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Three Positive Solutions for p-Laplacian Functional Dynamic Equations on Time Scales

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Abstract

In this paper, existence criteria of three positive solutions to the followimg p-Laplacian functional dynamic equation on time scales

$$\begin{cases} \left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + a(t)f(u(t), u(\mu(t))) = 0,\ t \in (0,T), \\ u_0(t) = \varphi(t),\ t \in [-r, 0],\ u(0) - B_0(u^{\triangle}(\eta)) = 0,\ u^{\triangle}(T) = 0, \end{cases}$$

are established by using the well-known Five Functionals Fixed Point Theorem.

Keywords: Time scale; p-Laplacian functional dynamic equation; Boundary value problem; Positive solution; fixed point theorem
MSC: 39K10, 34B15

1 Introduction

The theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [1-5]) since it was initiated by Hilger [6]. At the same time, boundary value problems (BVPs) for dynamic equation on time scales have received considerable attention [7-15]. However, to the best of our knowledge, few papers can be found in the literature for BVPs of p-Laplacian dynamic equations on time scales, especially for p-Laplacian functional dynamic equations on time scales [12, 14].

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let 0, T be points in $\mathbf{T}$, an interval $(0, T)$ denote time scales interval, that is, $(0, T) := (0, T) \cap \mathbf{T}$. Other types of intervals are defined similarly. Some definitions concerning time scales can be found in [2-4].

In this paper, we are concerned with the existence of positive solutions for the p-Laplacian functional dynamic equation on time scale

$$\begin{cases} \left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + a(t)f(u(t), u(\mu(t))) = 0,\ t \in (0,T), \\ u_0(t) = \varphi(t),\ t \in [-r, 0],\ u(0) - B_0(u^{\triangle}(\eta)) = 0,\ u^{\triangle}(T) = 0, \end{cases} \tag{1.1}$$

where $\Phi_p(s)$ is p-Laplacian operator, i.e., $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in (0, \rho(T))$ and

(C_1) $f : (R^+)^2 \to R^+$ is continuous ;

(C_2) $a : \mathbf{T} \to R^+$ is left dense continuous (i.e., $a \in C_{\mathbf{ld}}(\mathbf{T}, R^+)$) and does not vanish identically on any closed subinterval of $[0, T]$, where $C_{\mathbf{ld}}(\mathbf{T}, R^+)$ denotes the set of all left dense continuous functions from $\mathbf{T}$ to R^+;

(C_3) $\varphi : [-r, 0] \to R^+$ is continuous and $r > 0$;
(C_4) $\mu : [0, T] \to [-r, T]$ is continuous, $\mu(t) \le 0$ for all t;
(C_5) $B_0 : R \to R$ is continuous and satisfies the condition that there are $A \ge B \ge 0$ such that

$$Bv \le B_0(v) \le Av, \text{ for all } v \ge 0.$$

We note that in [16], Li and Shen studied the problem (1.1) when $\mathbf{T} = R$, $\varphi(t) = 0, t \in [-r, 0]$ and the nonlinear term is not involved $u(\mu(t))$. They imposed conditions on f to yield at least three positive solutions to the problem (1.1), by applying the Five Functionals Fixed Point Theorem [17] (which is a generalization of the Leggett-Williams Fixed-Point Theorem [18]).

In [14], Song and Xiao considered the problem (1.1), by using a double fixed-point theorem due to Avery et al. [19] in a cone, and obtained the existence of two positive solutions.

Motivated by [14] and [16], we shall show that the problem (1.1), has at least three positive solutions by means of the Five Functionals Fixed Point Theorem.

Let γ, β, θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P. Then, for nonnegative real numbers h, a, b, d and c, we define the convex sets

$$\begin{aligned}
P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\
P(\gamma, \alpha, a, c) &= \{x \in P : a \le \alpha(x),\ \gamma(x) \le c\}, \\
Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \le d,\ \gamma(x) \le c\}, \\
P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \le \alpha(x),\ \theta(x) \le b,\ \gamma(x) \le c\} \text{ and} \\
Q(\gamma, \beta, \psi, h, d, c) &= \{x \in P : h \le \psi(x),\ \beta(x) \le d,\ \gamma(x) \le c\}.
\end{aligned}$$

To prove our main results, we need the following Five Functionals Fixed Point Theorem[17].

Theorem 1.1. Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative, continuous, concave functionals α and ψ on P, and nonnegative, continuous, convex functionals γ, β and θ on P, with

$$\alpha(x) \le \beta(x) \text{ and } \|x\| \ \le M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$F : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that:

(i) $\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \ne \phi$ and $\alpha(Fx) > b$ for $x \in P(\gamma, \theta, \alpha, b, k, c)$;
(ii) $\{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \ne \phi$ and $\beta(Fx) < a$ for $x \in Q(\gamma, \beta, \psi, h, a, c)$;
(iii) $\alpha(Fx) > b$ for $x \in P(\gamma, \alpha, b, c)$ with $\theta(Fx) > k$;
(iv) $\beta(Fx) < a$ for $x \in Q(\gamma, \beta, a, c)$ with $\psi(Fx) < h$.

Then F has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < a,\ b < \alpha(x_2), \text{ and } a < \beta(x_3) \text{ with } \alpha(x_3) < b.$$

2 Existence of Three Positive Solutions

We note that $u(t)$ is a solution of the BVP (1.1) if and only if

$$u(t)=\begin{cases} B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right) & \\ \quad+\int_0^t \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s, & t\in[0,T],\\ \varphi(t), & t\in[-r,0].\end{cases}$$

Let $E=C_{\mathbf{ld}}([0,T],R)$ be endowed with $\|u\|=\sup_{t\in[0,T]}|u(t)|$, so E is a Banach space. Define cone $P\subset E$ by

$$P=\left\{u\in E: u \text{ is concave and nonnegative valued on } [0,T],\text{ and } u^{\triangle}(T)=0\right\}.$$

For each $u\in E$, extend $u(t)$ to $[-r,T]$ with $u(t)=\varphi(t)$ for $t\in[-r,0]$.
Define F: $P\to E$ by

$$\begin{aligned}(Fu)(t)= &\ B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right)\\ &+\int_0^t \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s,\ t\in[0,T].\end{aligned}$$

We seek a point, u_1, of F in the cone P. Define

$$u(t)=\begin{cases} u_1(t), & t\in[0,T],\\ \varphi(t), & t\in[-r,0].\end{cases}$$

Then $u(t)$ denotes a positive solution of the BVP (1.1).
We have the following results

Lemma 2.1. Let $u\in P$, then

(1) $u(t)\geq \frac{t}{T}\|u\|$ for $t\in[0,T]$, and
(2) $\tau u(\varsigma)\leq \varsigma u(\tau)$ for $0<\tau<\varsigma<T$ and $\tau,\varsigma\in\mathbf{T}$.

Proof. (1) is Lemma 3.1 of [12]. It is easy to conclude that (2) is satisfied by the concavity of u.

Set

$$Y_1=\{t\in[0,T]:\mu(t)<0\};\ Y_2=\{t\in[0,T]:\mu(t)\geq 0\};\ Y_3=Y_1\cap[l,T].$$

Throughout this paper, we assume $Y_3\neq\phi$ and $\int_{Y_3}a(r)\nabla r>0$.
Let $l\in\mathbf{T}$ be fixed such that $0<\eta<l<T$, and define the nonnegative, continuous, concave functionals α, ψ and the nonnegative, continuous, convex functionals β, θ, γ on the cone P respectively as

$$\begin{aligned}\gamma(u)&=\theta(u)=\max_{t\in[0,\eta]}u(t)=u(\eta),\ \alpha(u)=\min_{t\in[l,T]}u(t)=u(l),\\ \beta(u)&=\max_{t\in[0,l]}u(t)=u(l),\ \psi(u)=\min_{t\in[\eta,T]}u(t)=u(\eta).\end{aligned}$$

We observe that $\alpha(u) = \beta(u)$ for each $u \in P$.
In addition, by Lemma 2.1, we have $\gamma(u) = u(\eta) \geq \frac{\eta}{T}\|u\|$. Hence $\|u\| \leq \frac{T}{\eta}\gamma(u)$ for all $u \in P$.
For convenience, we define

$$\mu = (A+\eta)\Phi_q\left(\int_0^T a(r)\nabla r\right), \delta = (B+l)\Phi_q\left(\int_{Y_3} a(r)\nabla r\right),$$

$$\lambda = (A+l)\Phi_q(\int_0^T a(r)\nabla r).$$

We now state growth conditions on f so that the BVP (1.1) has at least three positive solutions.

Theorem 2.1. Let $0 < a < \frac{l}{T}b < \frac{\eta l}{T^2}c$, $\mu b < \delta c$, and suppose that f satisfies the following conditions:

(H_1) $f(u,\varphi(s)) < \Phi_p\left(\frac{c}{\mu}\right)$, if $0 \leq u \leq \frac{T}{\eta}c$, uniformly in $s \in [-r,0]$, and $f(u_1,u_2) < \Phi_p(\frac{c}{\mu})$, if $0 \leq u_i \leq \frac{T}{\eta}c, i = 1,2$;

(H_2) $f(u,\varphi(s)) > \Phi_p\left(\frac{b}{\delta}\right)$, if $b \leq u \leq \left(\frac{T}{\eta}\right)^2 b$, uniformly in $s \in [-r,0]$;

(H_3) $f(u,\varphi(s)) < \Phi_p\left(\frac{a}{\lambda}\right)$, if $0 \leq u \leq \frac{T}{l}a$, uniformly in $s \in [-r,0]$, and $f(u_1,u_2) < \Phi_p(\frac{a}{\lambda})$, if $0 \leq u_i \leq \frac{T}{l}a, i = 1,2$.

Then the BVP (1.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0,T], \ i = 1,2,3, \\ \varphi(t), & t \in [-r,0], \end{cases}$$

where $\max_{t\in[0,l]} u_1(t) < a, \min_{t\in[l,T]} u_2(t) > b$, and $a < \max_{t\in[0,l]} u_3(t)$ with $\min_{t\in[l,T]} u_3(t) < b$.

Proof. By [12], it is known that $F : P \to P$ is completely continuous.
Let $u \in \overline{P(\gamma,c)}$, then $\gamma(u) = \max_{t\in[0,\eta]} u(t) = u(\eta) \leq c$, consequently, $0 \leq u(t) \leq c$ for $t \in [0,\eta]$. Since $u(\eta) \geq \frac{\eta}{T}u(T)$, so $\|u\| = u(T) \leq \frac{T}{\eta}u(\eta) \leq \frac{T}{\eta}c$, this implies

$$0 \leq u(t) \leq \frac{T}{\eta}c, \text{ for } t \in [0,T].$$

From (H_1), we have

$$\begin{aligned}
\gamma(Fu) &= (Fu)(\eta) \\
&= B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right) + \int_0^\eta \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s \\
&\leq A\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right) + \eta\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&= (A+\eta)\Phi_q\left[\int_{Y_1} a(r)f(u(r),\varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)fu(r),u(\mu(r)))\nabla r\right] \\
&< (A+\eta)\Phi_q\left(\int_0^T a(r)\nabla r\right)\frac{c}{\mu} \\
&= c.
\end{aligned}$$

Therefore

$$Fu \in \overline{P(\gamma, c)}.$$

We now turn to property (i) of Theorem 1.1. Choosing $u \equiv \frac{T}{\eta}b$, $k = \frac{T}{\eta}b$, it follows that

$$\alpha(u) = u(l) = \frac{T}{\eta}b > b,\ \theta(u) = u(\eta) = \frac{T}{\eta}b = k,\ \gamma(u) = u(\eta) = \frac{T}{\eta}b < c,$$

which shows that $\{u \in P(\gamma, \theta, \alpha, b, k, c) :\ \alpha(u) > b\} \neq \phi$, and for $u \in P(\gamma, \theta, \alpha, b, \frac{T}{\eta}b, c)$, we have

$$b \leq u(t) \leq \left(\frac{T}{\eta}\right)^2 b,\ \text{for } t \in [l, T].$$

From (H_2), we have

$$\begin{aligned}
\alpha(Fu) &= (Fu)(l) \\
&= B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r\right)\right) + \int_0^l \Phi_q\left(\int_s^T a(r)f(u(r), u(\mu(r)))\nabla r\right)\triangle s \\
&\geq B\Phi_q\left(\int_l^T a(r)f(u(r), u(\mu(r)))\nabla r\right) + l\Phi_q\left(\int_l^T a(r)f(u(r), u(\mu(r)))\nabla r\right) \\
&\geq (B + l)\Phi_q\left(\int_{Y_3} a(r)f(u(r), \varphi(\mu(r)))\nabla r\right) \\
&> (B + l)\Phi_q\left(\int_{Y_3} a(r)\nabla r\right)\frac{b}{\delta} \\
&= b.
\end{aligned}$$

We conclude that (i) of Theorem 1.1 is satisfied.

We next address (ii) of Theorem 1.1. If we take $u \equiv \frac{\eta}{T}a$, $h = \frac{\eta}{T}a$, then

$$\gamma(u) = u(\eta) = \frac{\eta}{T}a < c,\ \psi(u) = u(\eta) = \frac{\eta}{T}a = h,\ \beta(u) = u(l) = \frac{\eta}{T}a < a.$$

From this we know that $\{u \in Q(\gamma, \beta, \psi, h, a, c) : \beta(u) < a\} \neq \phi$. If $u \in Q(\gamma, \beta, \psi, \frac{\eta}{T}a, a, c)$, then

$$0 \leq u(t) \leq \frac{T}{l}a,\ \text{for } t \in [0, T].$$

From (H_3), we have

$$\begin{aligned}
\beta(Fu) &= (Fu)(l) \\
&= B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r\right)\right) + \int_0^l \Phi_q\left(\int_s^T a(r)f(u(r), u(\mu(r)))\nabla r\right)\triangle s \\
&\leq A\Phi_q\left(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r\right) + l\Phi_q\left(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r\right) \\
&= (A + l)\Phi_q\left[\int_{Y_1} a(r)f(u(r), \varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r)))\nabla r\right] \\
&< (A + l)\Phi_q\left(\int_0^T a(r)\nabla r\right)\frac{a}{\lambda} \\
&= a.
\end{aligned}$$

Now we show that (iii) of Theorem 1.1 is satisfied. If $u \in P(\gamma, \alpha, b, c)$ and $\theta(Fu) = Fu(\eta) > \frac{T}{\eta}b$, then

$$\alpha(Fu) \geq (Fu)(l) = \frac{l}{T}Fu(l) \geq \frac{l}{T}Fu(\eta) > \frac{l}{\eta}b > b.$$

Finally, if $u \in Q(\alpha, \beta, a, c)$ and $\psi(Fu) = Fu(\eta) < \frac{\eta}{T}a$, then from (2) of the Lemma 2.1 we have

$$\beta(Fu) = Fu(l) \leq \frac{T}{l}Fu(l) \leq \frac{T}{\eta}Fu(\eta) < a.$$

which shows that condition (iv) of Theorem 1.1 is fulfilled.

Thus, all the conditions of Theorem 1.1 are satisfied. Hence, F has at least three fixed points u_1, u_2, u_3 satisfying

$$\beta(u_1) < a, \;\; b < \alpha(u_2), \;\text{ and } a < \beta(u_3) \text{ with } \alpha(u_3) < b.$$

Let

$$u(t) = \begin{cases} u_i(t), & t \in [0, T]\,, \;\; i = 1, 2, 3, \\ \varphi(t), & t \in [-r, 0]\,, \end{cases}$$

which are three positive solutions of the BVP (1.1).

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