Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **23**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

EXPONENTIAL STABILITY OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH BOUNDED DELAY AND THE W-TRANSFORM

RAMAZAN I. KADIEV AND ARCADY PONOSOV

Abstract. We demonstrate how the method of auxiliary ('reference') equations, also known as N. V. Azbelev's W-transform method, can be used to derive efficient conditions for the exponential Lyapunov stability of linear delay equations driven by a vector-valued Wiener process. For the sake of convenience the W-method is briefly outlined in the paper, its justification is however omitted. The paper contains a general stability result, which is specified in the last section in the form of seven corollaries providing sufficient stability conditions for some important classes of Itô equations with delay.

1. Introduction

Stability of solutions to delayed systems with random parameters is a popular topic in the mathematical literature. A rather complete, but definitely not exhaustive, list of publications can e. g. be found in [7], [8], [9], [11]. Most authors apply the method of auxiliary *functionals* (the Lyapunov-Krasovskii-Razumikhin method). On the other hand, an alternative method (Azbelev's W-transform method) to study asymptotic properties of linear functional-differential equations, based on auxiliary *equations* and developed in [2], [4], [5], [6], proved to be efficient as well, especially in the case of non-diffusion equations driven by semimartingales.

While in the case of the Lyapunov-Krasovskii-Razumikhin method a successful stability analysis depends on a skillful choice of an auxiliary functional with certain properties, the W-transform method is very much dependent on a suitable auxiliary equation that has the required asymptotic properties. The auxiliary equation gives rise to a certain integral transform which is applied to the equation in question, and the challenge is to find conditions under which the latter equation would inherit the asymptotic properties of the auxiliary equation. By this reason the auxiliary equation may be called *a reference equation.* Some hints on what kind of reference equations can be suitable for certain classes of linear stochastic functional differential equations, are discussed in [2], [4], [5], [6].

In the present paper we use the W-transform method to study the exponential Lyapunov $2p$–stability ($1 \leq p < \infty$) of the zero solution of linear and scalar differential equations with bounded delay, which are driven by a vector-valued Wiener process. We apply the general framework developed in our previous papers [2], [4], [5], [6]. The framework provides sufficient stability conditions in terms of the parameters of the given equation.

Note that all the results of the paper are new (the only exception is the lemma in Section 4, which was proved by the authors in [6]). We found it difficult to compare our stability results with those proved by the Lyapunov-Krasovskii-Razumikhin method (see e.g. [8] and references therein), as in most cases the results are almost independent of each other. We feel that a combination of these two methods will give better results, but this was beyond the scope of this particular paper.

2. Notation and definitions

In this paper we always assume that the real number p satisfies $1 \leq p < \infty$.

Below we introduce some basic notation to be used in the sequel. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a given stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-algebras on it, satisfying the usual assumptions (see [1]). By $\mathbb{E}$ we denote the expectation on this probability space. The stochastic $(m-1)$-vector process $\mathcal{B}$ consists of jointly independent Wiener processes

1991 *Mathematics Subject Classification.* 34K50, 34D20.
Key words and phrases. Stochastic differential equations, aftereffect, exponential stability, integral transforms.

$\mathcal{B}_i,\ i=2,...,m$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. The linear space K consists of functions $\eta:\Omega\to\mathbf{R}\equiv(-\infty,\infty)$ (random variables) which are $\mathcal{F}_0$-measurable (in our previous papers we used the notation k for this space).

In the sequel we will also use the universal constants c_p $(1\leq p<\infty)$, which come from the celebrated Burkholder-Davis-Gandy inequalities and help to estimate stochastic integrals. These constants are involved in the stability conditions of Sections 4 and 5. From [10] it is known that $c_p=2\sqrt{12p}$. However, other sources give other values (see e.g. [1]). All these values are not optimal. For instance, in the results presented below we may assume that $c_1=1$. Indeed, we estimate $\sup\limits_t \mathbb{E}|x(t)|^2$ and not $\mathbb{E}\sup\limits_t |x(t)|^2$ as in the Burkholder-Davis-Gandy inequalities.

The following scalar Itô-type equation with delay and the 'prehistory' condition is considered in this paper:

$$dx(t)=\sum_{j=0}^{m_1}a_{1j}(t)x(h_{1j}(t))dt+\sum_{i=2}^{m}\sum_{j=0}^{m_i}a_{ij}(t)x(h_{ij}(t))d\mathcal{B}_i(t)\ (t\geq 0),\tag{1}$$

$$x(\nu)=\varphi(\nu)\ (\nu<0),\tag{2}$$

where φ is a measurable stochastic process, which is independent of the vector-valued process $\mathcal{B}=(\mathcal{B}_2,...,\mathcal{B}_m)$ and which almost surely (a. s.) has trajectories that are essentially bounded, i. e. belong to L^∞. The functions $a_{ij},h_{ij},i=1,...,m,j=0,...,m_i$ are Lebesgue-measurable on $[0,\infty)$ and satisfy the following assumptions:

1) $a_{1j},j=0,...,m_1$ are locally integrable functions;
2) $a_{ij},i=2,...,m,j=0,...,m_i$ are locally square-integrable functions;
3) $h_{ij}(t)\leq t$ for $t\in[0,\infty)$ a. s., $\operatorname{vraisup}\limits_{t\geq 0}(t-h_{ij}(t))<\infty$ for $i=1,...,m,j=0,...,m_i$.

In what follows these hypotheses are always assumed to be fulfilled.

A solution to the equation (1), (2) is a stochastic process $x(t)$ $(t\in(-\infty,\infty))$ satisfying (1) and (2). This means, in particular, that $x(t)$ must be $(\mathcal{F}_t)$-adapted and have continuous trajectories for $t\geq 0$. We observe also that varying $x(0)$ (which is not fixed in (1), (2)) we obtain different solutions.

It can be proved (see [3]) that the assumptions **1)-3)** above guarantee that for any given $x_0\in k$ there exists a unique (up to the $\mathbb{P}$–equivalence) solution of (1), (2). We will denote this solution by $x(t,x_0,\varphi)$.

Remark 1. *Unlike most works on stochastic functional differential equations, we do not require here that the trajectories $x(t)$ $(t\geq 0)$ should be continuous extensions of the trajectories of the 'prehistory' process φ. This allows for use discontinuous φ. But the solution may even make a jump at $t=0$, when the initial function φ is continuous. This property seems to be more natural in the case of discontinuous stochastic processes. But even in the continuous case it may be convenient to allow jumps at $t=0$. For instance, in the conventional framework it is sometimes desirable to extend the set of initial functions from the space of continuous functions C to the space of square-integrable functions L_2 (see e.g.* [9]*), which necessarily requires a new, extended phase space $L_2\times\mathbf{R}$ (where $\varphi\in L^2$ and $x_0\in\mathbf{R}=(-\infty,\infty)$ can be chosen arbitrarily).*

The stochastic process

$$\begin{cases}\varphi(\nu),\ if\,\nu<0,\\ x_0,\ if\,\nu=0,\end{cases}$$

will be conventionally addressed as the initial function for the delay equation (1).

The equation (1), (2) is called homogeneous if $\varphi(\nu)\equiv 0$ $(\nu<0)$:

$$dx(t)=\sum_{j=0}^{m_1}a_{1j}(t)x(h_{1j}(t))dt+\sum_{i=2}^{m}\sum_{j=0}^{m_i}a_{ij}(t)x(h_{ij}(t))d\mathcal{B}_i(t)\ (t\geq 0),$$

$$x(\nu)=0\ (\nu<0).$$

Remark 2. *It is important to stress that our interpretation of homogeneous delay equations is different from the conventional one (see e.g.* [5]*). Usually, one calls the equation (1) homogeneous, while (2) is interpreted as the initial condition. Our definition (see e. g.* [5]*) is more restrictive, as we in addition require that $\varphi(\nu)=0$ a. s. for all $\nu<0$, so that the only initial assumption left is $x(0)=x_0$. The reason* *for that is technical: it gives us an opportunity to treat the homogeneous equation as finite dimensional, and thus exploit the most essential features of the W-transform. The "prehistory function" $\varphi(\nu)$ $(\nu < 0)$ does not disappear, but becomes a part of the (inhomogeneous) equation (1), (2). This will be demonstrated in the beginning of Section 4.*

In any case, the final output in our approach - the exponential estimate (EXP) - is the same as in the conventional theory. This is summarized in the definition below.

Definition. *The zero solution $x(t, 0, 0) \equiv 0$ of the homogeneous equation corresponding to the equation (1), (2) is called exponentially (Lyapunov) $2p$–stable w.r.t. the initial function, if there exist positive numbers $\bar{c}$, β such that*

$$\mathbb{E}|x(t, x_0, \varphi)|^{2p} \le \bar{c}(\mathbb{E}|x_0|^{2p} + \underset{\nu<0}{\operatorname{vraisup}}\, \mathbb{E}|\varphi(\nu)|^{2p}) \exp\{-\beta t\}\ (t \ge 0) \qquad (EXP)$$

for any $x_0 \in k$ and any measurable stochastic process φ, which is independent of the vector-valued process $\mathcal{B} = (\mathcal{B}_2, ..., \mathcal{B}_m)$ and which almost surely (a. s.) has trajectories belonging to L^∞.

Remark 3. *Usually, the exponential Lyapunov stability w.r.t. the initial function is simply called the exponential Lyapunov stability. However, in this paper we would like to use the terminology which is more consistent with that used in our previous papers* [2]-[6], *where we also study (Lyapunov) stability w.r.t. the initial value x_0 (keeping φ fixed).*

Of course, $2p$ can be replaced by another letter (say q) giving the usual definition of the stochastic exponential q-stability. However, in this paper we only intend to study stability of order 2 and higher. That is why the notation $2p$-stability with $p \ge 1$ is more convenient for our purposes.

3. Two auxiliary results

The first result provides a uniform estimate on the solutions of the equation (1), (2). The second result gives a convenient technical tool for deriving exponential estimates in the forthcoming sections.

Theorem 1. *For any given $s \in [0, \infty)$ and any admissible φ, x_0 such that* $\underset{\nu<0}{\operatorname{vraisup}}\, \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$ *the solution $x(t, x_0, \varphi)$ of (1), (2) satisfies* $\sup_{0\le t\le s} \mathbb{E}|x(t, x_0, \varphi)|^{2p} < \infty$.

Proof. Let s be an arbitrary number from $[0, \infty)$ and φ, x_0 be such that $\underset{\nu<0}{\operatorname{vraisup}}\, \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$. Given a natural number k we put $s_k = inf\{\zeta : \sum_{j=0}^{m_1} \int_0^\zeta |a_{1j}(\tau)|d\tau + \sum_{i=2}^{m} \sum_{j=0}^{m_i} c_p(\int_0^\zeta (a_{ij}(\tau))^2 d\tau)^{0.5} \ge k/2\}$. From this definition we immediately obtain that $s_l \ge s$ for $l = 2[\sum_{j=0}^{m_1} \int_0^s |a_{1j}(\tau)|d\tau + \sum_{i=2}^{m} \sum_{j=0}^{m_i} c_p\ (\int_0^s (a_{ij}(\tau))^2 d\tau)^{0.5}] + 1$. Here $[r]$ stands for the integer part of r.

Put

$$(S_h x)(t) = \begin{cases} x(h(t)), \, if\, h(t) \ge 0, \\ 0, \, if\, h(t) < 0, \end{cases}$$

$$\varphi_h(t) = \begin{cases} 0, \, if\, h(t) \ge 0, \\ \varphi(h(t)), \, if\, h(t) < 0. \end{cases}$$

To simplify calculations we introduce the following notation:

$$Z(t) = (t, \mathcal{B}_2(t), ..., \mathcal{B}_m(t))^T,\ (V x)(t) = ((V_1 x)(t), ..., (V_m x)(t))\,,\ (V_i x)(t) = \sum_{j=0}^{m_i} a_{ij}(t)(S_{h_{ij}} x)(t)\ (i = 1, ...m).$$

Clearly, (1)-(2) is equivalent to the integral equation

$$x(t) = x_0(t) + \int_0^t (V x)(\tau) dZ(\tau) \quad (t > 0),$$

where

$$x_0(t) = x_0 + \sum_{j=0}^{m_1} \int_0^t a_{1j}(\tau)\varphi_{h_{1j}}(\tau)d\tau + \sum_{i=2}^{m}\sum_{j=0}^{m_i} \int_0^t a_{ij}(\tau)\varphi_{h_{ij}}(\tau)d\mathcal{B}_i(\tau).$$

Consider the finite sequence of equations

$$x_k(t) = x_{k-1}(t) + \int_0^t I_k(\tau)(Vx_k)(\tau)dZ(\tau) \qquad (t \geq 0,\ k = 1, 2, ...), \tag{3}$$

where $I_k(\tau)$ is the indicator of the interval $[s_{k-1}, s_k]$, $x_0(t)$ is the same as above, and $x_{k-1}(t)$ is a solution of the $(k-1)$-th equation in the sequence (3).

Let us show by induction that the solution of the k-th equation in the sequence (3) coincides with the solution of (1), (2) on the interval $[0, s_k]$, i. e. $x_k(t) = x(t, x_0, \varphi) \equiv x(t)$ for $0 \leq t \leq s_k$, $k \geq 1$.

If $k = 1$ and $t \in [0, s_1]$, then the definition of $I_1(\tau)$ gives

$$x_1(t) = x_0(t) + \int_0^t (Vx_1)(\tau)dZ(\tau).$$

The integral representation of $x(t)$ and the uniqueness of solutions to (1)-(2) imply $x_1(t) = x(t)$ for $t \in [0, s_1]$.

Assume now that $x_n(t) = x(t)$ for $t \in [0, s_n]$, $n = 1, ..., k-1$ and divide the interval $[0, s_k]$ in two subintervals: $[0, s_{k-1}]$ and $[s_{k-1}, s_k]$.

For $t \in [0, s_{k-1}]$ we immediately obtain that $x_k(t) = x_{k-1}(t) = x(t)$, as the integral in (3) is zero by the definition of $I_k(\tau)$.

Assume that $t \in [s_{k-1}, s_k]$. Then

$$\begin{aligned} x_k(t) &= x_{k-1}(t) + \int_0^t I_k(\tau)(Vx_k)(\tau)dZ(\tau) = x_{k-1}(t) + \int_{s_{k-1}}^t (Vx_k)(\tau)dZ(\tau) \\ &= x_{k-2}(t) + \int_0^t I_{k-1}(\tau)(Vx_{k-1})(\tau)dZ(\tau) + \int_{s_{k-1}}^t (Vx_k)(\tau)dZ(\tau) \\ &= x_{k-2}(t) + \int_{s_{k-2}}^{s_{k-1}} (Vx_{k-1})(\tau)dZ(\tau) + \int_{s_{k-1}}^t (Vx_k)(\tau)dZ(\tau) = ... \\ ... &= x_0(t) + \int_0^{s_1} (Vx_1)(\tau)dZ(\tau) + ... + \int_{s_{k-2}}^{s_{k-1}} (Vx_{k-1})(\tau)dZ(\tau) + \int_{s_{k-1}}^t (Vx_k)(\tau)dZ(\tau). \end{aligned}$$

We have already proved that $x_k(t) = x(t)$ if $t \in [0, s_{k-1}]$. On the other hand, $x_n(t) = x(t)$ for $t \in [0, s_n]$, $n = 1, ..., k-1$. Hence $x_k(t) = x_n(t)$ for $t \in [0, s_n]$, $n = 1, ..., k-1$ and we obtain

$$\begin{aligned} x_k(t) &= x_0(t) + \int_0^{s_1} (Vx_k)(\tau)dZ(\tau) + ... + \int_{s_{k-2}}^{s_{k-1}} (Vx_k)(\tau)dZ(\tau) + \int_{s_{k-1}}^t (Vx_k)(\tau)dZ(\tau) \\ &= x_0(t) + \int_0^t (Vx_k)(\tau)dZ(\tau) \qquad (t \in [s_{k-1}, s_k]). \end{aligned}$$

Again uniqueness of solutions gives the equality $x_k(t) = x(t)$ for $t \in [s_{k-1}, s_k]$, and the induction is completed.

In particular, $x_l(t) = x(t) = x(t, x_0, \varphi)$ for $0 \leq t \leq s_l$ and hence for $0 \leq t \leq s_l$, as $s \leq s_l$. If we now show that $\sup\limits_{t \geq 0}(\mathbb{E}|x_k(t)|^{2p})^{1/2p} < \infty$ for $k = 1, ..., l$ and any φ, x_0 such that $\operatorname*{vraisup}\limits_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$, then it would give us $\sup\limits_{0 \leq t \leq s} \mathbb{E}|x(t, x_0, \varphi)|^{2p} < \infty$ for any φ, x_0 such that $\operatorname*{vraisup}\limits_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$.

To see it, we first of all prove that $\sup\limits_{t\geq 0}(\mathbb{E}|x_0(t)|^{2p})^{1/2p}<\infty$. Let $\chi_h(t)$ be a function on $[0,\infty)$ defined by $\chi_h(t)=\begin{cases}0,\, if\, h(t)\geq 0,\\ 1,\, if\, h(t)<0.\end{cases}$ Put $\bar{a}_{ij}(t)=\chi_{h_{ij}}(t)a_{ij}(t)$ for $i=1,...,m,\ j=0,...m_i$. Evidently, $x_0(t)=x_0+\sum\limits_{j=0}^{m_1}\int\limits_0^t \bar{a}_{1j}(\tau)\varphi_{h_{1j}}(\tau)d\tau+\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i}\int\limits_0^t \bar{a}_{ij}(\tau)\varphi_{h_{ij}}(\tau)d\mathcal{B}_i(\tau)$.

Thus we obtain

$$\sup_{t\geq 0}(\mathbb{E}|x_0(t)|^{2p})^{1/2p}\leq(\mathbb{E}|x_0|^{2p})^{1/2p}+\sum_{j=0}^{m_1}\sup_{t\geq 0}(\mathbb{E}|\int\limits_0^t\bar{a}_{1j}(\tau)\varphi_{h_{1j}}(\tau)d\tau|^{2p})^{1/2p}+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i}\sup_{t\geq 0}(\mathbb{E}|\int\limits_0^t\bar{a}_{ij}(\tau)\varphi_{h_{ij}}(\tau)d\mathcal{B}_i(\tau)|^{2p})^{1/2p}\leq$$

$$(\mathbb{E}|x_0|^{2p})^{1/2p}+\sum_{j=0}^{m_1}\sup_{t\geq 0}((\int\limits_0^t|\bar{a}_{1j}(\tau)|d\tau)^{(2p-1)/2p}(\mathbb{E}\int\limits_0^t|\bar{a}_{1j}(\tau)||\varphi_{h_{1j}}(\tau)|^{2p}d\tau)^{1/2p})+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i}c_p\sup_{t\geq 0}(\mathbb{E}(\int\limits_0^t(\bar{a}_{ij}(\tau)\varphi_{h_{ij}}(\tau))^2d\tau)^p)^{1/2p}\leq(\mathbb{E}|x_0|^{2p})^{1/2p}+$$

$$\sum_{j=0}^{m_1}\sup_{t\geq 0}\int\limits_0^t|\bar{a}_{1j}(\tau)|d\tau\ \underset{t\geq 0}{\mathrm{vraisup}}\ (\mathbb{E}|\varphi_{h_{1j}}(t)|^{2p})^{1/2p}+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i}c_p\sup_{t\geq 0}((\int\limits_0^t(\bar{a}_{ij}(\tau))^2d\tau)^{(p-1)/2p}(\mathbb{E}\int\limits_0^t(\bar{a}_{ij}(\tau))^2|\varphi_{h_{ij}}(\tau)|^{2p}d\tau)^{1/2p})\leq$$

$$(\mathbb{E}|x_0|^{2p})^{1/2p}+(\sum_{j=0}^{m_1}\sup_{t\geq 0}\int\limits_0^t|\bar{a}_{1j}(\tau)|d\tau+\sum_{i=2}^{m}\sum_{j=0}^{m_i}c_p\sup_{t\geq 0}(\int\limits_0^t(\bar{a}_{ij}(\tau))^2d\tau)^{0.5})\ \underset{\nu<0}{\mathrm{vraisup}}\ (\mathbb{E}|\varphi_(\nu)|^{2p})^{1/2p}.$$

From this and from the assumptions **1), 2), 3)** on the parameters in (1), (2) we conclude that $\sup\limits_{t\geq 0}(\mathbb{E}|x_0(t)|^{2p})^{1/2p}<\infty$ for any φ, x_0 such that $\underset{\nu<0}{\mathrm{vraisup}}\,\mathbb{E}|\varphi(\nu)|^{2p}<\infty$, $\mathbb{E}|x_0|^{2p}<\infty$.

From (3) we obtain

$$\sup_{t\geq 0}(\mathbb{E}|x_k(t)|^{2p})^{1/2p}\leq\sup_{t\geq 0}(\mathbb{E}|x_{k-1}(t)|^{2p})^{1/2p}+\sum_{j=0}^{m_1}\sup_{t\geq 0}(\mathbb{E}|\int\limits_0^t I_k(\tau)a_{1j}(\tau)(S_{h_{1j}}x_k)(\tau)d\tau|^{2p})^{1/2p}+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i}\sup_{t\geq 0}(\mathbb{E}|\int\limits_0^t I_k(\tau)a_{ij}(\tau)(S_{h_{ij}}x_k)(\tau)d\mathcal{B}_i(\tau)|^{2p})^{1/2p}\leq$$

$$\sup_{t\geq 0}(\mathbb{E}|x_{k-1}(t)|^{2p})^{1/2p}+\sum_{j=0}^{m_1}\sup_{t\geq 0}((\int\limits_0^t|I_k(\tau)a_{1j}(\tau)|d\tau)^{(2p-1)/2p}(\mathbb{E}\int\limits_0^t|I_k(\tau)a_{1j}(\tau)||(S_{h_{1j}}x_k)(\tau)|^{2p}d\tau)^{1/2p})+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i}c_p\sup_{t\geq 0}(\mathbb{E}(\int\limits_0^t(I_k(\tau)a_{ij}(\tau)(S_{h_{ij}}x_k)(\tau))^2d\tau)^p)^{1/2p}\leq$$

$$\sup_{t\ge 0}(\mathbb{E}|x_{k-1}(t)|^{2p})^{1/2p} + \sum_{j=0}^{m_1}\sup_{t\ge 0}\int_0^t |I_k(\tau)a_{1j}(\tau)|d\tau \sup_{t\ge 0}(\mathbb{E}|(S_{h_{1j}}x_k)(t)|^{2p})^{1/2p}+$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i} c_p \sup_{t\ge 0}((\int_0^t (I_k(\tau)a_{ij}(\tau))^2 d\tau)^{(p-1)/2p})(\mathbb{E}\int_0^t (I_k(\tau)a_{ij}(\tau))^2 |(S_{h_{ij}}x_k)(\tau)|^{2p} d\tau)^{1/2p}) \le$$

$$\sup_{t\ge 0}(\mathbb{E}|x_{k-1}(t)|^{2p})^{1/2p} + (\sum_{j=0}^{m_1}\sup_{t\ge 0}\int_0^t |I_k(\tau)a_{1j}(\tau)|d\tau +$$

$$\sum_{i=2}^{m}\sum_{j=0}^{m_i} c_p \sup_{t\ge 0}(\int_0^t (I_k(\tau)a_{ij}(\tau))^2 d\tau)^{0.5}) \sup_{t\ge 0}(\mathbb{E}|x_k(t)|^{2p})^{1/2p} \le \sup_{t\ge 0}(\mathbb{E}|x_{k-1}(t)|^{2p})^{1/2p} + \rho \sup_{t\ge 0}(\mathbb{E}|x_k(t)|^{2p})^{1/2p},$$

where $\rho < 1$ in accordance with the choice of s_k $(k = 1, ..., l)$.

This implies that for any φ, x_0 such that $\operatorname{vraisup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$ one has $\sup_{t\ge 0}(\mathbb{E}|x_k(t))|^{2p})^{1/2p} < \infty$ for $k = 1, ..., l$. This is exactly what we needed. By this, the theorem is proved.

Let T be an arbitrary number from the interval $[0, \infty)$. In addition to (1), (2) let us consider the following equation

$$dy(t) = \sum_{j=0}^{m_1} a_{1j}(t)y(h_{1j}(t))dt + \sum_{i=2}^{m}\sum_{j=0}^{m_i} a_{ij}(t)y(h_{ij}(t))d\mathcal{B}_i(t)\ (t \ge T), \tag{4}$$

$$y(\nu) = \psi(\nu)\ (\nu < T), \tag{5}$$

where $\psi(\nu)$ $(\nu < T)$ is a measurable stochastic process which is independent of the Wiener process $\mathcal{B}(t) = (B_2(t), ..., B_m(t))$, $(t \ge T)$ and which a. s. has trajectories from L^∞. Note that the other parameters are the same as in (1), (2). The following result will be used in the next section.

Theorem 2. *The zero solution of the homogeneous equation corresponding to (1), (2) is exponentially $2p$–stable w.r.t. the initial function (i.e. exponentially Lyapunov $2p$–stable) if for some $T \in [0, \infty)$ the zero solution of the homogeneous equation corresponding to (4), (5) is exponentially $2p$–stable w.r.t. the initial function.*

Proof. Let $y(t, y_0, \psi)$ be solution of (4), (5) satisfying $y(T, y_0, \psi) = y_0$.

Assume that the zero solution $y(t, 0, 0)$ of the homogeneous equation corresponding to (4), (5) is exponentially $2p$–stable w.r.t. the initial function. This means that there exist positive numbers $\hat{c}, \beta_1$, for which $\mathbb{E}|y(t, y_0, \psi)|^{2p} \le \hat{c}(\mathbb{E}|y_0|^{2p} + \operatorname{vraisup}_{\nu<T} \mathbb{E}|\psi(\nu)|^{2p}) \exp\{-\beta_1 t\}\ (t \ge T)$. We are to show that $\mathbb{E}|x(t, x_0, \varphi)|^{2p} \le \bar{c}(\mathbb{E}|x_0|^{2p} + \operatorname{vraisup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p}) \exp\{-\beta t\}\ (t \ge 0)$ for some positive constants $\bar{c}, \beta$. In the equation (4), (5) we define $\psi(\nu)$ to be $\varphi(\nu)$ if $\nu < 0$ and $x(\nu, x_0, \varphi)$ if $0 \le \nu < T$. Then taking $y_0 = x(T, x_0, \varphi)$ implies $x(t, x_0, \varphi) = y(t, y_0, \psi)$ for $t \ge T$. From Theorem 1 we obtain $\mathbb{E}|y_0|^{2p} < \infty$, $\operatorname{vraisup}_{\nu<T} \mathbb{E}|\psi(\nu)|^{2p} < \infty$ for any φ, x_0 such that $\operatorname{vraisup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p} < \infty$, $\mathbb{E}|x_0|^{2p} < \infty$. This and the exponential $2p$–stability of the zero solution $y(t, 0, 0)$ w.r.t. the initial function yields the exponential $2p$–stability of the zero solution $x(t, 0, 0)$ of the equation (1), (2) w.r.t. the initial function.

The theorem is proved.

4. The main theorem

We start with a brief description of the W-transform method referring the reader to the paper [6] for further details. However, one of the results from [6] will be formulated explicitly (in a simplified form adjusted to our objectives).

We have already mentioned that any W-transform comes from an auxiliary equation, which is called *a reference equation*. Like in the Lyapunov function(al) approach, there is no precise algorithm describing how to choose the reference equation. Roughly speaking, this is an equation, which is similar to the equation in question, but "simpler". In addition, the reference equation must a priori have the asymptotic properties which we want the original equation to have as well.

Below we again use the notation $Z(t) = (t, \mathcal{B}_2(t), ..., \mathcal{B}_m(t))^T$. The stochastic process $Z(t)$ is a continuous m-dimensional semimartingale (see e. g. [1]). The equation (1), (2) can then be rewritten as follows:

$$dx(t) = [(Vx)(t) + f(t)]dZ(t) \quad (t \geq 0),$$

To do it, we use the notation from Section 3:

$$(Vx)(t) = ((V_1x)(t), ..., (V_mx)(t)), \quad (V_ix)(t) = \sum_{j=0}^{m_i} a_{ij}(t)(S_{h_{ij}}x)(t) \quad (i = 1, ...m),$$

$$f(t) = (f_1(t), ..., f_m(t)), \quad f_i(t) = \sum_{j=0}^{m_i} a_{ij}(t)\varphi_{h_{ij}}(t) \quad (i = 1, ...m).$$

Let the reference equation have the form

$$dx(t) = [(Qx)(t) + g(t)]dZ(t) \quad (t \geq 0),$$

where Q is a K-linear Volterra operator (see e. g. [6] where the notation k was used instead of K).

Assume further that the reference equation admits "the Cauchy representation" $x(t) = U(t)x(0) + (Wg)(t)$ $(t \geq 0)$, where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy (also called Green's) operator. This representation gives rise to the W-transform, which is applied to the original equation in the following manner:

$$dx(t) = [(Qx)(t) + ((V - Q)x)(t) + f(t)]dZ(t) \quad (t \geq 0),$$

or, alternatively,

$$x(t) = U(t)x(0) + (W(V - Q)x)(t) + (Wf)(t) \quad (t \geq 0).$$

Denoting $W(V - Q) = \Theta$, we obtain the operator equation

$$((I - \Theta)x)(t) = U(t)x(0) + (Wf)(t) \quad (t \geq 0).$$

Put

$$M_{2p,T} \equiv \{x : x \in C, ||x||_{M_{2p,T}} := (\sup_{t \geq T} \mathbb{E}|x(t)|^p)^{1/2p} < \infty\},$$

where C denotes the set of all $(\mathcal{F}_t)_{t \geq T}$-adapted stochastic processes with a. s. continuous trajectories. The norm in this linear space is defined by $\sup_{t \geq T}(\mathbb{E}|x(t)|^{2p})^{1/2p}$. The main idea of the W-transform approach is to check invertibility of the operator $I - \Theta$ in the space $M_{2p} \equiv M_{2p,0}$ with the norm $||x||_{M_{2p,0}} \equiv ||x||_{M_{2p}}$, as the theory [6] says that this would imply exponential $2p$-stability.

To justify this reduction, we need some additional hypotheses.

R1. The fundamental solution $U(t)$ satisfies $|U(t)| \leq \bar{c}$, where $\bar{c} \in \mathbf{R}_+$.

R2. The W-operator is integral:

$$(Wg)(t) = \int_0^t C(t, s)g(s)dZ(s) \quad (t \geq 0),$$

where the function $C(t, s)$ is defined on $G := \{(t, s) : t \in [0, \infty),\ 0 \leq s \leq t\}$, and satisfies

$$|C(t, s)| \leq \bar{\bar{c}} \exp\{-\alpha(t - s)\}$$

for some $\alpha > 0, \bar{\bar{c}} > 0$.

Remark 4. *In our paper* [5] *we showed that asymptotic properties of linear stochastic equations can be deduced, as in the deterministic case, from the property of admissibility of certain pairs of functional spaces. We exploit this idea, however somewhat implicitly, in the present paper as well. This partly explains why we claim that the reference equation should have the same asymptotic properties as the equation to be studied,* *while obviously the estimate analogous to (EXP) is not valid for an arbitrary $g(t)$ in the reference equation. In fact, we mean that as soon as perturbations $f(t)$ and $g(t)$ are of the same kind, then the asymptotic properties of the two equations will be the same, too. For instance, if we assume that $g(t)$ is similar to $f(t)$ above, i.e. $g(t)$ has a compact support, then the solutions of the reference equation would satisfy a similar exponential estimate. Likewise, if $f(t)$ is as general as $g(t)$ is assumed to be, then the solutions of neither equation would satisfy the estimate (EXP) or simliar, in general.*

Lemma *Let $Z(t) = (t, \mathcal{B}_2(t), ..., \mathcal{B}_m(t))^T$, the conditions* **1)-3)** *be fulfilled and the reference equation satisfy the hypotheses* **R1**-**R2**. *Assume that the operator $(I - \Theta) : M_{2p} \to M_{2p}$ has a bounded inverse. Then the solutions to the equation (1), (2) satisfies the estimate (EXP) for some $\beta > 0$.*

Proof. The lemma is just a particular case of our previous result [6, Corollary 4.2], where we should put $\xi = 1$ and observe that Condition D2 follows from our assumption **3)**.

Remark 5. *The above lemma holds true if we replace the interval $[0, \infty)$ with the interval $[T, \infty)$ where $T > 0$. Accordingly, we should also replace $M_{2p} = M_{2p,0}$ with $M_{2p,T}$. This observation will be used in the course of the proof of Theorem 3 below.*

We let $h^T(t)$ be a function on $[T, \infty)$, which is defined for a given function $h(t)(t \in [T, \infty))$ in the following way

$$h^T(t) = \begin{cases} h(t), \, if \, h(t) \geq T, \\ T, \, if \, h(t) < T. \end{cases}$$

Theorem 3. *Let for some subset $I \subset \{0, ..., m_1\}$ and some numbers $a_0 > 0$, $\gamma_i > 0, i = 1, 2$, $T \in [0, \infty)$ the estimates $\sum\limits_{k \in I} a_{1k}(t) \leq -a_0$, $\sum\limits_{k \in I} |a_{1k}(t)| [\sum\limits_{j=0}^{m_1} \int\limits_{h_{1k}^T(t)}^{t} |a_{1j}(s)| ds + c_p \sum\limits_{i=2}^{m} \sum\limits_{j=0}^{m_i} (\int\limits_{h_{1k}^T(t)}^{t} (a_{ij}(s))^2 ds)^{0.5}] + \sum\limits_{k \bar{\in} I} |a_{1k}(t)| \leq -\gamma_1 \sum\limits_{k \in I} a_{1k}(t)$, $\sum\limits_{i=2}^{m} \sum\limits_{j=0}^{m_i} (a_{ij}(t))^2 \leq -2\gamma_2 \sum\limits_{k \in I} a_{1k}(t)$ be valid for $t \geq T$ a.s. If, in addition, $\gamma_1 + c_p \sqrt{\gamma_2} < 1$, where c_p are specified in Section 2, then the solutions to the equation (1), (2) satisfy the estimate (EXP) for some $\beta > 0$.*

Proof. First of all, we notice that without loss of generality we may assume that the 'prehistory' process $\varphi(\nu)$ $(\nu < 0)$ is $\mathcal{F}_0$-measurable. Indeed, it is independent of the Wiener process in (1), so that adding the σ-algebra generated by $\varphi(\nu)$ $(\nu < 0)$ to the filtration $(\mathcal{F}_t)_{t \geq 0}$, results in an extended filtration, under which the Wiener process preserves its properties, while $\varphi(\nu)$ $(\nu < 0)$ becomes $\mathcal{F}_0$-measurable.

Due to Theorem 2 it suffices to show that under the assumptions of the present theorem the zero solution of the homogeneous equation corresponding to the equation (4), (5) is exponentially $2p$–stable w.r.t. the initial function for some $T \in [0, \infty)$.

To do this, we make use of Lemma above as well as Remark 4.

We introduce the operator S_h^T putting

$$(S_h^T x)(t) = \begin{cases} x(h(t)), \, if \, h(t) \geq T, \\ 0, \, if \, h(t) < T \end{cases}$$

(so that $S_h^0 = S_h$).

The reference equation is chosen as follows:

$$dy(t) = [\sum_{k \in I} a_{1k}(t) y(t) + g_0(t)] dt + \sum_{i=1}^{m} g_i(t) d\mathcal{B}_i(t) \ (t \geq T),$$

where g_0 is an $(\mathcal{F}_t)_{t \geq T}$-adapted stochastic process with a.s. locally integrable trajectories, g_i $(i = 2, ..., m)$ is an $(\mathcal{F}_t)_{t \geq T}$-adapted stochastic process with a.s. locally square-integrable trajectories, and the other parameters are defined in (1), (2). This is an ordinary differential equation with stochastic perturbations. Thus, it satisfies the assumptions **R1**–**R2** (with $\alpha = a_0$), which means that the theorem will be proved if the operator $(I - \Theta) : M_{2p,T} \to M_{2p,T}$ is shown to be invertible.

To see this, let us write down the operator Θ explicitly:

$$(\Theta y)(t) = \int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}\{[\sum_{k\in I} a_{1k}(s) \int_{h_{1k}^T(s)}^s (\sum_{j=0}^{m_1} a_{1j}(\tau)(S_{h_{1j}}^T y)(\tau)d\tau+$$

$$\sum_{i=2}^m \sum_{j=0}^{m_i} a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau)d\mathcal{B}_i(\tau)) + \sum_{k\bar{\in} I} a_{1k}(s)(S_{h_{1k}}^T y)(s)]ds + \sum_{i=2}^m \sum_{j=0}^{m_i} a_{ij}(s)(S_{h_{ij}}^T y)(s)d\mathcal{B}_i(s)\},$$

and $M_{2p,T}$ is described right before the lemma.

Now, to prove the property of invertibility of the linear operator $I - \Theta$ we estimate the norm of the operator Θ in the space $M_{2p,T}$. We have

$$||\Theta y||_{M_{2p,T}} \leq \sup_{t\geq T}(\mathbb{E}|\int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}[\sum_{k\in I} a_{1k}(s)(\sum_{j=0}^{m_1} \int_{h_{1k}^T(s)}^s a_{1j}(\tau)(S_{h_{1j}}^T y)(\tau)d\tau+$$

$$\sum_{i=2}^m \sum_{j=0}^{m_i} \int_{h_{1k}^T(s)}^s a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau)d\mathcal{B}_i(\tau)) + \sum_{k\bar{\in} I} a_{1k}(s)(S_{h_{1k}}^T y)(s)]ds|^{2p})^{1/2p}+$$

$$c_p \sum_{i=2}^m \sum_{j=0}^{m_i} \sup_{t\geq T}(\mathbb{E}(\int_T^t \exp\{2\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(a_{ij}(s)(S_{h_{ij}^T} y)(s))^2 ds)^p)^{1/2p} \leq$$

$$\sup_{t\geq T}(\mathbb{E}|\sum_{k\in I}\sum_{j=0}^{m_1} \int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\} a_{1k}(s) \int_{h_{1k}^T(s)}^s a_{1j}(\tau)(S_{h_{1j}}^T y)(\tau)d\tau ds+$$

$$\sum_{k\in I}\sum_{i=2}^m \sum_{j=0}^{m_i} \int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\} a_{1k}(s) \int_{h_{1k}^T(s)}^s a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau)d\mathcal{B}_i(\tau)ds+$$

$$\sum_{k\bar{\in} I} \int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\} a_{1k}(s)(S_{h_{1k}}^T y)(s)ds|^{2p})^{1/2p}+$$

$$c_p \sum_{i=2}^m \sum_{j=0}^{m_i} \sup_{t\geq T}((\int_T^t \exp\{2\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(a_{ij}(s))^2 ds)^{(p-1)/2p}\times$$

$$(\mathbb{E}\int_T^t \exp\{2\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(a_{ij}(s))^2|(S_{h_{ij}}^T y)(s)|^{2p}ds)^{1/2p}) \leq$$

$$\sum_{k\in I}\sum_{j=0}^{m_1} \sup_{t\geq T}((\int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)| \int_{h_{1k}^T(s)}^s |a_{1j}(\tau)|d\tau ds)^{(2p-1)/2p}\times$$

$$(\mathbb{E}\int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)|(\int_{h_{1k}^T(s)}^s |a_{1j}(\tau)|d\tau)^{1-2p}|\int_{h_{1k}^T(s)}^s a_{1j}(\tau)(S_{h_{1j}} y)(\tau)d\tau|^{2p}ds)^{1/2p})+$$

$$\sum_{k\in I}\sum_{i=2}^m \sum_{j=0}^{m_i} \sup_{t\geq T}((\int_T^t \exp\{\int_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)|(\int_{h_{1k}^T(s)}^s |a_{ij}(\tau)|^2 d\tau)^{0.5} ds)^{(2p-1)/2p}\times$$

$$(\mathbb{E}\int\limits_T^t \exp\{\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)|(\int\limits_{h_{1k}^T(s)}^s |a_{ij}(\tau)|^2 d\tau)^{0.5-p}|\int\limits_{h_{1k}^T(s)}^s a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau)d\mathcal{B}_i(\tau)|^{2p}ds)^{1/2p})+$$

$$\sum_{k\bar{\in} I}\sup_{t\geq T}((\int\limits_T^t \exp\{\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)|ds)^{(2p-1)/2p}\times$$

$$(\mathbb{E}\int\limits_T^t \exp\{\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}|a_{1k}(s)|\,|(S_{h_{1k}}^T y)(s)|^{2p}ds)^{1/2p})+$$

$$c_p\sum_{i=2}^m\sum_{j=0}^{m_i}\sup_{t\geq T}((\int\limits_T^t \exp\{2\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(a_{ij}(s))^2 ds)^{(p-1)/2p}\times$$

$$(\mathbb{E}\int\limits_T^t \exp\{2\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(a_{ik}(s))^2|(S_{h_{ij}}^T y)(s)|^{2p}ds)^{1/2p}).$$

Taking into account that

$$\mathbb{E}|\int\limits_{h_{1k}^T(s)}^s a_{1j}(\tau)(S_{h_{1j}}^T y)(\tau)d\tau|^{2p} \leq (\int\limits_{h_{1k}^T(s)}^s |a_{1j}(\tau)|d\tau)^{2p-1}\mathbb{E}\int\limits_{h_{1k}^T(s)}^s |a_{1j}(\tau)|\,|(S_{h_{1j}}^T y)(\tau)|^{2p}d\tau \leq$$

$$(\int\limits_{h_{1k}^T(s)}^s |a_{1j}(\tau)|d\tau)^{2p}(||y||_{M_{2p,T}})^{2p}$$

for $k\in I$, $j=0,...,m_1$ and

$$\mathbb{E}|\int\limits_{h_{1k}^T(s)}^s a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau)d\mathcal{B}_i(\tau)|^{2p} \leq c_p\mathbb{E}(\int\limits_{h_{1k}^T(s)}^s (a_{ij}(\tau)(S_{h_{ij}}^T y)(\tau))^2 d\tau)^p \leq$$

$$c_p(\int\limits_{h_{1k}^T(s)}^s (a_{ij}(\tau))^2 d\tau)^{p-1}\mathbb{E}\int\limits_{h_{1k}^T(s)}^s (a_{ij}(\tau))^2|(S_{h_{ij}}^T y)(\tau)|^{2p}d\tau \leq (\int\limits_{h_{1k}^T(s)}^s (a_{ij}(\tau))^2 d\tau)^p(||y||_{M_{2p,T}})^{2p}$$

for $k\in I$, $i=2,...,m$, $j=0,...,m_i$ we obtain

$$||\Theta y||_{M_{2p,T}} \leq [\sup_{t\geq T}\int\limits_T^t \exp\{\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(\sum_{k\in I}|a_{1k}(s)|[\sum_{j=0}^{m_1}\int\limits_{h_{1k}^T(s)}^s |a_{1j}(\tau)|d\tau+$$

$$c_p\sum_{i=2}^m\sum_{j=0}^{m_i}(\int\limits_{h_{1k}^T(s)}^s (a_{ij}(\tau))^2 d\tau)^{0.5}]+\sum_{k\bar{\in} I}|a_{1k}(s)|)ds+$$

$$c_p\sup_{t\geq T}(\int\limits_T^t \exp\{2\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}\sum_{i=2}^m\sum_{k=0}^{m_i}(a_{ik}(s))^2 ds)^{0.5}]||y||_{M_{2p,T}} \leq$$

$$[\sup_{t\geq T}\int\limits_T^t \exp\{\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(-\gamma_1\sum_{k\in I} a_{1k}(s))ds+$$

$$c_p \sup_{t\geq T}(\int\limits_T^t \exp\{2\int\limits_s^t (\sum_{k\in I} a_{1k}(\tau))d\tau\}(-2\gamma_2 \sum_{k\in I} a_{1k}(s))ds)^{0.5}]]||y||_{M_{2p,T}} \leq [\gamma_1 + c_p\sqrt{\gamma_2}]||y||_{M_{2p,T}}.$$

As we assumed that $\gamma_1 + c_p\sqrt{\gamma_2} < 1$, the estimate $||\Theta||_{M_{2p,T}} < 1$ is proved, and so is the theorem.

In the last section we show how Theorem 3 provides sufficient conditions for exponential $2p$–stability of the zero solution to specific stochastic equations.

5. Some corollaries

In the corollaries below we again use the universal constants c_p $(1 \leq p < \infty)$ which are described in [1, p. 65].

Corollary 1. *Let $h_{10}(t) \equiv t$ for $t \in [0,\infty)$. Assume that there exist numbers $a_0 > 0$, $\gamma_i > 0, i = 1, 2$, $T \in [0,\infty)$ such that $a_{10}(t) \leq -a_0$, $\sum\limits_{j=1}^{m_1} |a_{1j}(t)| \leq -\gamma_1 a_{10}(t)$, $\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i}(a_{ij}(t))^2 \leq -2\gamma_2 a_{10}(t)$ for $t \geq T$ a.s. If, in addition, $\gamma_1 + c_p\sqrt{\gamma_2} < 1$, then the solutions to the equation (1), (2) satisfy the estimate (EXP) for some $\beta > 0$.*

Corollary 1 follows directly from Theorem 3 if we put $I = \{0\}$.

Corollary 2. *Assume that there exist numbers $a_0 > 0$, $\gamma_i > 0, i = 1, 2$, $T \in [0,\infty)$ such that $\sum\limits_{k=0}^{m_1} a_{1k}(t) \leq -a_0$, $\sum\limits_{k=0}^{m_1} |a_{1k}(t)|[\sum\limits_{j=0}^{m_1}\int\limits_{h_{1k}^T(t)}^{t} |a_{1j}(s)|ds + c_p\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i}(\int\limits_{h_{1k}^T(t)}^{t} (a_{ij}(s))^2 ds)^{0.5}] \leq -\gamma_1\sum\limits_{k=0}^{m_1} a_{1k}(t)$, $\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i}(a_{ij}(t))^2 \leq -2\gamma_2\sum\limits_{k=0}^{m_1} a_{1k}(t)$ for $t \geq T$ a.s. If now $\gamma_1 + c_p\sqrt{\gamma_2} < 1$. Then the solutions to the equation (1), (2) satisfy the estimate (EXP) for some $\beta > 0$.*

Corollary 2 follows directly from Theorem 3 if we put $I = \{0, ..., m_1\}$.

Corollary 3. *Assume that there exists a number $T \in [0,\infty)$ such that the coefficients in (1), (2) satisfy*

$$a_{1j}(t) = A_{1j}r(t), k = 0, ..., m_1, a_{ij}(t) = A_{ij}\sqrt{r(t)}, i = 2, ..., m, j = 0, ..., m_i,$$

$$r(t) \geq r_0 > 0 \ (t \in [T,\infty)) \quad a.s.$$

Assume also that for some $I \subset \{0, ..., m_1\}$ the estimates $\sum\limits_{k\in I} A_{1k} < 0$ and $\gamma_1 + c_p\sqrt{\gamma_2} < 1$ are valid, where $\gamma_1 = \lim\limits_{t\to\infty}\sup\limits_{T\leq\tau\leq t}[\sum\limits_{k\in I}|A_{1k}[\int\limits_{h_{1k}^T(\tau)}^{\tau} r(s)ds\sum\limits_{j=0}^{m_1}|A_{1j}| + c_p(\int\limits_{h_{1k}^T(\tau)}^{\tau} r(s)ds)^{0.5}\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i}|A_{ij}|] + \sum\limits_{k\bar{\in} I}|A_{1k}]/(-\sum\limits_{k\in I} A_{1k})$, $\gamma_2 = (\sum\limits_{i=2}^{m}\sum\limits_{j=0}^{m_i} A_{ij}^2)/(-2\sum\limits_{k\in I} A_{1k})$. Then the solutions to the equation (1), (2) satisfy the estimate (EXP) for some $\beta > 0$.

The next two corollaries provide sufficient stability conditions for a particular case of (1), (2) given by the following scalar stochastic differential equation

$$dx(t) = (-a(t)x(t) - b(t)x(h(t)))dt + c(t)x(g(t))d\mathcal{B}(t) \ (t \geq 0), \tag{6}$$

$$x(\nu) = \varphi(\nu) \ (\nu < 0), \tag{7}$$

where φ is a stochastic process which is independent of the standard scalar Wiener process $\mathcal{B}$ and which a. s. has trajectories from L^∞. The functions a, b, c, g, h in (6) are all Lebesgue-measurable, a, b are, in addition, locally integrable, c is locally square-integrable, $h(t) \leq t$, $g(t) \leq t$ for $t \in [0,\infty)$ a.s., $\operatorname{vraisup}\limits_{t\geq 0}(t - h(t)) < \infty$, $\operatorname{vraisup}\limits_{t\geq 0}(t - h(t)) < \infty$.

Corollary 4. *Assume that there exist numbers $a_0 > 0$, $\gamma_i > 0, i = 1, 2$, $T \in [0,\infty)$ such that for the equation (6), (7) one of the following conditions holds:*

$$1)\ a(t) \geq a_0, |b(t)| \leq \gamma_1 a(t), (c(t))^2 \leq 2\gamma_2 a(t) \ (t \geq T) \quad a.s.,$$

$$2)\, a(t)+b(t)\geq a_0, |b(t)|[\int\limits_{h^T(t)}^{t}(|a(s)|+|b(s)|)ds+c_p(\int\limits_{h^T(t)}^{t}(c(s))^2ds)^{0.5}]\leq$$

$$\gamma_1(a(t)+b(t)), (c(t))^2\leq 2\gamma_2(a(t)+b(t))\ (t\geq T)\quad a.s.$$

If, in addition, $\gamma_1+c_p\sqrt{\gamma_2}<1$, *then the solutions to the equation (6), (7) satisfy the estimate (EXP) for some* $\beta>0$.

Corollary 5. *Assume that there exists a number* $T\in[0,\infty)$ *such that the coefficients in (6), (7) satisfy*

$$a(t)=Ar(t), b(t)=Br(t), c(t)=C\sqrt{r(t)}, r(t)\geq r_0>0\ (t\in[T,\infty))\quad a.s.$$

Assume also that one of the following conditions holds:

$$1)\, A>0, |B|/A+c_p|C|/\sqrt{2A}<1,$$

$$2)\, A+B>0, \lim_{t\to\infty}\sup_{T\leq\tau\leq t}|B|[\int\limits_{h^T(\tau)}^{\tau}r(s)ds(|A|+|B|)+c_p(\int\limits_{h^T(\tau)}^{\tau}r(s)ds)^{0.5}|C|]/(A+B)+c_p|C|/\sqrt{2(A+B)}<1,$$

$$3)\, A>0, B>0, \lim_{t\to\infty}\sup_{T\leq\tau\leq t}B[\int\limits_{h^T(\tau)}^{\tau}r(s)ds+c_p(\int\limits_{h^T(\tau)}^{\tau}r(s)ds)^{0.5}|C|/(A+B)]+c_p|C|/\sqrt{2(A+B)}<1.$$

Then the solutions to the equation (6), (7) satisfy the estimate (EXP) for some $\beta>0$.

Finally, we consider another particular case of the equation (1), (2) given by

$$dx(t)=(-a(t)x(t)-b(t)x(h(t))-d(t)x(l(t)))dt+c(t)x(g(t))d\mathcal{B}(t)\ (t\geq 0),\tag{8}$$

$$x(\nu)=\varphi(\nu)\ (\nu<0),\tag{9}$$

where φ is a stochastic process which is independent of the standard scalar Wiener process $\mathcal{B}$ and which a. s. has trajectories from L^∞. The functions a,b,c,d,g,h,l in (8) are all assumed to be Lebesgue-measurable, where a,b,d are, in addition, locally integrable, c is locally square-integrable, $h(t)\leq t$, $l(t)\leq t$, $g(t)\leq t$ for $t\in[0,\infty)$ a. s., $\operatorname{vraisup}_{t\geq 0}(t-h(t))<\infty$, $\operatorname{vraisup}_{t\geq 0}(t-l(t))<\infty$, $\operatorname{vraisup}_{t\geq 0}(t-g(t))<\infty$.

Corollary 6. *Assume that there exist numbers* $a_0>0$, $\gamma_i>0, i=1,2$, $T\in[0,\infty)$ *such that for the equation (8), (9) one of the following conditions holds:*

$$1)\, a(t)\geq a_0, |b(t)|+|d(t)|\leq\gamma_1 a(t), (c(t))^2\leq 2\gamma_2 a(t)\ (t\geq T)\quad a.s.,$$

$$2)\, a(t)+b(t)\geq a_0, |b(t)|[\int\limits_{h^T(t)}^{t}(|a(s)|+|b(s)|+|d(s)|)ds+c_p(\int\limits_{h^T(t)}^{t}(c(s))^2ds)^{0.5}]\leq$$

$$\gamma_1(a(t)+b(t)), (c(t))^2\leq 2\gamma_2(a(t)+b(t))\ (t\geq T)\ \ a.s.,$$

$$3)\, a(t)+d(t)\geq a_0, |d(t)|[\int\limits_{l^T(t)}^{t}(|a(s)|+|b(s)|+|d(s)|)ds+c_p(\int\limits_{l^T(t)}^{t}(c(s))^2ds)^{0.5}]\leq$$

$$\gamma_1(a(t)+d(t)), (c(t))^2\leq 2\gamma_2(a(t)+d(t))\ (t\geq T)\quad a.s.,$$

$$4)\, a(t)+b(t)+d(t)\geq a_0, |b(t)|[\int\limits_{h^T(t)}^{t}(|a(s)|+|b(s)|+|d(s)|)ds+c_p(\int\limits_{h^T(t)}^{t}(c(s))^2ds)^{0.5}]+$$

$$|d(t)|[\int\limits_{l^T(t)}^{t}(|a(s)|+|b(s)|+|d(s)|)ds+c_p(\int\limits_{l^T(t)}^{t}(c(s))^2ds)^{0.5}]\leq$$

$$\gamma_1(a(t)+b(t)+d(t)), (c(t))^2\leq 2\gamma_2(a(t)+b(t)+d(t))\ (t\geq T)\quad a.s.$$

If, in addition, $\gamma_1 + c_p\sqrt{\gamma_2} < 1$, then the solutions to the equation (8), (9) satisfy the estimate (EXP) for some $\beta > 0$.

Corollary 7. *Assume that there exists a number $T \in [0,\infty)$ such that the coefficients in (8), (9) satisfy*

$$a(t) = Ar(t), b(t) = Br(t), d(t) = Dr(t), c(t) = C\sqrt{r(t)}, r(t) \geq r_0 > 0 \ (t \in [T,\infty)) \ \ a.s.$$

Assume also that one of the following conditions holds:

$$1)\ A > 0, (|B| + |D|)/A + c_p|C|/\sqrt{2A} < 1,$$

$$2)\ A+B>0, \lim_{t\to\infty} \sup_{T\leq\tau\leq t} |B|[\int_{h^T(\tau)}^{\tau} r(s)ds(|A|+|B|+|D|)+c_p(\int_{h^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|]/(A+B)+c_p|C|/\sqrt{2(A+B)}<1,$$

$$3)\ A+D>0, \lim_{t\to\infty} \sup_{T\leq\tau\leq t} |D|[\int_{l^T(\tau)}^{\tau} r(s)ds(|A|+|B|+|D|)+c_p(\int_{l^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|]/(A+D)+c_p|C|/)\sqrt{2(A+D)}<1,$$

$$4)\ A+B+D > 0, \lim_{t\to\infty} \sup_{T\leq\tau\leq t} |B|[\int_{h^T(\tau)}^{\tau} r(s)ds(|A|+|B|+|D|) + c_p(\int_{h^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|]/(A+B+D)+$$

$$\lim_{t\to\infty} \sup_{T\leq\tau\leq t} |D|[\int_{l^T(\tau)}^{\tau} r(s)ds(|A|+|B|+|D|)+c_p(\int_{l^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|]/(A+B+D)+c_p|C|/\sqrt{2(A+B+D)} < 1,$$

$$5)\ A > 0, B > 0, D > 0, \lim_{t\to\infty} \sup_{T\leq\tau\leq t} B[\int_{h^T(\tau)}^{\tau} r(s)ds + c_p(\int_{h^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|/(A+B+D)]+$$

$$\lim_{t\to\infty} \sup_{T\leq\tau\leq t} D[\int_{l^T(\tau)}^{\tau} r(s)ds + c_p(\int_{l^T(\tau)}^{\tau} r(s)ds)^{0.5}|C|/(A+B+D)] + c_p|C|/\sqrt{2(A+B+D)} < 1.$$

Then the solutions to the equation (8), (9) satisfy the estimate (EXP) for some $\beta > 0$.

Acknowledgments. The authors thank an anonymous referee for the valuable comments and suggestions which helped us to improve the paper.

References

[1] R. S. Liptser and A. N. Shirjaev. Theory of Martingales. Dordrecht: Kluwer. 1989. 808 p.
[2] R. I. Kadiev. Sufficient conditions for stability of stochastic systems with aftereffect, Differential Equations. 1994. V. 30. P. 555-564 (translated from Russian).
[3] R. I. Kadiev. Existence and uniqueness of solutions in the initial value problem for stochastic functional differential equations with a driven semimartingale, Izvestija Vuzov. Matematica. 1995. V. 10. P. 35-40 (in Russian).
[4] R. I. Kadiev. Stability of solutions to stochastic functional differential equations, Habilitation thesis. Jekaterinburg. 2000. 231 p. (in Russian).
[5] R. I. Kadiev and A. V. Ponosov. Stability of stochastic functional differential equations under constantly acting perturbations, Differential Equations. 1992. V. 28. P. 198-207 (translated from Russian).
[6] R. I. Kadiev and A. V. Ponosov. Stability of stochastic functional differential equations and the W-transform, Electron. J. Diff. Eqns. 2004. V. 2004, no. 92. P. 1-36.
[7] V. B. Kolmanovskii and V. R. Nosov. Stability of Functional Differential Equations. New York: Academic Press. 1986. 217 p.
[8] X. Mao. Stochastic Differential Equations and Applications. Chichester: Horwood Publishing ltd. 1997. 360 p.
[9] S.-E. A. Mohammed. Stochastic Functional Differential Equations. Boston - London Melbourne: Pitman Advanced Publishing Program. 1984. Research notes in Mathematics, no. 99. 245 p.
[10] I. Neveu. Discrete Parameter Martingales. 1975. North-Holland, Amsterdam.
[11] E. F. Tsarkov. Random Perturbations of Functional Differential Equations. Riga: Zinatne 1989. 421 p. (in Russian).

(Received January 30, 2007)

Dagestan Scientific Center, Russian Academy of Sciences, Makhachkala 367005 (Russia)
E-mail address: dgu@dgu.ru

CIGENE & Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, NO-1432 Ås (Norway)
E-mail address: arkadi@umb.no