\ x_0, \end{cases}$$

was studied, where $F\ :\ J \times \Re \rightarrow 2^\Re$ is a set-valued map, $J\ =\ [0,1]$, $x_0\ \in \Re$ is given, $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.

Our objective in this paper is to investigate, by using the Banach contraction fixed point theorem, the existence of a unique solution of the following fractional-order differential equation:

$$D^\alpha\ x(t)\ =\ c(t)\ f(x(t))\ +\ b(t), \tag{1}$$

with the nonlocal condition:

$$x(0)\ +\ \sum_{k=1}^m\ a_k\ x(t_k)\ =\ x_0, \tag{2}$$

where $x_0 \in \Re$ and $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$. Then we will prove that this solution is uniformly stable.

2 Existence of solution

Here the space $C[0,1]$ denotes the space of all continuous functions on the interval $[0,1]$ with the supremum norm $||y|| = \sup_{t\in[0,1]} |y(t)|$.

To facilitate our discussion, let us first state the following assumptions:

(i) $|\frac{\partial f}{\partial x}| \leq k$,

(ii) $c(t)$ is a function which is absolutely continuous,

(iii) $b(t)$ is a function which is absolutely continuous.

Definition 2.1 *By a solution of the initial value Problem (1) - (2) we mean a function* $x \in C[0,1]$ *with* $\frac{dx}{dt} \in L_1[0,1]$.

Theorem 2.1 *If the above assumptions (i) - (iii) are satisfied such that*

$$1 + \sum_{k=1}^{m} a_k \neq 0 \quad \text{and} \quad A < \frac{\Gamma(1+\alpha)}{k\ ||c||},$$

$$\text{where } A = 1 + |a| \sum_{k=1}^{m} |a_k| \quad \text{and} \quad a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1},$$

then the initial value Problem (1) - (2) has a unique solution.

Proof: For simplicity let $c(t)f(x(t)) + b(t) = g(t, x(t))$.
If $x(t)$ satisfies (1) - (2), then by using the definitions and properties of the fractional-order integration and fractional-order differentiation equation (1) can be written as

$$I^{1-\alpha}\, x'(t) = g(t, x(t)).$$

Operating by I^{α} on both sides of the last equation, we obtain

$$x(t) - x(0) = I^{\alpha}\, g(t, x(t)),$$

by substituting for the value of $x(0)$ from (2), we get

$$x(t) = x_0 - \sum_{k=1}^{m} a_k\, x(t_k) + I^{\alpha}\, g(t, x(t)). \tag{3}$$

If we put $t = t_k$ in (3), we obtain

$$x(t_k) = x_0 - \sum_{k=1}^{m} a_k\, x(t_k) + I^{\alpha}\, g(t, x(t))|_{t=t_k}. \tag{4}$$

Then subtract (3) from (4) to get

$$x(t_k) = x(t) - I^{\alpha}\, g(t, x(t)) + I^{\alpha}\, g(t, x(t))|_{t=t_k}. \tag{5}$$

Substitute from (5) in (3), we get

$$\begin{aligned}
x(t) &= x_0 + I^{\alpha}\, g(t, x(t)) \\
&\quad - \sum_{k=1}^{m} a_k \left(x(t) - I^{\alpha}\, g(t, x(t)) + I^{\alpha}\, g(t, x(t))|_{t=t_k} \right) \\
&= x_0 + I^{\alpha}\, g(t, x(t)) \\
&\quad - \sum_{k=1}^{m} a_k\, x(t) + \sum_{k=1}^{m} a_k\, I^{\alpha}\, g(t, x(t)) - \sum_{k=1}^{m} a_k\, I^{\alpha}\, g(t, x(t))|_{t=t_k}, \\
\left(1 + \sum_{k=1}^{m} a_k\right) x(t) &= x_0 - \sum_{k=1}^{m} a_k\, I^{\alpha}\, g(t, x(t))|_{t=t_k} + \left(1 + \sum_{k=1}^{m} a_k\right) I^{\alpha}\, g(t, x(t)), \\
x(t) &= a \left(x_0 - \sum_{k=1}^{m} a_k\, I^{\alpha}\, g(t, x(t))|_{t=t_k} \right) + I^{\alpha}\, g(t, x(t)). \qquad (6)
\end{aligned}$$

Now define the operator $T : C \to C$ by

$$Tx(t) \;=\; a\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \{c(s)f(x(s)) + b(s)\}\ ds\right) + I^\alpha\{c(s)f(x(s)) + b(s)\}. \tag{7}$$

Let $x,\ y\ \in\ C$, then

$$\begin{aligned}
Tx(t) - Ty(t) \;&=\; -\,a\ \sum_{k=1}^{m} a_k\ \int_0^{t_k} \frac{(t_k\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ f(x(s))\ ds \\
&+\ a\ \sum_{k=1}^{m} a_k\ \int_0^{t_k} \frac{(t_k\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ f(y(s))\ ds \\
&+\ \int_0^{t} \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ \{f(x(s))\ -\ f(y(s))\}\ ds \\
&=\; -\,a\ \sum_{k=1}^{m} a_k\ \int_0^{t_k} \frac{(t_k\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ \{f(x(s))\ -\ f(y(s))\}\ ds \\
&+\ \int_0^{t} \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ \{f(x(s))\ -\ f(y(s))\}\ ds, \\
|Tx(t) - Ty(t)| \;&\le\; k\ |a|\ \sum_{k=1}^{m} |a_k|\ \int_0^{t_k} \frac{(t_k\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ |c(s)|\ |x(s)\ -\ y(s)|\ ds \\
&+\ k\ \int_0^{t} \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ |c(s)|\ |x(s)\ -\ y(s)|\ ds \\
&\le\ k\ |a|\ \sum_{k=1}^{m} |a_k|\ \sup_t |c(t)|\ \sup_t |x(t)\ -\ y(t)|\ \int_0^{t_k} \frac{(t_k\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ ds \\
&+\ k\ \sup_t |c(t)|\ \sup_t |x(t)\ -\ y(t)|\ \int_0^{t} \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ ds \\
&\le\ k\ |a|\ \sum_{k=1}^{m} |a_k|\ \|c\|\ \|x\ -\ y\|\ \frac{t_k^\alpha}{\Gamma(1+\alpha)} \\
&+\ k\ \|c\|\ \|x\ -\ y\|\ \frac{t^\alpha}{\Gamma(1+\alpha)} \\
&\le\ \frac{k}{\Gamma(1+\alpha)}\ \left(1\ +\ |a|\ \sum_{k=1}^{m} |a_k|\right)\ \|c\|\ \|x\ -\ y\| \\
&\le\ \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\ \|x\ -\ y\|\ =\ K||x\ -\ y||.
\end{aligned}$$

but since $K = \frac{kA\|c\|}{\Gamma(1+\alpha)} < 1$, then we get

$$||Tx\ -\ Ty||\ <\ K\ ||x\ -\ y||,$$

which proves that the map $T : C \to C$ is contraction. Applying the Banach contraction fixed point theorem we deduce that (7) has a unique fixed point $x \in C[0, 1]$.

Now, differentiate (6) to obtain

$$\begin{aligned}
x'(t) &= \frac{d}{dt} I^{\alpha} \ (\ c(t) \ f(x(t)) \ + \ b(t) \ ) \\
&= (c(t) \ f(x(t)) \ + \ b(t))|_{t=0} \ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \ + \ I^{\alpha} \ \frac{d}{dt} \ (c(t) \ f(x(t)) \ + \ b(t)) \\
&= K_1 \ \frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha} \ (c'(t) \ f(x(t)) \ + \ \frac{\partial f}{\partial x} \ x'(t) \ c(t) \ + \ b'(t)),
\end{aligned}$$

$$\begin{aligned}
\int_0^1 |x'(t)| \ dt &\leq \frac{K_1}{\Gamma(1+\alpha)} \ t^{\alpha}|_0^1 \\
&+ \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| c'(s) \ f(x(s)) \ + \ \frac{\partial f}{\partial x} \ x'(s) \ c(s) \ + \ b'(s) \right| ds \ dt \\
&= \frac{K_1}{\Gamma(1+\alpha)} + \int_0^1 \left| c'(s) \ f(x(s)) + \frac{\partial f}{\partial x} \ x'(s) \ c(s) + b'(s) \right| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ dt \ ds \\
&\leq \frac{K_1}{\Gamma(1+\alpha)} \ + \ \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| c'(s) \ f(x(s)) \ + \ \frac{\partial f}{\partial x} \ x'(s) \ c(s) \ + \ b'(s) \right| ds, \\
\|x'\|_{L_1} &\leq \frac{K_1}{\Gamma(1+\alpha)} \ + \ \frac{1}{\Gamma(1+\alpha)} \ (\|c'\|_{L_1} \ \|f\| \ + \ k \ \|x'\|_{L_1} \ \|c\| \ + \ \|b'\|_{L_1}) \, , \\
\left(1 - \frac{k\|c\|}{\Gamma(1+\alpha)}\right) \|x'\|_{L_1} &\leq \frac{K_1}{\Gamma(1+\alpha)} \ + \ \frac{1}{\Gamma(1+\alpha)} \ (\|c'\|_{L_1} \ \|f\| \ + \ \|b'\|_{L_1}) \, , \\
\|x'\|_{L_1} &\leq \left(1 - \frac{k\|c\|}{\Gamma(1+\alpha)}\right)^{-1} \left(\frac{K_1}{\Gamma(1+\alpha)} \ + \ \frac{1}{\Gamma(1+\alpha)} \ (\|c'\|_{L_1} \ \|f\| \ + \ \|b'\|_{L_1}) \right).
\end{aligned}$$

Therefore we obtain that $x' \in L_1[0,1]$.
To complete the equivalence of equation (6) with the initial value problem (1) - (2), let $x(t)$ be a solution of (6), differentiate both sides, and get

$$\begin{aligned}
x'(t) &= \frac{d}{dt} \ I^{\alpha} \ g(t, x(t)) \\
&= g(t, x(t))|_{t=0} \ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \ + \ I^{\alpha} \ \frac{d}{dt} \ g(t, x(t)).
\end{aligned}$$

Then operate by $I^{1-\alpha}$ on both sides to obtain

$$D^{\alpha} \ x(t) \ = \ g(t, x(t)).$$

And if $t = 0$ we find that the nonlocal condition (2) is satisfied. Which proves the equivalence. ■

3 Stability

In this section we study the uniform stability (see [1], [4] and [6]) of the solution of the initial-value problem (1) - (2).

Theorem 3.1 *The solution of the initial-value problem (1) - (2) is uniformly stable*

Proof: Let $x(t)$ be a solution of

$$x(t) \;=\; a\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\{c(s)f(x(s))+b(s)\}\ ds\right) + I^{\alpha}\{c(s)f(x(s))+b(s)\} \tag{8}$$

and let $\widetilde{x}(t)$ be a solution of equation (8) such that $\widetilde{x}(0)=\widetilde{x}_0-\sum_{k=1}^{m} a_k\ \widetilde{x}(t_k)$. Then

$$\begin{aligned}
x(t)-\widetilde{x}(t) \;&=\; a\ (x_0 - \widetilde{x}_0) - a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ f(x(s))\ ds \\
&+\; a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ f(\widetilde{x}(s))\ ds \\
&+\; \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}\ c(s)\ \{f(x(s)) - f(\widetilde{x}(s))\}\ ds, \\
|x(t) - \widetilde{x}(t)| \;&\leq\; |a|\ |x_0 - \widetilde{x}_0| \\
&+\; |a| \sum_{k=1}^{m} |a_k| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}\ |c(s)|\ |f(x(s)) - f(\widetilde{x}(s))|\ ds \\
&+\; \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}\ |c(s)|\ |f(x(s)) - f(\widetilde{x}(s))|\ ds \\
&\leq\; |a|\ |x_0 - \widetilde{x}_0| \\
&+\; k\ |a| \sum_{k=1}^{m} |a_k|\ \sup_t |c(t)| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}\ |(x(s) - \widetilde{x}(s)|\ ds \\
&+\; k\ \sup_t |c(t)| \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}\ |x(s) - \widetilde{x}(s)|\ ds \\
&\leq\; |a|\ |x_0 - \widetilde{x}_0| \\
&+\; k\ |a|\ \|c\| \sum_{k=1}^{m} |a_k|\ \sup_t |x(t) - \widetilde{x}(t)| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}\ ds \\
&+\; k\ \|c\|\ \sup_t |x(t) - \widetilde{x}(t)| \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}\ ds, \\
\|x - \widetilde{x}\| \;&\leq\; |a|\ |x_0 - \widetilde{x}_0| + k\ |a|\ \|c\| \sum_{k=1}^{m} |a_k|\ \|x - \widetilde{x}\| \frac{t_k^{\alpha}}{\Gamma(1+\alpha)} \\
&+\; k\ \|c\|\ \|x - \widetilde{x}\| \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&\leq\; |a|\ |x_0 - \widetilde{x}_0| + \frac{k\ \|c\|}{\Gamma(1+\alpha)}\left(1 + |a| \sum_{k=1}^{m} |a_k|\right) \|x - \widetilde{x}\|
\end{aligned}$$

$$
\begin{aligned}
&= |a|\,|x_0 - \widetilde{x}_0| + \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\ \|x - \widetilde{x}\|, \\
\left(1 - \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\right) ||x - \widetilde{x}|| &\leq |a|\,|x_0 - \widetilde{x}_0|, \\
||x - \widetilde{x}|| &\leq \left(1 - \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\right)^{-1} |a|\,|x_0 - \widetilde{x}_0|.
\end{aligned}
$$

Therefore, if $|\ x_0 - \widetilde{x}_0\ | < \delta(\varepsilon)$, then $||\ x - \widetilde{x}\ || < \varepsilon$, which complete the proof of the theorem. ■

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