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Triple positive solutions for a boundary value problem of nonlinear fractional differential equation

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Abstract

In this paper, we investigate the existence of three positive solutions for the nonlinear fractional boundary value problem

$$D^{\alpha}_{0+}u(t) + a(t)f(t, u(t), u''(t)) = 0, \quad 0 < t < 1, \quad 3 < \alpha \le 4,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0,$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. The method involves applications of a new fixed-point theorem due to Bai and Ge. The interesting point lies in the fact that the nonlinear term is allowed to depend on the second order derivative u''.

Keywords: Fractional derivative; Boundary value problem; Positive solution; Fixed point theorem

1. Introduction

Many papers and books on fractional calculus and fractional differential equations have appeared recently, see for example [1-3, 7-12]. Very recently, El-Shahed [5] used the Krasnoselskii's fixed-point theorem on cone expansion and compression to show the existence and non-existence of positive solutions of nonlinear fractional boundary value problem :

$$D^{\alpha}_{0+}u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \le 3,$$

$$u(0) = u'(0) = u'(1) = 0,$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. Kaufmann and Mboumi [6] studied the existence and multiplicity of positive solutions of nonlinear fractional boundary value problem :

$$D^{\alpha}_{0+}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \le 2,$$

$$u(0) = u'(1) = 0.$$

Motivated by the above works, in this paper we study the existence of three positive solutions for the following nonlinear fractional boundary value problem :

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$$D_{0+}^{\alpha}u(t) + a(t)f(t, u(t), u''(t)) = 0, \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \tag{1.1}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{1.2}$$

by using a new fixed-point theorem due to Bai and Ge [4]. Here, the interesting point lies in the fact that the nonlinear term f is allowed to depend on the second order derivative u''. To the best of the authors knowledge, no one has studied the existence of positive solutions for nonlinear fractional boundary value problems (1.1)-(1.2).

Throughout this paper, we assume that the following conditions hold.

(H_1) $f : [0, 1] \times [0, \infty) \times (-\infty, +\infty) \to [0, \infty)$ is continuous;

(H_2) $a \in C([0, 1], [0, +\infty))$ and there exists $0 < \omega < 1$ such that $\int_{\omega}^{1}[(1-s)^{\alpha-3}-(1-s)^{\alpha-1}]a(s)ds > 0$.

The rest of this paper is organized as follows: In section 2, we present some preliminaries and lemmas. Section 3 is devoted to prove the existence of three positive solutions for BVP (1.1) and (1.2).

2. Preliminaries

For the convenience of the reader, we present some definitions from the cone theory on ordered Banach spaces.

Definition 2.1. The map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y), \quad \forall x, y \in P, \quad 0 \leq t \leq 1.$$

Similarly, we say the map ϕ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\phi : P \to [0, \infty)$ is continuous and

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y), \quad \forall x, y \in P, \quad 0 \leq t \leq 1.$$

Definition 2.2. Let $r > a > 0$, $L > 0$ be given and ψ be a nonnegative continuous concave functional and γ, β be nonnegative continuous convex functionals on the cone P. Define convex sets:

$$P(\gamma, r; \beta, L) = \{x \in P \mid \gamma(x) < r, \beta(x) < L\},$$

$$\overline{P}(\gamma, r; \beta, L) = \{x \in P \mid \gamma(x) \leq r, \beta(x) \leq L\},$$

$$P(\gamma, r; \beta, L; \psi, a) = \{x \in P \mid \gamma(x) < r, \beta(x) < L, \psi(x) > a\},$$

$$\overline{P}(\gamma, r; \beta, L; \psi, a) = \{x \in P \mid \gamma(x) \leq r, \beta(x) \leq L, \psi(x) \geq a\}.$$

Suppose that the nonnegative continuous convex functionals γ, β on the cone P satisfy

(A_1) there exists $M > 0$ such that $\|x\| \le M \max\{\gamma(x), \beta(x)\}$, for $x \in P$;

(A_2) $P(\gamma, r; \beta, L) \neq \emptyset$, for any $r > 0$, $L > 0$.

Lemma 2.1.[4] *Let P be a cone in a real Banach space E and $r_2 \ge d > b > r_1 > 0$, $L_2 \ge L_1 > 0$ constants. Assume that γ, β are nonnegative continuous convex functionals on P such that* (A_1) *and* (A_2) *are satisfied. ψ is a nonnegative continuous concave functional on P such that $\psi(x) \le \gamma(x)$ for all $x \in \overline{P}(\gamma, r_2; \beta, L_2)$ and let $T : \overline{P}(\gamma, r_2; \beta, L_2) \to \overline{P}(\gamma, r_2; \beta, L_2)$ be a completely continuous operator. Suppose*

(i) $\{x \in \overline{P}(\gamma, d; \beta, L_2; \psi, b) \mid \psi(x) > b\} \neq \emptyset$, $\psi(Tx) > b$ *for* $x \in \overline{P}(\gamma, d; \beta, L_2; \psi, b)$,

(ii) $\gamma(Tx) < r_1, \beta(Tx) < L_1$ *for all* $x \in \overline{P}(\gamma, r_1; \beta, L_1)$,

(iii) $\psi(Tx) > b$ *for all* $x \in \overline{P}(\gamma, r_2; \beta, L_2; \psi, b)$ *with* $\gamma(Tx) > d$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P}(\gamma, r_2; \beta, L_2)$. Further,

$$x_1 \in P(\gamma, r_1; \beta, L_1), \qquad x_2 \in \{\overline{P}(\gamma, r_2; \beta, L_2; \psi, b) \mid \psi(x) > b\},$$

and

$$x_3 \in \overline{P}(\gamma, r_2; \beta, L_2) \setminus (\overline{P}(\gamma, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\gamma, r_1; \beta, L_1)).$$

The above fixed-point theorem is fundamental in the proof of our main result.

Next, we give some definitions from the fractional calculus.

Definition 2.3. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is defined as

$$I^{\alpha}_{0+} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.4. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is

$$D^{\alpha}_{0+} u(t) = \frac{d^n}{dt^n} \left[I^{n-\alpha}_{0+} u(t) \right],$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

The following lemma is crucial in finding an integral representation of the boundary value problem (1), (2).

Lemma 2.2. [3] *Suppose that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

From Lemmas 2.2, we now give an integral representation of the solution of the linearized problem.

Lemma 2.3. *If* $y \in C[0, 1]$, *then the boundary value problem*

$$D^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1, \quad 3 < \alpha \le 4, \tag{2.1}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1 - s)^{\alpha-3} - (t - s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1 - s)^{\alpha-3}, & 0 \le t < s \le 1. \end{cases}$$

Proof. From Lemma 2.2, we get

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4} - \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

By (2.2), there are $c_2 = c_3 = c_4 = 0$, and $c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-3} y(s)ds$. Hence, the unique solution of BVP (2.1), (2.2) is

$$u(t) = \int_0^1 \frac{t^{\alpha-1}(1 - s)^{\alpha-3}}{\Gamma(\alpha)} y(s)ds - \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds = \int_0^1 G(t, s)y(s)ds.$$

The proof is complete.

Lemma 2.4. $G(t, s)$ *has the following properties.*

(i) $0 \le G(t, s) \le h(s), \quad t, s \in [0, 1]$,

where

$$h(s) = \frac{(1 - s)^{\alpha-3} - (1 - s)^{\alpha-1}}{\Gamma(\alpha)};$$

(ii) $G(t, s) \ge \frac{1}{2} t^{\alpha-1} h(s), \quad$ *for* $0 \le t, s \le 1$.

Proof. It is easy to check that (i) holds. Next, we prove (ii) holds. If $t \ge s$, then

$$\begin{aligned} \frac{G(t, s)}{h(s)} &= \frac{t^{\alpha-1}(1 - s)^{\alpha-3} - (t - s)^{\alpha-1}}{(1 - s)^{\alpha-3} - (1 - s)^{\alpha-1}} \\ &\ge \frac{t^2(t - ts)^{\alpha-3} - (t - s)^2(t - ts)^{\alpha-3}}{(1 - s)^{\alpha-3} - (1 - s)^{\alpha-1}} = \frac{(t - ts)^{\alpha-3} s(2t - s)}{(1 - s)^{\alpha-3} s(2 - s)} \\ &= \frac{t^{\alpha-3}(2t - s)}{2 - s} \ge \frac{t^{\alpha-2}}{2 - s} \ge \frac{t^{\alpha-1}}{2}. \end{aligned}$$

If $t \le s$, then

$$\frac{G(t,s)}{h(s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-s)^{\alpha-3}-(1-s)^{\alpha-1}} \ge \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-s)^{\alpha-3}} \ge \frac{t^{\alpha-1}}{2}.$$

The proof is complete.

3. Main results

Let $X = \{u \in C^2[0,1] : u(0) = u'(0) = 0\}$. Then we have the following lemma.

Lemma 3.1. *For* $u \in X$, $\|u\|_0 \le \|u'\|_0 \le \|u''\|_0$, *where* $\|u\|_0 = \max_{0\le t\le 1} |u(t)|$.

By Lemma 3.1, X is a Banach space when it is endowed with the norm $\|u\| = \|u''\|_0$.

It is easy to know that

$$\frac{\partial}{\partial t}G(t,s) = \frac{\alpha-1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}, & 0 \le s \le t \le 1,\\ t^{\alpha-2}(1-s)^{\alpha-3}, & 0 \le t < s \le 1.\end{cases} \tag{3.1}$$

$$\frac{\partial^2}{\partial t^2}G(t,s) = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\begin{cases} (t(1-s))^{\alpha-3}-(t-s)^{\alpha-3}, & 0 \le s \le t \le 1,\\ (t(1-s))^{\alpha-3}, & 0 \le t < s \le 1.\end{cases} \tag{3.2}$$

We define the operator T by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds, \quad 0 \le t \le 1. \tag{3.3}$$

From (3.1) and (3.3) , we have $Tu(0) = (Tu)'(0) = 0$. Moreover, we obtain by (H_1), (H_2) and (3.3) that

$$(Tu)''(t) = \int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)a(s)f(s,u(s),u''(s))ds \in C^2[0,1], \quad \forall u \in X.$$

Thus, $T : X \to X$. By Lemma 2.3, $u(t)$ is a solution of the fractional boundary value problem (1.1)-(1.2) if and only if $u(t)$ is a fixed point of the operator T.

Define the cone $P \subset X$ by

$$P = \left\{u \in X : u(t) \ge 0,\ \forall t \in [0,1],\ \min_{\omega \le t \le 1} u(t) \ge \frac{1}{2}\omega^{\alpha-1}\|u\|_0\right\},$$

where $0 < \omega < 1$ as in (H_2).

Let the nonnegative continuous convex functionals γ, β and the nonnegative continuous concave functional ψ be defined on the cone P by

$$\gamma(u) = \max_{0\le t\le 1} |u(t)|, \quad \beta(u) = \max_{0\le t\le 1} |u''(t)|, \quad \psi(u) = \min_{\omega\le t\le 1} |u(t)|.$$

Then $\gamma, \beta, \psi : P \to [0,\infty)$ are three continuous nonnegative functionals such that $\|u\| = \max\{\gamma(u), \beta(u)\}$, and ($A_1$), ($A_2$) hold; γ, β are convex, ψ is concave and there holds $\psi(u) \le \gamma(u)$, for all $u \in P$.

For $3 < \alpha \le 4$, it is rather straightforward that

$$\max_{0\le t\le 1} t^{\alpha-3}(1-t^{\alpha-2}) = \frac{\alpha-2}{2\alpha-5}\left(\frac{\alpha-3}{2\alpha-5}\right)^{\frac{\alpha-3}{\alpha-2}}, \quad \max_{0\le t\le 1} t^{\alpha-1}(1-t^{\alpha-2}) = \frac{\alpha-2}{2\alpha-3}\left(\frac{\alpha-1}{2\alpha-3}\right)^{\frac{\alpha-1}{\alpha-2}}. \tag{3.4}$$

For convenience, we denote

$$M = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)(2\alpha-5)}\left(\frac{\alpha-3}{2\alpha-5}\right)^{\frac{\alpha-3}{\alpha-2}}, \quad N = \frac{1}{\Gamma(\alpha)(2\alpha-3)}\left(\frac{\alpha-1}{2\alpha-3}\right)^{\frac{\alpha-1}{\alpha-2}},$$

$$m = \frac{\omega^{\alpha-1}}{2\Gamma(\alpha)}\int_\omega^1 \left[(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\right] a(s)ds > 0, \text{ (by (H}_2\text{))}$$

where $3 < \alpha \le 4$.

We are now in a position to present and prove our main result.

Theorem 3.2. *Assume that* (H_1) *and* (H_2) *hold. Suppose there exist constants* $r_2 \ge \frac{2b}{\omega^{\alpha-1}} > b > r_1 > 0$, $L_2 \ge L_1 > 0$ *such that* $\frac{b}{m} \le \min\{\frac{r_2}{N\|a\|_0}, \frac{L_2}{M\|a\|_0}\}$. *If the following assumptions hold*

(H_3) $f(t,u,v) < \min\left\{\frac{r_1}{N\|a\|_0}, \frac{L_1}{M\|a\|_0}\right\}$, *for* $(t,u,v) \in [0,1]\times[0,r_1]\times[-L_1, L_1]$;

(H_4) $f(t,u,v) > \frac{b}{m}$, *for* $(t,u,v) \in [\omega,1]\times\left[b, \frac{2b}{\omega^{\alpha-1}}\right]\times[-L_2, L_2]$;

(H_5) $f(t,u,v) \le \min\left\{\frac{r_2}{N\|a\|_0}, \frac{L_2}{M\|a\|_0}\right\}$, *for* $(t,u,v) \in [0,1]\times[0,r_2]\times[-L_2, L_2]$,

then BVP (1.1)-(1.2) has at least three positive solutions u_1, u_2, *and* u_3 *such that*

$$\max_{0\le t\le 1} u_1(t) \le r_1, \quad \max_{0\le t\le 1} |u_1''(t)| \le L_1;$$

$$b < \min_{\omega\le t\le 1} u_2(t) \le \max_{0\le t\le 1} u_2(t) \le r_2, \quad \max_{0\le t\le 1} |u_2''(t)| \le L_2;$$

$$\max_{0\le t\le 1} u_3(t) \le \frac{2b}{\omega^{\alpha-1}}, \quad \max_{0\le t\le 1} |u_3''(t)| \le L_2.$$

Proof. By (H_1), (H_2), Lemma 2.4 and (3.3), for $u \in P$, we have $Tu(t) \ge 0$, $\forall t \in [0,1]$, and

$$\begin{aligned}
\min_{\omega\le t\le 1} Tu(t) &= \min_{\omega\le t\le 1}\int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds \\
&\ge \int_0^1 \min_{\omega\le t\le 1} G(t,s)a(s)f(s,u(s),u''(s))ds \\
&\ge \frac{1}{2}\omega^{\alpha-1}\int_0^1 h(s)a(s)f(s,u(s),u''(s))ds
\end{aligned}$$

$$\geq \frac{1}{2}\omega^{\alpha-1} \max_{0\leq t\leq 1} \int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds = \frac{1}{2}\omega^{\alpha-1}\|Tu\|_0, \quad t \in [0,1].$$

Thus, $T(P) \subset P$. Moreover, it is easy to check by the Arzela-Ascoli theorem that the operator T is completely continuous. We now show that all the conditions of Lemma 2.1 are satisfied.

If $u \in \overline{P}(\gamma, r_2; \beta, L_2)$, then $\gamma(u) = \max_{0\leq t\leq 1} |u(t)| \leq r_2$, $\beta(u) = \max_{0\leq t\leq 1} |u''(t)| \leq L_2$, and assumption (H_5) implies

$$f(t,u(t),u''(t)) \leq \min\left\{\frac{r_2}{N\|a\|_0}, \frac{L_2}{M\|a\|_0}\right\}, \quad \forall t \in [0,1]. \tag{3.5}$$

Thus, by (3.4), (3.2) and (3.3), we get

$$\begin{aligned}
\beta(Tu) &= \max_{0\leq t\leq 1} |(Tu)''(t)| = \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)a(s)f(s,u(s),u''(s))ds\right| \\
&\leq \|a\|_0 \frac{L_2}{M\|a\|_0} \max_{0\leq t\leq 1} \int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)ds \\
&= \frac{L_2}{M}\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \max_{0\leq t\leq 1}\left[\int_0^t \left(t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right)ds + \int_t^1 t^{\alpha-3}(1-s)^{\alpha-3}ds\right] \\
&= \frac{L_2}{M}\frac{\alpha-1}{\Gamma(\alpha)} \max_{0\leq t\leq 1} t^{\alpha-3}(1-t^{\alpha-2}) = \frac{L_2}{M}\frac{\alpha-1}{\Gamma(\alpha)}\frac{\alpha-2}{2\alpha-5}\left(\frac{\alpha-3}{2\alpha-5}\right)^{\frac{\alpha-3}{\alpha-2}} \\
&= \frac{L_2}{M}\cdot M = L_2.
\end{aligned}$$

Moreover, we obtain by (3.5) that

$$\begin{aligned}
\gamma(Tu) &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds\right| \\
&\leq \|a\|_0 \frac{r_2}{N\|a\|_0} \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds \\
&= \frac{r_2}{N}\frac{1}{\Gamma(\alpha)} \max_{0\leq t\leq 1}\left[\int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right)ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-3}ds\right] \\
&= \frac{r_2}{N}\frac{1}{\Gamma(\alpha)(\alpha-2)} \max_{0\leq t\leq 1} t^{\alpha-1}(1-t^{\alpha-2}) = \frac{r_2}{N}\frac{1}{\Gamma(\alpha)}\frac{1}{2\alpha-3}\left(\frac{\alpha-1}{2\alpha-3}\right)^{\frac{\alpha-1}{\alpha-2}} \\
&= \frac{r_2}{N}\cdot N = r_2.
\end{aligned}$$

Hence, $T : \overline{P}(\gamma, r_2; \beta, L_2) \to \overline{P}(\gamma, r_2; \beta, L_2)$. Similarly, if $u \in \overline{P}(\gamma, r_1; \beta, L_1)$, then assumption (H_3) yields $f(t,u(t),u''(t)) < \min\left\{\frac{r_1}{N\|a\|_0}, \frac{L_1}{M\|a\|_0}\right\}$ for $t \in [0,1]$. As in the argument above, we can obtain that $T : \overline{P}(\gamma, r_1; \beta, L_1) \to P(\gamma, r_1; \beta, L_1)$. Hence, condition (ii) of Lemma 2.1 is satisfied.

To check condition (i) of Lemma 2.1, we choose $u(t) = \frac{2b}{\omega^{\alpha-1}}$, $0 \le t \le 1$. It is easy to see that $u(t) = \frac{2b}{\omega^{\alpha-1}} \in \overline{P}(\gamma, \frac{2b}{\omega^{\alpha-1}}; \beta, L_2; \psi, b)$, $\psi(u) = \psi(\frac{2b}{\omega^{\alpha-1}}) > b$, and so $\{u \in \overline{P}(\gamma, \frac{2b}{\omega^{\alpha-1}}; \beta, L_2; \psi, b) \mid \psi(u) > b\} \neq \emptyset$. Hence, if $u \in \overline{P}(\gamma, \frac{2b}{\omega^{\alpha-1}}; \beta, L_2; \psi, b)$, then $b \le u(t) \le \frac{2b}{\omega^{\alpha-1}}$, $|u''(t)| \le L_2$ for $\omega \le t \le 1$. From assumption (H_4), we have $f(t, u(t), u''(t)) > \frac{b}{m}$ for $\omega \le t \le 1$. Thus, by Lemma 2.4 and (3.2), we have

$$\begin{aligned}
\psi(Tu) &= \min_{\omega \le t \le 1} \left| \int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds \right| \\
&\ge \int_0^1 \min_{\omega \le t \le 1} G(t,s)a(s)f(s,u(s),u''(s))ds \ge \frac{\omega^{\alpha-1}}{2} \int_0^1 h(s)a(s)f(s,u(s),u''(s))ds \\
&\ge \frac{\omega^{\alpha-1}}{2} \int_\omega^1 h(s)a(s)f(s,u(s),u''(s))ds > \frac{\omega^{\alpha-1}}{2} \frac{b}{m} \int_\omega^1 h(s)a(s)ds \\
&= \frac{\omega^{\alpha-1}}{2} \frac{b}{m} \frac{1}{\Gamma(\alpha)} \int_\omega^1 \left[(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\right] a(s)ds \\
&= b,
\end{aligned}$$

i.e.,

$$\psi(Tu) > b, \quad \forall u \in \overline{P}(\gamma, \tfrac{2b}{\omega^{\alpha-1}}; \beta, L_2, \psi, b).$$

This shows that condition (i) of Lemma 2.1 is satisfied. We finally show that (iii) of Lemma 2.1 also holds. Suppose that $u \in \overline{P}(\gamma, r_2; \beta, L_2; \psi, b)$ with $\gamma(Tu) > \frac{2b}{\omega^{\alpha-1}}$. Then, by the definition of ψ and $Tu \in P$, we have

$$\begin{aligned}
\psi(Tu) &= \min_{\omega \le t \le 1} \left| \int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds \right| \\
&\ge \frac{\omega^{\alpha-1}}{2} \int_0^1 h(s)a(s)f(s,u(s),u''(s))ds \ge \frac{\omega^{\alpha-1}}{2} \max_{0 \le t \le 1} \int_0^1 G(t,s)a(s)f(s,u(s),u''(s))ds \\
&= \frac{\omega^{\alpha-1}}{2} \gamma(Tu) > \frac{\omega^{\alpha-1}}{2} \cdot \frac{2b}{\omega^{\alpha-1}} = b.
\end{aligned}$$

So, the condition (iii) of Lemma 2.1 is satisfied. Therefore, an application of Lemma 2.1 implies that the boundary value problem (1.1)-(1.2) has at least three positive solutions u_1, u_2, and u_3 in $\overline{P}(\alpha, r_2; \beta, L_2)$ such that

$$\begin{aligned}
&\max_{0 \le t \le 1} u_1(t) < r_1, \quad \max_{0 \le t \le 1} |u_1''(t)| < L_1; \\
&b \le \min_{\omega \le t \le 1} u_2(t) \le \max_{0 \le t \le 1} u_2(t) \le r_2, \quad \max_{0 \le t \le 1} |u_2''(t)| \le L_2; \\
&\max_{0 \le t \le 1} u_3(t) \le \frac{2b}{\omega^{\alpha-1}}, \quad \max_{0 \le t \le 1} |u_3''(t)| \le L_2.
\end{aligned}$$

The proof is complete.

Finally, we give an example to illustrate the effectiveness of our result.

Example 3.1. Consider the nonlinear fractional boundary value problem :

$$D_{0+}^{3.6}u(t) + a(t)f(t, u(t), u''(t)) = 0, \quad 0 < t < 1, \tag{3.6}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{3.7}$$

where $a(t) = 100(1-t)$, and

$$f(t,u,v) = \begin{cases} \frac{1}{10^3}|cost| + \frac{1}{380}3^{u^3} + \left(\frac{|v|}{800}\right)^3, & u \le 2, \\ \frac{1}{10^3}|cost| + \frac{6561}{380} + \left(\frac{|v|}{800}\right)^3, & u \ge 2. \end{cases}$$

Set $\alpha = 3.6$, $\omega = \frac{1}{2}$, we have

$$M = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)(2\alpha-5)}\left(\frac{\alpha-3}{2\alpha-5}\right)^{\frac{\alpha-3}{\alpha-2}} = 0.3125, \quad N = \frac{1}{\Gamma(\alpha)(2\alpha-3)}\left(\frac{\alpha-1}{2\alpha-3}\right)^{\frac{\alpha-1}{\alpha-2}} = 0.8397,$$

$$\|a\|_0 = 100, \quad m = \frac{\omega^{\alpha-1}}{2\Gamma(\alpha)}\int_\omega^1 \left[(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\right] a(s)ds = 0.12.$$

Obviously, $a(t)$ satisfies condition (H_2). Choose $r_1 = 1$, $b = 2$, $r_2 = 1500$, $L_1 = 30$, and $L_2 = 600$, then

$$\frac{2b}{\omega^{\alpha-1}} = 24.2515, \quad \frac{b}{m} = 16.6667,$$

$$\min\left\{\frac{r_1}{N\|a\|_0}, \frac{L_1}{M\|a\|_0}\right\} = 0.0119, \quad \min\left\{\frac{r_2}{N\|a\|_0}, \frac{L_2}{M\|a\|_0}\right\} = 17.8635.$$

Consequently, $f(t,u,v)$ satisfy

$$f(t,u,v) < 0.0119, \text{ for } (t,u,v) \in [0,1]\times[0,1]\times[-30,30];$$

$$f(t,u,v) > 16.6667, \text{ for } (t,u,v) \in [1/2,1]\times[2,24.2515]\times[-600,600];$$

$$f(t,u,v) < 17.8635, \text{ for } (t,u,v) \in [0,1]\times[0,1500]\times[-600,600],$$

Then all the assumptions of Theorem 3.2 hold. Hence, with Theorem 3.2, nonlinear fractional boundary value problem (3.6), (3.7) has at least three positive solutions u_1, u_2, u_3, such that

$$\max_{0\le t\le 1} u_1(t) \le 1, \quad \max_{0\le t\le 1} |u_1''(t)| \le 30;$$

$$2 < \min_{1/2\le t\le 1} u_2(t) \le \max_{0\le t\le 1} u_2(t) \le 1500, \quad \max_{0\le t\le 1} |u_2''(t)| \le 600;$$

$$\max_{0\le t\le 1} u_3(t) \le 24.2515, \quad \max_{0\le t\le 1} |u_3''(t)| \le 600.$$

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