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EXISTENCE OF $\Psi-$ BOUNDED SOLUTIONS FOR LINEAR DIFFERENCE EQUATIONS ON $\mathbb{Z}$

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Abstract

In this paper[1], we give a necessary and sufficient condition for the existence of $\Psi-$ bounded solutions for the nonhomogeneous linear difference equation $x(n + 1) = A(n)x(n) + f(n)$ on $\mathbb{Z}$. In addition, we give a result in connection with the asymptotic behavior of the $\Psi-$ bounded solutions of this equation.

1. Introduction

The problem of boundedness of the solutions for the system of ordinary differential equations $x' = A(t)x + f(t)$ was studied by Coppel in [2]. In [3], [4], [5], the author proposes a novel concept, $\Psi-$ boundedness of solutions (Ψ being a matrix function), which is interesting and useful in some practical cases and presents the existence condition for such solutions. Also, in [1], the author associates this problem with the concept of $\Psi-$ dichotomy on $\mathbb{R}$ of the system $x' = A(t)x$.

Naturally, one wonders whether there are any similar concepts and results on the solutions of difference equations, which can be seen as the discrete version of differential equations.

In [7], the authors extend the concept of $\Psi-$ boundedness to the solutions of difference equation

$$x(n + 1) = A(n)x(n) + f(n) \tag{1}$$

(via $\Psi-$ bounded sequence) and establish a necessary and sufficient condition for existence of $\Psi-$ bounded solutions for the nonhomogeneous linear difference equation (1) in case f is a $\Psi-$ summable sequence on $\mathbb{N}$.

In [6], the author proved a necessary and sufficient condition for the existence of $\Psi-$ bounded solutions of (1) in case f is a $\Psi-$ bounded sequence on $\mathbb{N}$.

Similarly, we can consider solutions of (1) which are bounded not only $\mathbb{N}$ but on the $\mathbb{Z}$.

In this case, the conditions for the existence of at least one $\Psi-$bounded solution are rather more complicated, as we will see below.

In this paper, we give a necessary and sufficient condition so that the nonhomogeneous linear difference equation (1) have at least one $\Psi-$bounded solution on $\mathbb{Z}$ for every $\Psi-$summable function f on $\mathbb{Z}$

Here, Ψ is a matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

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2. Preliminaries

Let $\mathbb{R}^d$ be the Euclidean d-space. For $x = (x_1, x_2,...,x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, ... |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = (a_{ij})$, the norm $|A|$ is defined by $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. It is well-known that $|A| = \max_{1 \leq i \leq d} \sum_{j=1}^{d} |a_{ij}|$.

Let $\Psi_i : \mathbb{Z} \longrightarrow (0,\infty)$, $i = 1, 2, ...d$ and let the matrix function

$$\Psi = \text{diag}\, [\Psi_1,\Psi_2,...\Psi_d].$$

Then, $\Psi(n)$ is invertible for each $n \in \mathbb{Z}$.

Definition 1. A function $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}^d$ is called $\Psi-$ bounded iff the function $\Psi\varphi$ is bounded (i.e. there exists $M > 0$ such that $\| \Psi(n)\varphi(n)\| \leq M$ for all $n \in \mathbb{Z}$).

Definition 2. A function $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}^d$ is called $\Psi-$ summable on $\mathbb{Z}$ if $\sum_{n=-\infty}^{\infty} \| \Psi(n)\varphi(n)\|$ is convergent (i.e. $\lim_{\substack{p\to-\infty \\ q\to+\infty}} \sum_{n=p}^{q} \| \Psi(n)\varphi(n)\|$ is finite).

Consider the nonautonomous difference linear equation

$$y(n + 1) = A(n)y(n) \tag{2}$$

where the $d \times d$ real matrix $A(n)$ is invertible at $n \in \mathbb{Z}$. Let Y be the fundamental matrix of (2) with $Y(0) = I_d$ (identity $d \times d$ matrix). It is well-known that

i). $Y(n) = \begin{cases} A(n-1)A(n-2)\cdots A(1)A(0), & n > 0 \\ I_d, & n = 0 \\ [A(-1)A(-2)\cdots A(n)]^{-1}, & n < 0 \end{cases}$,

ii). $Y(n + 1) = A(n)Y(n)$ for all $n \in \mathbb{Z}$

iii). the solution of (2) with the initial condition $y(0) = y_0$ is

$$y(n) = Y(n)y_0,\ n \in \mathbb{Z};$$

iv). Y is invertible for each $n \in \mathbb{Z}$ and

$$Y^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\cdots A^{-1}(n-1), & n > 0 \\ I_d, & n = 0 \\ A(-1)A(-2)\cdots A(n), & n < 0 \end{cases}$$

Let the vector space $\mathbb{R}^d$ represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution y of (2) is $\Psi-$ bounded on $\mathbb{Z}$ if and only if $y(0) \in X_0$ and $\Psi-$ bounded on $\mathbb{Z}_+ = \{0,1,2,\cdots\}$ if and only if $y(0) \in X_- \oplus X_0$. Also let P_-, P_0, P_+ denote the corresponding projection of $\mathbb{R}^d$ onto X_-, X_0, X_+ respectively.

3. Main result

The main result of this paper is the following.

Theorem 1. The equation (1) has at least one $\Psi-$ bounded solution on $\mathbb{Z}$ for every $\Psi-$ summable function f on $\mathbb{Z}$ if and only if there is a positive constant K such that

$$\begin{cases} \mid \Psi(n)Y(n)P\_Y^{-1}(k+1)\Psi^{-1}(k) \mid & \leq K, \quad k+1 \leq \min\{0,n\} \\ \mid \Psi(n)Y(n)(P_0+P_+)Y^{-1}(k+1)\Psi^{-1}(k) \mid & \leq K, \quad n < k+1 \leq 0 \\ \mid \Psi(n)Y(n)(P_0+P_-)Y^{-1}(k+1)\Psi^{-1}(k) \mid & \leq K, \quad 0 < k+1 \leq n \\ \mid \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k) \mid & \leq K, \quad k+1 > \max\{0,n\} \end{cases} \tag{3}$$

Proof. First, we prove the "only if" part. We define the sets:

$B_\Psi = \{x : \mathbb{Z} \longrightarrow \mathbb{R}^d \mid x \text{ is } \Psi- \text{ bounded}\}$,

$B = \{x : \mathbb{Z} \longrightarrow \mathbb{R}^d \mid x \text{ is } \Psi- \text{ summable on } \mathbb{Z}\}$,

$D = \{x : \mathbb{Z} \longrightarrow \mathbb{R}^d \mid x \in B_\Psi,\ x(0) \in X_- \oplus X_+,\ (x(n+1)-A(n)x(n)) \in B\}$

Obviously, B_Ψ, B and D are vector spaces over $\mathbb{R}$ and the functionals

$x \longmapsto \|x\|_{B_\Psi} = \sup_{n\in\mathbb{Z}} \|\Psi(n)x(n)\|$,

$x \longmapsto \|x\|_B = \sum_{n=-\infty}^{\infty} \|\Psi(n)x(n)\|$,

$x \longmapsto \|x\|_D = \|x\|_{B_\Psi} + \|x(n+1) - A(n)x(n)\|_B$

are norms on B_Ψ, B and D respectively.

Step 1. It is a simple exercise that $(B_\Psi, \|\cdot\|_{B_\Psi})$ and $(B, \|\cdot\|_B)$ are Banach spaces.

Step 2. $(D, \|\cdot\|_D)$ is a Banach space.

Let $(x_p)_{p\in\mathbb{N}}$ be a fundamental sequence in D. Then, $(x_p)_{p\in\mathbb{N}}$ is a fundamental sequence in B_Ψ. Therefore, there exists a $\Psi-$ bounded function $x : \mathbb{Z} \longrightarrow \mathbb{R}^d$ such that $\lim_{p\to\infty} \Psi(n)x_p(n) = \Psi(n)x(n)$, uniformly on $\mathbb{Z}$. From

$$\|x_p(n) - x(n)\| \leq |\Psi^{-1}(n)| \|\Psi(n)(x_p(n) - x(n))\|,$$

it follows that the sequence $(x_p)_{p\in\mathbb{N}}$ is almost uniformly convergent to function x on $\mathbb{Z}$. Because $x_p(0) \in X_- \oplus X_+$, $p \in \mathbb{N}$, it follows that $x(0) \in X_- \oplus X_+$.

On the other hand, the sequence $(f_p)_{p\in\mathbb{N}}$, $f_p(n) = x_p(n+1) - A(n)x_p(n)$, $n \in \mathbb{Z}$, is a fundamental sequence in B. Therefore, there exists a function $f \in B$ such that

$$\sum_{n=-\infty}^{\infty} \|\Psi(n)f_p(n) - \Psi(n)f(n)\| \longrightarrow 0 \text{ as } p \longrightarrow \infty.$$

It follows that $\Psi(n)f_p(n) \longrightarrow \Psi(n)f(n)$ and $f_p(n) \longrightarrow f(n)$ for each $n \in \mathbb{Z}$.

For a fixed but arbitrary $n \in \mathbb{Z}$, $n > 0$, we have

$$x(n+1) - x(0) = \lim_{p\to\infty} [x_p(n+1) - x_p(0)] =$$

$$= \lim_{p\to\infty} \sum_{i=0}^{n} [x_p(i+1) - x_p(i)] =$$

$$= \lim_{p\to\infty} \sum_{i=0}^{n} [x_p(i+1) - A(i)x_p(i) + A(i)x_p(i) - x_p(i)] =$$

$$= \lim_{p\to\infty} \sum_{i=0}^{n} [f_p(i) - f(i) + f(i) + A(i)x_p(i) - x_p(i)] =$$

$$= \sum_{i=0}^{n} [f(i) + A(i)x(i) - x(i)] =$$
$$= \sum_{i=0}^{n-1} [f(i) + A(i)x(i) - x(i)] + f(n) + A(n)x(n) - x(n) =$$
$$= x(n) - x(0) + f(n) + A(n)x(n) - x(n) = A(n)x(n) + f(n) - x(0).$$

Similarly, we have

$$x(1) - x(0) = A(0)x(0) + f(0) - x(0)$$

and, for $n \in \mathbb{Z}$, $n < 0$,

$$x(n) - x(0) = \lim_{p\to\infty} [x_p(n) - x_p(0)] = \lim_{p\to\infty} \sum_{i=n}^{-1} [x_p(i) - x_p(i+1)] =$$
$$= \lim_{p\to\infty} \sum_{i=n}^{-1} [x_p(i) - A(i)x_p(i) + A(i)x_p(i) - x_p(i+1)] =$$
$$= \lim_{p\to\infty} \sum_{i=n}^{-1} [x_p(i) - A(i)x_p(i) - f_p(i)] =$$
$$= \sum_{i=n}^{-1} [x(i) - A(i)x(i) - f(i)] =$$
$$= \sum_{i=n+1}^{-1} [x(i) - A(i)x(i) - f(i)] + x(n) - A(n)x(n) - f(n) =$$
$$= x(n+1) - x(0) + x(n) - A(n)x(n) - f(n).$$

By the above relations, we have that

$$x(n+1) - A(n)x(n) = f(n),\ n \in \mathbb{Z}.$$

It follows that $x \in D$.

Now, from the relations

$$\sum_{n=-\infty}^{\infty} \| \Psi(n)(x_p - x)(n+1) - \Psi(n)A(n)(x_p - x)(n)\| \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$
$$\|x_p - x\|_{B_\Psi} \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$

it follows that $\|x_p - x\|_D \longrightarrow 0$ as $p \longrightarrow +\infty$.

Thus, $(D, \|\cdot\|_D)$ is a Banach space.

Step 3. There exists a positive constant K such that, for every $f \in B$ and for corresponding solution $x \in D$ of (1), we have

$$\|x\|_{B_\Psi} \leq K\cdot \|f\|_B. \tag{4}$$

We define the operator $T : D \longrightarrow B$, $(Tx)(n) = x(n+1) - A(n)x(n)$, $n \in \mathbb{Z}$.

Clearly, T is linear and bounded, with $\|T\| \leq 1$. Let $Tx = 0$ be. Then, $x \in D$ and $x(n+1) = A(n)x(n)$. This shows that x is a $\Psi-$ bounded solution of (2) with $x(0) \in X_- \oplus X_+$. From the Definition of X_0, we have $x(0) \in X_0$. Thus, $x(0) \in X_0 \cap (X_- \oplus X_+) = \{0\}$. It follows that $x = 0$. This means that the operator T is one-to-one.

Now, for $f \in B$, let x be a $\Psi-$ bounded solution of the equation (1). Let z be the solution of the Cauchy problem

$$z(n+1) = A(n)z(n) + f(n),\ z(0) = (P_- + P_+)x(0).$$

Then, the function $u = x - z$ is a solution of the equation (2) with

$$u(0) = x(0) - z(0) = P_0x(0) \in X_0.$$

It follows that the function u is $\Psi-$ bounded on $\mathbb{Z}$. Thus, the function z is $\Psi-$ bounded on $\mathbb{Z}$. It follows that $z \in D$ and $Tz = f$. Consequently, T is onto.

From a fundamental result of Banach "If T is a bounded one-to-one linear operator from a Banach space onto another, then the inverse operator T^{-1} is also bounded", we have that

$$\|T^{-1}f\|_D \le \|T^{-1}\|\|f\|_B, \text{ for } f \in B.$$

Denoting $T^{-1}f = x$, we have $\|x\|_D = \|x\|_{B_\Psi} + \|f\|_B \le \|T^{-1}\|\|f\|_B$ and then

$$\|x\|_{B_\Psi} \le (\|T^{-1}\| - 1)\|f\|_B .$$

Thus, we have (4), where $K = \|T^{-1}\| - 1$.

Step 4. The end of the proof.

For a fixed but arbitrary $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^d$, we consider the function $f : \mathbb{Z} \longrightarrow \mathbb{R}^d$ defined by

$$f(n) = \begin{cases} \Psi^{-1}(n)\xi, & \text{if } n = k \\ 0, & \text{elsewhere} \end{cases} .$$

Obviously, $f \in B$ and $\|f\|_B = \| \xi \|$. The corresponding solution $x \in D$ of (1) is $x(n) = G(n,k+1)f(k)$, where

$$G(n,k) = \begin{cases} Y(n)P_-Y^{-1}(k) & k \le \min\{0,n\} \\ -Y(n)(P_0 + P_+)Y^{-1}(k) & n < k \le 0 \\ Y(n)(P_0 + P_-)Y^{-1}(k) & 0 < k \le n \\ -Y(n)P_+Y^{-1}(k) & k > \max\{0,n\} \end{cases} .$$

Indeed, we prove this in more cases:

Case $k \le -1$. a). for $k + 1 \le n \le 0$,

$x(n+1) = G(n+1,k+1)f(k) = Y(n+1)P_-Y^{-1}(k+1)f(k) =$
$= A(n)Y(n)P_-Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n)$ (because $f(n) = 0$);

b). for $n = k$,

$x(n+1) = G(n+1,k+1)f(k) = Y(n+1)P_-Y^{-1}(k+1)f(k) =$
$= Y(k+1)(I-P_0-P_+)\cdot Y^{-1}(k+1)f(k) = f(k)-A(k)Y(k)(P_0+P_+)Y^{-1}(k+1)f(k) =$
$= f(k) + A(k)G(k,k+1)f(k) = A(n)x(n) + f(n)$;

c). for $n < k$,

$x(n+1) = G(n+1,k+1)f(k) = -Y(n+1)(P_0 + P_+)Y^{-1}(k+1)f(k) =$
$= -A(n)Y(n)(P_0 + P_+)Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n)$;

d). for $n > 0$,

$x(n+1) = G(n+1,k+1)f(k) = Y(n+1)P_-Y^{-1}(k+1)f(k) =$
$= A(n)Y(n)P_-Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n)$;

Case $k > -1$. α). for $n < 0$,

$x(n+1) = G(n+1,k+1)f(k) = -Y(n+1)P_+Y^{-1}(k+1)f(k) =$
$= - A(n)Y(n)P_+Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n)$;

β). for $n = 0$ and $k = 0$,

$x(1) = G(1,1)f(0) = Y(1)(P_0 + P_-)Y^{-1}(1)f(0) = Y(1)(I - P_+)Y^{-1}(1)f(0) =$
$= f(0) - A(0)Y(0)P_+Y^{-1}(1)f(0) = A(0)x(0) + f(0)$;

γ). $n = 0$ and $k > 0$,

$x(1) = G(1,k+1)f(k) = -Y(1)P_+Y^{-1}(k+1)f(k) = -A(0)Y(0)P_+Y^{-1}(k+1)f(k) =$
$= A(0)G(0,k+1)f(k) = A(0)x(0) + f(0)$;

δ). for $0 < n = k$,

$x(n+1) = G(k+1,k+1)f(k) = Y(k+1)(P_0 + P_-)Y^{-1}(k+1)f(k) =$
$= Y(k+1)(I - P_+)Y^{-1}(k+1)f(k) = f(k) - A(k)Y(k)P_+Y^{-1}(k+1)f(k) =$

$= A(n)x(n) + f(n);$

ε). for $0 < n < k$,

$x(n+1) = G(n+1,k+1)f(k) = -Y(n+1)P_+Y^{-1}(k+1)f(k) =$

$= - A(n)Y(n)P_+Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n);$

ζ). for $n \geq k + 1$,

$x(n+1) = G(n+1,k+1)f(k) = Y(n+1)(P_0 + P_-)Y^{-1}(k+1)f(k) =$

$= A(n)Y(n)(P_0 + P_-)Y^{-1}(k+1)f(k) = A(n)x(n) = A(n)x(n) + f(n).$

On the other hand, $x(0) \in X_- \oplus X_+$, because

$$x(0) = G(0,k+1)f(k) = \begin{cases} +P_-Y^{-1}(k+1)f(k), & k + 1 \leq 0 \\ -P_+Y^{-1}(k+1)f(k), & k + 1 > 0 \end{cases}.$$

Finally, we have

$$x(n) = G(n,k+1)f(k) = \begin{cases} -Y(n)(P_0 + P_+)Y^{-1}(k+1)f(k), & n < k + 1 \leq 0 \\ Y(n)(P_0 + P_-)Y^{-1}(k+1)f(k), & n \geq k + 1 \geq 0 \end{cases}.$$

From the Definitions of X_-, X_0 and X_+, it follows that the function x is $\Psi-$bounded on $\mathbb{Z}_-$ and $\mathbb{N}$. Thus, x is the solution of (1) in D.

Now, we have, $\| \Psi(n)x(n)\| = \| \Psi(n)G(n,k+1)f(k)\| = \| \Psi(n)G(n,k+1)\Psi^{-1}(k)\xi \|$.

The inequality (4) becomes

$$\| \Psi(n)G(n,k+1)\Psi^{-1}(k)\xi \| \leq K\| \xi \|, \text{ for all } k, n \in \mathbb{Z}, \xi \in \mathbb{R}^d.$$

It follows that $| \Psi(n)G(n,k+1)\Psi^{-1}(k)| \leq K$, for all $k, n \in \mathbb{Z}$, which is equivalent with (3).

Now, we prove the "if" part.

For a given $\Psi-$ summable function $f : \mathbb{Z} \longrightarrow \mathbb{R}^d$, consider $u : \mathbb{Z} \longrightarrow \mathbb{R}^d$ defined by

$$u(n) = \begin{cases} \sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k) - \sum_{k=n}^{-1} Y(n)P_0Y^{-1}(k+1)f(k) - \\ \qquad - \sum_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k), & n < 0 \\ \sum_{k=-\infty}^{-1} Y(0)P_-Y^{-1}(k+1)f(k) - \sum_{k=0}^{\infty} Y(0)P_+Y^{-1}(k+1)f(k), & n = 0 \\ \sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k) + \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k) - \\ \qquad - \sum_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k), & n > 0 \end{cases}$$

Step 5. The function u is well-defined.

For $p, q \in \mathbb{Z}$, $q < 0 < p$, we have

$$\sum_{k=q}^{-1} \|Y(0)P_-Y^{-1}(k+1)f(k)\| + \sum_{k=0}^{p} \|Y(0)P_+Y^{-1}(k+1)f(k)\| \leq$$

$$\leq | \Psi^{-1}(0)| \sum_{k=q}^{-1} | \Psi(0)Y(0)P_-Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| +$$

$$+ | \Psi^{-1}(0)| \sum_{k=0}^{p} | \Psi(0)Y(0)P_+Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| \leq$$

$$\leq K| \Psi^{-1}(0)| \left(\sum_{k=q}^{p} \| \Psi(k)f(k) \| \right),$$

and then, $\sum\limits_{k=-\infty}^{-1} Y(0)P_-Y^{-1}(k+1)f(k)$ and $\sum\limits_{k=0}^{\infty} Y(0)P_+Y^{-1}(k+1)f(k)$ are absolutely convergent series. Thus, u(0) is well-defined.

For m, n $\in \mathbb{Z}$, $m \geq n > 0$, we have

$$\sum_{k=n}^{m} \|Y(n)P_+Y^{-1}(k+1)f(k)\| =$$
$$= \sum_{k=n}^{m} \| \Psi^{-1}(n)(\Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k))(\Psi(k)f(k))\| \leq$$
$$\leq | \Psi^{-1}(n)| \sum_{k=n}^{m} | \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| \leq$$
$$\leq K| \Psi^{-1}(n)| \left(\sum_{k=n}^{m} \| \Psi(k)f(k) \| \right),$$

and then, $\sum\limits_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k)$ is an absolutely convergent series for $n > 0$.

For $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $m < n-1$, we have

$$\sum_{k=m}^{n-1} \|Y(n)P_-Y^{-1}(k+1)f(k)\| =$$
$$= \sum_{k=m}^{n-1} \| \Psi^{-1}(n)(\Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k))(\Psi(k)f(k))\|$$
$$\leq | \Psi^{-1}(n)| \sum_{k=m}^{n-1} | \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| \leq$$
$$\leq K| \Psi^{-1}(n)| \sum_{k=m}^{n-1} \| \Psi(k)f(k)\|,$$

and then, $\sum\limits_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k)$ is an absolutely convergent series for $n > 0$.

Thus, the function u is well defined for $n \geq 0$.

Similarly, the function u is well defined for $n < 0$.

Step 6. The function u is a solution of the equation (1).

Indeed, using the expresion of the function u, we obtain:

- $u(1) = \sum\limits_{k=-\infty}^{0} Y(1)P_-Y^{-1}(k+1)f(k) + Y(1)P_0Y^{-1}(1)f(0) -$
  $- \sum\limits_{k=1}^{\infty} Y(1)P_+Y^{-1}(k+1)f(k) = A(0)[\sum\limits_{k=-\infty}^{0} Y(0)P_-Y^{-1}(k+1)f(k) +$
  $+ Y(0)P_0Y^{-1}(1)f(0) - \sum\limits_{k=1}^{\infty} Y(0)P_+Y^{-1}(k+1)f(k)] =$
  $= A(0)[\sum\limits_{k=-\infty}^{-1} Y(0)P_-Y^{-1}(k+1)f(k)+Y(0)P_-Y^{-1}(1)f(0)+Y(0)P_0Y^{-1}(1)f(0)$
  $- \sum\limits_{k=0}^{\infty} Y(0)P_+Y^{-1}(k+1)f(k) + Y(0)P_+Y^{-1}(1)f(0)] =$
  $= A(0)u(0) + A(0)Y(0)(P_- + P_0 + P_+)Y^{-1}(1)f(0) = A(0)u(0) + f(0);$
- for $n > 0$, $u(n+1) = \sum\limits_{k=-\infty}^{n} Y(n+1)P_-Y^{-1}(k+1)f(k) +$
  $+ \sum\limits_{k=0}^{n} Y(n+1)P_0Y^{-1}(k+1)f(k) - \sum\limits_{k=n+1}^{\infty} Y(n+1)P_+Y^{-1}(k+1)f(k) =$

$$= A(n)[\sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k) + Y(n)P_-Y^{-1}(n+1)f(n) +$$
$$+ \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k) + Y(n)P_0Y^{-1}(n+1)f(n) -$$
$$- \sum_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k) + Y(n)P_+Y^{-1}(n+1)f(n)] =$$
$$= A(n)u(n) + Y(n+1)(P_- + P_0 + P_+)Y^{-1}(n+1)f(n) = A(n)u(n) + f(n);$$

- $u(0) = \sum_{k=-\infty}^{-1} Y(0)P_-Y^{-1}(k+1)f(k) - \sum_{k=0}^{\infty} Y(0)P_+Y^{-1}(k+1)f(k) =$

$$= A(-1)[\sum_{k=-\infty}^{-1} Y(-1)P_-Y^{-1}(k+1)f(k) - \sum_{k=0}^{\infty} Y(-1)P_+Y^{-1}(k+1)f(k)] =$$
$$= A(-1)[\sum_{k=-\infty}^{-2} Y(-1)P_-Y^{-1}(k+1)f(k) + Y(-1)P_-Y^{-1}(0)f(-1) -$$
$$- \sum_{k=-1}^{-1} Y(-1)P_0Y^{-1}(k+1)f(k) + Y(-1)P_0Y^{-1}(0)f(-1) -$$
$$- \sum_{k=-1}^{\infty} Y(-1)P_+Y^{-1}(k+1)f(k) + Y(-1)P_+Y^{-1}(0)f(-1)] =$$
$$= A(-1)u(-1) + Y(0)(P_- + P_0 + P_+)Y^{-1}(0)f(-1) =$$
$$= A(-1)u(-1) + f(-1);$$

- for $n < -1$, $u(n+1) = \sum_{k=-\infty}^{n} Y(n+1)P_-Y^{-1}(k+1)f(k) -$

$$- \sum_{k=n+1}^{-1} Y(n+1)P_0Y^{-1}(k+1)f(k) - \sum_{k=n+1}^{\infty} Y(n+1)P_+Y^{-1}(k+1)f(k) =$$
$$= A(n)[\sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k) + Y(n)P_-Y^{-1}(n+1)f(n) -$$
$$- \sum_{k=n}^{-1} Y(n)P_0Y^{-1}(k+1)f(k) + Y(n)P_0Y^{-1}(n+1)f(n) -$$
$$- \sum_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k) + Y(n)P_+Y^{-1}(n+1)f(n)] =$$
$$= A(n)u(n) + Y(n+1)(P_- + P_0 + P_+)Y^{-1}(n+1)f(n) = A(n)u(n) + f(n).$$

These relations show that the function u is a solution of the equation (1).

Step 7. The function u is $\Psi-$ bounded on $\mathbb{Z}$.

Indeed, for $n > 0$ we have

$$\| \Psi(n)u(n)\| = \| \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) +$$
$$+ \sum_{k=0}^{n-1} \Psi(n)Y(n)P_0Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$
$$- \sum_{k=n}^{\infty} \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k)\| =$$
$$= \| \sum_{k=-\infty}^{-1} \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) +$$
$$+ \sum_{k=0}^{n-1} \Psi(n)Y(n)(P_0 + P_-)Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$

$$- \sum_{k=n}^{\infty} \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k)\| \le$$
$$\le \sum_{k=-\infty}^{-1} | \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| +$$
$$+ \sum_{k=0}^{n-1} | \Psi(n)Y(n)(P_0 + P_-)Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| +$$
$$+ \sum_{k=n}^{\infty} | \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| \le$$
$$\le K\left(\sum_{k=-\infty}^{-1} \| \Psi(k)f(k) \| + \sum_{k=0}^{n-1} \| \Psi(k)f(k) \| + \sum_{k=n}^{\infty} \| \Psi(k)f(k) \| \right) =$$
$$= K \sum_{k=-\infty}^{+\infty} \| \Psi(k)f(k)\| = K\|f\|_B .$$

For $n < 0$, we have

$$\| \Psi(n)u(n)\| = \| \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$
$$- \sum_{k=n}^{-1} \Psi(n)Y(n)P_0Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$
$$- \sum_{k=n}^{\infty} \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k)\| =$$
$$= \| \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$
$$- \sum_{k=n}^{-1} \Psi(n)Y(n)(P_0 + P_+)Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) -$$
$$- \sum_{k=0}^{\infty} \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k)\| \le$$
$$\le \sum_{k=-\infty}^{n-1} | \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| +$$
$$+ \sum_{k=n}^{-1} | \Psi(n)Y(n)(P_0 + P_+)Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| +$$
$$+ \sum_{k=0}^{\infty} | \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)|\| \Psi(k)f(k)\| \le$$
$$\le K\left(\sum_{k=-\infty}^{n-1} \| \Psi(k)f(k) \| + \sum_{k=n}^{-1} \| \Psi(k)f(k) \| + \sum_{k=0}^{\infty} \| \Psi(k)f(k) \| \right) =$$
$$= K \sum_{k=-\infty}^{+\infty} \| \Psi(k)f(k)\| = K\|f\|_B .$$

Similarly, $\| \Psi(0)u(0)\| \le K\|f\|_B$.
Therefore, $\| \Psi(n)u(n)\| \le K\|f\|_B$, for all $n \in \mathbb{Z}$.
Thus, the solution u of the equation (1) is $\Psi-$ bounded on $\mathbb{Z}$.
The proof is now complete.

Corollary 1. If the homogeneous equation (2) has no nontrivial $\Psi-$ bounded solution on $\mathbb{Z}$, then, the equation (1) has a unique $\Psi-$ bounded solution on $\mathbb{Z}$ for every $\Psi-$ summable function f on $\mathbb{Z}$ if and only if there exists a positive constant K such that, for $k, n \in \mathbb{Z}$,

$$\begin{cases} \mid \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k) \mid \ \le \ K, & \text{for } k+1 \le n \\ \mid \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k) \mid \ \le \ K, & \text{for } n < k+1 \end{cases} \tag{5}$$

Proof. Indeed, in this case, $P_0 = 0$. Now, the Corollary follows from the above Theorem.

Finally, we give a result in which we will see that the asymptotic behavior of $\Psi-$ bounded solutions of (1) is determined completely by the asymptotic behavior of the fundamental matrix Y of (2).

Theorem 2. Suppose that:

1°. the fundamental matrix Y of (2) satisfies the conditions (3) for some $K > 0$ and the conditions

i). $\lim\limits_{n\to\pm\infty} \mid \Psi(n)Y(n)P_0 \mid = 0$;

ii). $\lim\limits_{n\to+\infty} \mid \Psi(n)Y(n)P_- \mid = 0$;

iii). $\lim\limits_{n\to-\infty} \mid \Psi(n)Y(n)P_+ \mid = 0$.

2°. the function $f : \mathbb{Z} \longrightarrow \mathbb{R}^d$ is $\Psi-$ summable on $\mathbb{Z}$.

Then, every $\Psi-$ bounded solution x of (1) satisfies the condition

$$\lim_{n\to\pm\infty} \| \Psi(n)x(n)\| = 0.$$

Proof. Let x be a $\Psi-$ bounded solution of (1). Let u be the $\Psi-$ bounded solution of (1) from the proof of Theorem 1 ("if" part).

Let the function $y(n) = x(n) - u(n) - Y(n)P_0(x(0) - u(0))$, $n \in \mathbb{Z}$.

It is easy to see that y is a $\Psi-$ bounded solution of (2) and then $y(0) \in X_0$.

On the other hand,

$$y(0) = (I - P_0)(x(0) - u(0)) = (P_- + P_+)(x(0) - u(0)) \in X_- \oplus X_+.$$

Thus, $y(0) \in (X_- \oplus X_+) \cap X_0 = \{0\}$. It follows that $y = 0$ and then

$$x(n) = u(n) + Y(n)P_0(x(0) - u(0)),\ n \in \mathbb{Z}.$$

Now, we prove that $\lim\limits_{n\to\pm\infty} \| \Psi(n)x(n)\| = 0$.

For $n > 0$, we have

$$x(n) = Y(n)P_0(x(0) - u(0)) + \sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k) + \\ + \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k) - \sum_{k=n}^{\infty} Y(n)P_+Y^{-1}(k+1)f(k)$$

and then

$$\Psi(n)x(n) = \Psi(n)Y(n)P_0(x(0) - u(0)) + \\ + \sum_{k=-\infty}^{-1} \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) + \\ + \sum_{k=0}^{n-1} \Psi(n)Y(n)(P_0 + P_-)Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k) - \\ - \sum_{k=n}^{\infty} \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k).$$

By the hypotheses, for a given $\varepsilon > 0$, there exist:

• $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$,

$$\sum_{k=-\infty}^{-n} \| \Psi(k)f(k)\| < \frac{\varepsilon}{5K} \text{ and } \sum_{k=n}^{\infty} \| \Psi(k)f(k)\| < \frac{\varepsilon}{5K};$$

• $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that, for $n \geq n_2$,

$$| \Psi(n)Y(n)P_- | < \frac{\varepsilon}{5}\left(1 + \sum_{k=-n_1+1}^{-1} \| Y^{-1}(k+1)f(k) \|\right)^{-1};$$

• $n_3 \in \mathbb{N}$, $n_3 > n_2$, such that, for $n \geq n_3$,

$$| \Psi(n)Y(n)P_0 | < \frac{\varepsilon}{5}(1 + \| x(0) - u(0) \|)^{-1};$$

• $n_4 \in \mathbb{N}$, $n_4 > n_3$, such that, for $n \geq n_4$,

$$| \Psi(n)Y(n)(P_0+P_-)| < \frac{\varepsilon}{5}\left(1 + \sum_{k=0}^{n_1} \| Y^{-1}(k+1)f(k) \|\right)^{-1}.$$

Then, for $n \geq n_4$ we have

$$\begin{aligned}
&\| \Psi(n)x(n) \| \leq | \Psi(n)Y(n)P_0 | \|x(0) - u(0)\| + \\
&+ \sum_{k=-\infty}^{-n_1} | \Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)| \| \Psi(k)f(k)\| + \\
&+ \sum_{k=-n_1+1}^{-1} | \Psi(n)Y(n)P_- | \|Y^{-1}(k+1)f(k)\| + \\
&+ \sum_{k=0}^{n_1} | \Psi(n)Y(n)(P_0 + P_-)| \|Y^{-1}(k+1)f(k)\| + \\
&+ \sum_{k=n_1+1}^{n-1} | \Psi(n)Y(n)(P_0 + P_-)Y^{-1}(k+1)\Psi^{-1}(k)| \| \Psi(k)f(k)\| + \\
&+ \sum_{k=n}^{\infty} | \Psi(n)Y(n)P_+Y^{-1}(k+1)\Psi^{-1}(k)| \| \Psi(k)f(k)\| < \\
&< \frac{\varepsilon}{5}(1 + \| x(0) - u(0) \|)^{-1} \|x(0) - u(0)\| + \\
&+ K\sum_{k=-\infty}^{-n_1} \| \Psi(k)f(k)\| + | \Psi(n)Y(n)P_- | \sum_{k=-n_1+1}^{-1} \|Y^{-1}(k+1)f(k)\| + \\
&+ | \Psi(n)Y(n)(P_0 + P_-)| \sum_{k=0}^{n_1} \|Y^{-1}(k+1)f(k)\| + \\
&+ K\sum_{k=n_1+1}^{n-1} \| \Psi(k)f(k)\| + K\sum_{k=n}^{\infty} \| \Psi(k)f(k)\| < \\
&< \frac{\varepsilon}{5} + K\frac{\varepsilon}{5K} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + K\frac{\varepsilon}{5K} = \varepsilon.
\end{aligned}$$

This shows that $\lim\limits_{n\to+\infty} \| \Psi(n)x(n)\| = 0$.

Similarly, $\lim\limits_{n\to-\infty} \| \Psi(n)x(n)\| = 0$.

The proof is now complete.

Corollary 2. Suppose that:
1°. the homogeneous equation (2) has no nontrivial $\Psi-$ bounded solution on $\mathbb{Z}$;
2°. the fundamental matrix Y of (2) satisfies:
   a). the conditions (5) for some $K > 0$;
   b). the conditions:
      i). $\lim\limits_{n\to+\infty} \mid \Psi(n)Y(n)P_- \mid = 0$
      ii). $\lim\limits_{n\to-\infty} \mid \Psi(n)Y(n)P_+ \mid = 0$.
2°. the function $f : \mathbb{Z} \longrightarrow \mathbb{R}^d$ is $\Psi-$ summable on $\mathbb{Z}$.

Then, the equation (1) has a unique solution x on $\mathbb{Z}$ such that

$$\lim_{n\to\pm\infty} \| \Psi(n)x(n)\| = 0.$$

Proof. It results from the above Corollary and Theorem 2.

Note that the Theorem 2 (and the Corollary 2) is no longer true if we require that the function f be $\Psi-$ bounded on $\mathbb{Z}$, instead of the condition 2° of the Theorem. This is shown by the next

Example 1. Consider the system (1) with

$$A(n) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$f(n) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & n = 0, 1, 2,... \\ \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & n = -1, -2... \end{cases}.$$

Then, $Y(n) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix}$ is the fundamental matrix of (2) with $Y(0) = I_2$.
Consider $\Psi(n) = \begin{pmatrix} 1 & 0 \\ 0 & 3^{-n} \end{pmatrix}$, $n \in \mathbb{Z}$.
The first hypothesis of the Theorem 2 is satisfied with

$$P_0 = O_2,\ P_- = I_2,\ P_+ = O_2 \text{ and } K = 1.$$

In addition, we have $\| \Psi(n)f(n)\| = 1$ for $n \geq 0$ and $\| \Psi(n)f(n)\| = 0$ for $n < 0$. The function f is not $\Psi-$ summable on $\mathbb{Z}$, but it is $\Psi-$ bounded on $\mathbb{Z}$.
On the other hand, the solutions on $\mathbb{Z}$ of the system (1) are

$$x(n) = \begin{cases} \begin{pmatrix} 2^{-n}c_1 \\ 2^n c_2 \end{pmatrix}, & \text{for } n < 0, \\ \\ \begin{pmatrix} 2^{-n}c_1 + 2 - 2^{1-n} \\ 2^n c_2 \end{pmatrix}, & \text{for } n \geq 0. \end{cases},\quad c_1,\ c_2 \in \mathbb{R}.$$

It results from this that there is no solution x for $\lim_{n\to\pm\infty} \| \Psi(n)x(n)\| = 0$.

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