Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **22**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

Existence of solutions of nth order impulsive integro-differential equations in Banach spaces*

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Abstract: In this paper, we prove the existence of solutions of initial value problems for nth order nonlinear impulsive integro-differential equations of mixed type on an infinite interval with an infinite number of impulsive times in Banach spaces. Our results are obtained by introducing a suitable measure of noncompactness.
Keywords: Impulsive integro-differential equation; Measure of noncompactness; Fixed point.

1 Introduction

The branch of modern applied analysis known as "impulsive" differential equations furnishes a natural framework to mathematically describe some "jumping processes". Consequently, the area of impulsive differential equations has been developing at a rapid rate, with the wide applications significantly motivating a deeper theoretical study of the subject(see [1-3]). But most of the works in this area discussed the first- and second- order equations (see [2-7]). The theory of nth order nonlinear impulsive integro-differential equations of mixed type has received attention and has been achieved significant development in recent years (see [8-10]). For instance, D. Guo [9] and [10] have established the existence of solutions for the above nth order problems on an infinite interval with an infinite number of impulsive times in Banach spaces by means of the Schauder fixed point theorem and the fixed point index theory of completely continuous operators, respectively. However, as we show in Example 2 below, these techniques do not cover interesting cases. In this paper, we will use the technique associated with measures of noncompactness to consider the boundary value problem (BVP) for nth-order

*Supported by Natural Science Foundation of Zhejiang Province (Y607178) and China(10771048)

nonlinear impulsive integro-differential equation of mixed type as follows:

$$\begin{cases} u^{(n)}(t) = f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t)), \qquad \forall t \in J' \\ \Delta u^{(i)}|_{t=t_k} = I_{ik}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k)) \\ (i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots), \\ u^{(i)}(0) = \theta\ (i = 0, 1, \cdots, n-2),\ u^{(n-1)}(\infty) = \beta u^{(n-1)}(0), \end{cases} \tag{1}$$

where $J = [0, \infty)$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $J' = J/\{t_1, \cdots, t_k \cdots\}$, $f \in C[J \times E \times \cdots \times E \times E \times E, E]$, $I_{ik} \in C[E \times E \times \cdots \times E, E]$ $(i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots)$, $(E, \|\cdot\|)$ is a Banach space, θ stands for zero element of E (so it is in all places where it appears), $\beta > 1, u^{(n-1)}(\infty) = \lim_{t\to\infty} u^{(n-1)}(t)$ and

$$(Tu)(t) = \int_0^t K(t,s)u(s)ds, \quad (Su)(t) = \int_0^\infty H(t,s)u(s)ds \tag{2}$$

with $K \in C[D, \mathbb{R}_+], D = \{(t,s) \in J \times J : t \geq s\}$ and $H \in C[J \times J, \mathbb{R}_+]$ (here $\mathbb{R}_+$ denotes the set of all nonnegative numbers). $\Delta u^{(i)}|_{t=t_k}$ denotes the jump of $u^{(i)}(t)$ at $t = t_k$, i. e.

$$\Delta u^{(i)}|_{t=t_k} = u^{(i)}(t_k^+) - u^{(i)}(t_k^-),$$

where $u^{(i)}(t_k^+)$ and $u^{(i)}(t_k^-)$ represent the right and left limits of $u^{(i)}(t)$ at $t = t_k$, respectively $(i = 0, 1, \cdots, n-1)$. Let $PC[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \cdots\}$, $B^*PC[J, E] = \{u \in PC[J, E] : e^{-t}\|u(t)\| \to 0$ as $t \to \infty$ $\}$ and $BPC[J, E] = \{u \in B^*PC[J, E] : u$ is bounded on J with respect to the norm $\|\cdot\|$ $\}$. [10] has shown that $B^*PC[J, E]$ is a Banach space with norm

$$\|u\|_B = \sup\{e^{-t}\|u(t)\| : t \in J\}.$$

In this case, it is easy to see that $BPC[J, E]$ is also a Banach space. Let $PC^{n-1}[J, E] = \{u \in PC[J, E] : u^{(n-1)}(t)$ exists and is continuous at $t \neq t_k$, and $u^{(n-1)}(t_k^+)$, $u^{(n-1)}(t_k^-)$ exist for $k = 1, 2, \cdots\}$. For $u \in PC^{n-1}[J, E]$, as shown in [10], $u^{(i)}(t_k^+)$ and $u^{(i)}(t_k^-)$ exist and $u^{(i)} \in PC[J, E]$, where $i = 1, 2, \cdots, n-2, k = 1, 2, \cdots$. We define $u^{(i)}(t_k) = u^{(i)}(t_k^-)$. Moreover, in (1) and in what follows, $u^{(i)}(t_k)$ is understood as $u^{(i)}(t_k^-)$ $(i = 1, 2, \cdots, n-1)$. Let $DPC[J, E] = \{u \in PC^{n-1}[J, E] : u^{(i)} \in BPC[J, E], i = 1, 2, \cdots, n-1\}$, then $DPC[J, E]$ is a Banach space (see [10]) with norm

$$\|u\|_D = \max\{\|u\|_B, \|u'\|_B, \cdots, \|u^{(n-1)}\|_B\}.$$

We verify the existence of solutions to BVP(1) for which the function f does not need to be completely continuous. The idea of the present paper has originated from the study of an analogous problem examined by J. Banaś and B. Rzepka [13] for a nonlinear functional-integral equation.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from the concept of a measure of noncompactness [11-13] which are used throughout this paper.

By $B(x,r)$ we denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta,r)$.

Let X be a subset of E and $\overline{X}$, conv$X$ denote the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E is denoted by $bf(E)$. The following definition of the concept of a measure of noncompactness is due to [12].

Definition. A mapping $\gamma : bf(E) \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(I) The family $\ker\gamma = \{X \in bf(E) : \gamma(X) = 0\}$ is nonempty and each of its numbers is a relatively compact subset of E;

(II) $X \subset Y \Rightarrow \gamma(X) \leq \gamma(Y)$;

(III) $\gamma(\mathrm{conv}X) = \gamma(X)$;

(IV) $\gamma(\overline{X}) = \gamma(X)$;

(V) $\gamma(\lambda X + (1-\lambda)Y) \leq \lambda\gamma(X) + (1-\lambda)\gamma(Y)$ for some $\lambda \in [0,1]$;

(VI) If $\{X_n\}$ is a sequence of sets from $bf(E)$ such that $X_{n+1} \subset X_n, \overline{X}_n = X_n$ $(n = 1, 2, \cdots)$, and if $\lim\limits_{n\to\infty} \gamma(X_n) = 0$, then the intersection $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

Remark 1. As shown in [12], the family $\ker\gamma$ described in (I) is called the kernel of the measure of noncompactness γ. A measure γ is said to be sublinear if it satisfies the following conditions.

(VII) $\gamma(\lambda X) = |\lambda|\gamma(X)$ for $\lambda \in \mathbb{R}$;

(VIII) $\gamma(X+Y) \leq \gamma(X) + \gamma(Y)$.

For our further purposes we will need the following fixed point theorem .

Lemma 1[12]. Let Q be nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous operator such that $\gamma(FX) \leq k\gamma(X)$ for any nonempty bounded subset X of Q, where $k \in [0,1)$ is a constant. Then F has a fixed point in the set Q.

Let us recall the following special measure of noncompactness which originates from [11] and will be used in our main results.

To do this let us fix a nonempty bounded subset X of $BPC[J,E]$ and a positive number $N > 0$. For any $x \in X$ and $\varepsilon \geq 0$, $\omega^N(x,\varepsilon)$ stands for the modulus of continuity of the function x on the interval $[0,N]$, namely,

$$\omega^N(x,\varepsilon) = \sup\{\|e^{-t}x(t) - e^{-s}x(s)\| : t, s \in [0,N], |t-s| \leq \varepsilon\}.$$

Define

$$\omega^N(X,\varepsilon) = \sup[\omega^N(x,\varepsilon) : x \in X],$$

$$\omega_0^N(X) = \lim_{\varepsilon\to 0} \omega^N(X,\varepsilon), \qquad \omega_0(X) = \lim_{N\to\infty} \omega_0^N(X),$$

and

$$\mathrm{diam}X(t) = \sup\{\|e^{-t}x(t) - e^{-t}y(t)\| : x, y \in X\}$$

with $X(t) = \{x(t) : x \in X\}$ for fixed $t \geq 0$.

Now we can introduce the measure of noncompactness by the formula

$$\gamma(X) = \omega_0(X) + \lim_{t\to\infty} \sup \operatorname{diam} X(t).$$

It can be shown similar to [11] that the function γ is a sublinear measure of noncompactness on the space $BPC[J, E]$.

For the sake of convenience, we impose the following hypotheses on the functions appearing in BVP(1).

(h1) $\sup\limits_{t\in J} \left(\int_0^t K(t,s)ds\right) < \infty$, $\sup\limits_{t\in J} \left(\int_0^\infty H(t,s)ds\right) < \infty$ and there exist positive constant k^*, h^* such that

$$\sup_{t\in J}\left(e^{-t}\int_0^t K(t,s)e^s ds\right) \le k^*, \quad \sup_{t\in J}\left(e^{-t}\int_0^\infty H(t,s)e^s ds\right) \le h^*.$$

(h2) The function $t \to f(t, 0, 0, \cdots, 0, 0, 0)$ is an element of the space $BPC[J, E]$ and satisfies

$$a^* = \int_0^\infty \|f(t, 0, 0, \cdots, 0, 0, 0)\| dt < \infty.$$

There exist functions $g \in C[J, \mathsf{R}_+]$ with

$$m^* = \int_0^\infty g(t)dt < \infty$$

such that

$$\begin{aligned}&\|f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t)) -\\ &f(t, v(t), v'(t), \cdots, v^{(n-1)}(t), (Tv)(t), (Sv)(t))\| \le g(t)\|u(t)e^{-t} - v(t)e^{-t}\|\end{aligned}$$

for any $t \in J$ and $u, v \in DPC^{n-1}[J, E]$.

(h3) $I_{ik}(0, 0, \cdots, 0)$ $(i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots)$ is an element of the space $BPC[J, E]$ and satisfies

$$d_i^* = \sup_{t\in J} \sum_{k=1}^{\infty} \|I_{ik}(0, 0, \cdots, 0)\| < \infty, \quad i = 0, 1, \cdots, n-1.$$

There exist nonnegative constants c_{ik} for $i = 0, 1, \cdots, n-1$, $k = 1, 2, \cdots$ with

$$c_i^* = \sum_{k=1}^{\infty} c_{ik} < \infty, \quad i = 0, 1, \cdots, n-1$$

such that

$$\|I_{ik}(u(t), u'(t), \cdots, u^{(n-1)}(t)) - I_{ik}(v(t), v'(t), \cdots, v^{(n-1)}(t))\| \le c_{ik}\|u(t)e^{-t} - v(t)e^{-t}\|$$

for any $t \in J$, $u, v \in DPC^{n-1}[J, E]$ and $i = 0, 1, \cdots, n-1$, $k = 1, 2, \cdots$.

3 Main Results

Throughout this section we will work in the Banach space $DPC^{n-1}[J,E]$ and our considerations are placed in the Banach space $DPC^{n-1}[J,E]$ considered previously.

We say that a map $u \in PC^{n-1}[J,E] \cap C^n[J',E]$ is called a solution of BVP(1) if $u(t)$ satisfies (1) for $t \in J$.

Theorem 1. Let conditions (h1)-(h3) be satisfied. Assume that

$$\tau = \frac{\beta}{\beta-1}(m^* + c^*_{n-1}) + \sum_{j=0}^{n-2} c^*_j < 1. \tag{3}$$

Then BVP(1) has at least one solution $x = x(t)$ which belongs to the space $DPC^{n-1}[J,E]$.

Proof. Define an operator A as follows:

$$\begin{aligned}(Au)(t) \;=\;& \frac{t^{n-1}}{(\beta-1)(n-1)!}\Bigg\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\ &+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\Bigg\} \\ &+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\ &+\sum_{0<t<t_k}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!} I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)), \quad \forall t \in J. \end{aligned} \tag{4}$$

[9, Lemma 3] has proved that $u \in DPC^{n-1}[J,E] \cap C^n[J,E]$ is a solution of BVP(1) if and only if u is a fixed point of A.

In what follows, we write $J_1 = [0,t_1]$, $J_k = (t_{k-1},t_k]$ for $k = 2,3,\cdots$.

We are now in a position to prove that the operator A has a fixed point by means of Lemma 1.

In virtue of our assumptions the function Au is continuous on the interval J for each function $u \in DPC^{n-1}[J,E]$. It is obvious from the condition (h1) that the operators T and S defined by (2) are bounded linear operators from $BPC[J,E]$ into itself and

$$\|T\| \le k^*, \; \|S\| \le h^*. \tag{5}$$

Under the assumptions (h2) and (h3) [10] has proved that the infinite integral

$$\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds$$

is convergent for any $u \in DPC^{n-1}[J,E]$. Differentiating (4) i times for $i = 0,1,\cdots,n-1$, we have

$$
\begin{aligned}
(A^{(i)}u)(t) &= \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!}\left\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right.\\
&\quad\left.+\sum_{k=1}^\infty I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\right\}\\
&\quad+\frac{1}{(n-i-1)!}\int_0^t (t-s)^{n-i-1}f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\\
&\quad+\sum_{0<t_k<t}\sum_{j=i}^{n-1}\frac{(t-t_k)^{j-i}}{(j-i)!}I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)),\quad \forall t\in J.
\end{aligned}
$$

and so

$$
\begin{aligned}
\|(A^{(i)}u)(t)\| &\le \frac{\beta}{\beta-1}\cdot\frac{t^{n-i-1}}{(n-i-1)!}\int_0^\infty \|f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))\|ds\\
&\quad+\frac{t^{n-i-1}}{(\beta-1)(n-i-1)!}\sum_{k=1}^\infty \|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|\\
&\quad+\sum_{j=i}^{n-1}\frac{t^{j-i}}{(j-i)!}\sum_{0<t_k<t}\|I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|,\quad \forall t\in J.
\end{aligned}
$$

This, together with (h2) and (h3), implies that

$$
\begin{aligned}
e^{-t}\|(A^{(i)}u)(t)\| &\le \frac{\beta}{\beta-1}\int_0^\infty \|f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))\|ds\\
&\quad+\frac{1}{\beta-1}\sum_{k=1}^\infty \|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|\\
&\quad+\sum_{j=0}^{n-1}\sum_{k=1}^\infty \|I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|\\
&\le \frac{\beta}{\beta-1}\int_0^\infty [g(s)\|u(s)e^{-s}\|+\|f(s,0,\cdots,0,0,0)\|]ds\\
&\quad+\frac{1}{\beta-1}\sum_{k=1}^\infty \left[c_{n-1k}\|u(t_k)e^{-t_k}\|+\|I_{n-1k}(0,0,\cdots,0)\|\right]\\
&\quad+\sum_{j=0}^{n-1}\sum_{k=1}^\infty \left[c_{jk}\|u(t_k)e^{-t_k}\|+\|I_{jk}(0,0,\cdots,0)\|\right]\\
&\le \frac{\beta}{\beta-1}\left[\|u\|_D\int_0^\infty g(s)ds+a^*\right]\\
&\quad+\frac{1}{\beta-1}[c^*_{n-1}\|u\|_D+d^*_{n-1}]+\sum_{j=0}^{n-1}[c^*_j\|u\|_D+d^*_j]\\
&= \left[\frac{\beta}{\beta-1}(m^*+c^*_{n-1})+\sum_{j=0}^{n-2}c^*_j\right]\|u\|_D+\frac{\beta}{\beta-1}(a^*+d^*_{n-1})+\sum_{j=0}^{n-2}d^*_j. \quad (6)
\end{aligned}
$$

In view of the assumptions (h2) and (h3) we have the following estimate:

$$\|Au\|_D \leq \tau \|u\|_D + \rho \tag{7}$$

with $\rho =: \frac{\beta}{\beta-1}(a^* + d^*_{n-1}) + \sum\limits_{j=0}^{n-2} d^*_j$. We deduce from this estimate that the operator A transforms the ball B_r into itself with $r = \rho/(1-\tau)$.

In what follows we show that A is continuous on the ball B_r. In order to do this let us take $u, v \in B_r$. Then for $t \in J$ we have

$$\begin{aligned}&\|f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t)) - \\ &f(t, v(t), v'(t), \cdots, v^{(n-1)}(t), (Tv)(t), (Sv)(t))\| \leq g(t)\|u(t)e^{-t} - v(t)e^{-t}\| \leq 2rg(t).\end{aligned}$$

This and the dominated convergence theorem guarantee that

$$\begin{aligned}&\lim_{u\to v}\int_0^\infty \|f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t)) - \\ &f(t, v(t), v'(t), \cdots, v^{(n-1)}(t), (Tv)(t), (Sv)(t))\|dt = 0.\end{aligned} \tag{8}$$

Similarly, from the condition (h3) we get

$$\lim_{u\to v}\sum_{k=1}^{\infty}\|I_{ik}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k)) - I_{ik}(v(t_k), v'(t_k), \cdots, v^{(n-1)}(t_k))\| = 0. \tag{9}$$

On the other hand, Similar to (6), it is easy to see

$$\begin{aligned}\|Au - Av\|_D \;\leq\; & \frac{\beta}{\beta-1}\int_0^\infty \|f(s, u(s), u'(s), \cdots, u^{(n-1)}(s), (Tu)(s), (Su)(s)) \\ & - f(s, v(s), v'(s), \cdots, v^{(n-1)}(s), (Tv)(s), (Sv)(s))\|ds \\ & + \frac{1}{\beta-1}\sum_{k=1}^{\infty}\|I_{n-1k}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k)) - I_{n-1k}(v(t_k), v'(t_k), \\ & \cdots, v^{(n-1)}(t_k))\| + \sum_{j=0}^{n-1}\sum_{k=1}^{\infty}\|I_{jk}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k)) \\ & - I_{jk}(v(t_k), v'(t_k), \cdots, v^{(n-1)}(t_k))\|.\end{aligned} \tag{10}$$

We conclude from (8), (9) and (10) that $\|Au - Av\|_D \to 0$, i.e., A is continuous on the ball B_r.

Let us take a nonempty set $X \subset B_r$. Then, for any $u, v \in X$ and for a fixed $t \in J$, from the conditions (h2) and (h3) we have the following estimate:

$$\begin{aligned}
& e^{-t}\|(Au)(t)-(Av)(t)\| \\
\leq\ & \frac{\beta}{\beta-1}\int_0^\infty g(t)\|u(t)e^{-t}-v(t)e^{-t}\|dt+\frac{1}{\beta-1}\sum_{k=1}^\infty c_{n-1k}\|(u(t_k)e^{-t_k}-v(t_k)e^{-t_k}\| \\
& +\sum_{j=0}^{n-1}\sum_{k=1}^\infty c_{jk}\|(u(t_k)e^{-t_k}-v(t_k)e^{-t_k}\| \\
\leq\ & \frac{\beta}{\beta-1}m^*\sup\operatorname{diam}(X(t))+\frac{1}{\beta-1}c^*_{n-1}\sup\operatorname{diam}(X(t))+\sum_{j=0}^{n-1}c^*_j\sup\operatorname{diam}(X(t)) \\
\leq\ & \tau\sup\operatorname{diam}(X(t)).
\end{aligned}$$

This implies that

$$\lim_{t\to\infty}\sup\operatorname{diam}((AX)(t))\leq\tau\lim_{t\to\infty}\sup\operatorname{diam}(X(t)). \tag{11}$$

Now, let us fix arbitrarily numbers $N>0$ and $\varepsilon>0$. Choose a function $u\in X$ and take $s,t\in[0,N]$ such that $|t-s|\leq\varepsilon$. Without loss of generality we assume that $s<t$. Then, in the light of (4) we get

$$\begin{aligned}
& \|(Au)(t)e^{-t}-(Au)(s)e^{-s}\| \\
\leq\ & \left[\frac{e^{-t}t^{n-1}}{(\beta-1)(n-1)!}-\frac{e^{-s}s^{n-1}}{(\beta-1)(n-1)!}\right]\left\{\int_0^\infty\|f(h,u(h),u'(h),\cdots,u^{(n-1)}(h),(Tu)(h),\right. \\
& \left.(Su)(h))\|dh+\sum_{k=1}^\infty\|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|\right\} \\
& +\frac{1}{(n-1)!}\left\{\int_s^t(t-h)^{n-1}\|f(h,u(h),u'(h),\cdots,u^{(n-1)}(h),(Tu)(h),(Su)(h))\|dh\right. \\
& \left.+\int_0^s[(t-h)^{n-1}-(s-h)^{n-1}]\|f(h,u(h),u'(h),\cdots,u^{(n-1)}(h),(Tu)(h),(Su)(h))\|dh\right\} \\
& +\sum_{0<t_k<s}\sum_{j=0}^{n-1}\left[\frac{(t-t_k)^j}{j!}-\frac{(s-t_k)^j}{j!}\right]\|I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\| \\
& +\sum_{s<t_k<t}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!}\|I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\|.
\end{aligned} \tag{12}$$

From the conditions (h2) and (h3) it follows that

$$\begin{aligned}
& \|f(h,u(h),u'(h),\cdots,u^{(n-1)}(h),(Tu)(h),(Su)(h))\| \\
\leq\ & g(h)\|e^{-h}u(h)\|+\|f(h,0,0,\cdots,0,0,0)\| \\
\leq\ & g(h)r+\|f(h,0,0,\cdots,0,0,0)\|, \\
& \|I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\| \\
\leq\ & c_{jk}\|u(t_k)e^{-t_k}\|+\|I_{jk}(0,0,\cdots,0)\|\leq rc_{jk}+\|I_{jk}(0,0,\cdots,0)\|.
\end{aligned}$$

When we load this into (12), we obtain

$$
\begin{aligned}
&\|(Au)(t)e^{-t}-(Au)(s)e^{-s}\| \\
\leq\ & \left[\frac{e^{-t}t^{n-1}}{(\beta-1)(n-1)!}-\frac{e^{-s}s^{n-1}}{(\beta-1)(n-1)!}\right](rm^*+a^*+rc^*_{n-1}+d^*_{n-1}) \\
&+\frac{1}{(n-1)!}\left\{\int_s^t (t-h)^{n-1}[g(h)r+\|f(h,0,0,\cdots,0,0,0)\|]dh\right. \\
&\left.+\int_0^s[(t-h)^{n-1}-(s-h)^{n-1}][g(h)r+\|f(h,0,0,\cdots,0,0,0)\|dh\right\} \\
&+\sum_{0<t_k<s}\sum_{j=0}^{n-1}\left[\frac{(t-t_k)^j}{j!}-\frac{(s-t_k)^j}{j!}\right][rc_{jk}+\|I_{jk}(0,0,\cdots,0)\|] \\
&+\sum_{s<t_k<t}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!}[rc_{jk}+\|I_{jk}(0,0,\cdots,0)\|].
\end{aligned}
$$

Hence we deduce that $\omega_0^N(AX,\varepsilon)\to 0$ as $\varepsilon\to 0$, that is,

$$\omega_0(AX)=0\leq\tau\omega_0(X). \tag{13}$$

Now, combining (11) with (13), and keeping in mind the definition of the measure of noncompactness γ in the above section, we have

$$\gamma(AX)\leq\tau\gamma(X).$$

Consequently, the conditions of Lemma 1 are fulfilled and Lemma 1 guarantees that operator A has at least one fixed point in $DPC^{n-1}[J,E]$. The proof is completed.

Remark 2. Similar to [13], we can define the concept of asymptotic stability of a solution of BVP(1) on the interval J, namely, for any $\varepsilon>0$, there exist $N>0$ and $r>0$ such that if $x,y\in B_r$ and $x=x(t),y=y(t)$ are solutions of BVP(1) then $\|x(t)-y(t)\|\leq\varepsilon$ for $t\geq N$. We infer easily from the proof of Theorem 1 that any solution of BVP(1) which belongs to B_r is asymptotically stable.

Example 1. consider the infinite system of scalar third order impulsive integro-differential equations

$$
\begin{cases}
u_n'''=\frac{e^{-2t}}{20n}[1+u_{n+1}+\sin(u_n'+u_{n+2}'')]-\frac{te^{-2t}}{6n^2}\left(1-\int_0^t\frac{u_n(s)ds}{1+ts}\right)^{\frac{1}{5}} \\
+\frac{e^{-3t}}{10\sqrt{n}}\int_0^\infty e^{-2s}\cos(t-s)u_{2n}(s)ds,\ \forall t\in J,t\neq k\ (k=1,2,\cdots); \\
\Delta u_n|_{t=k}=\frac{1}{n4^{2k}}u_{n+1}(k)-\frac{1}{(n+1)^{k^2}}, \\
\Delta u_n'|_{t=k}=\frac{1}{\sqrt{n}5^{2k}}[u_n(k)-u_{2n}''(k)], \\
\Delta u_n''|_{t=k}=\frac{1}{n^25^{2k}}u_{n+2}'(k)\quad(k=1,2,\cdots) \\
u_n(0)=u_n'(0)=0,\ 2u_n''(\infty)=3u_n''(0)\quad(n=1,2,\cdots).
\end{cases} \tag{14}
$$

Conclusion. Infinite system (14) has a solution $\{u_n(t)\}$ with $u_n\in C^3[J',\mathbb{R}]$ for $n=1,2,\cdots$, where $J'=[0,\infty)/\{1,2,\cdots\}$, such that $u_n(t)\to 0$ as $n\to\infty$ for $0\leq t<\infty$ and $e^{-t}\sup_n|u_n^{(i)}(t)|\to$

0 as $t \to \infty$ $(i = 1, 2, 3)$.

In fact, let $J = [0, \infty)$, $E = C_0 = \{u = (u_1, u_2, \cdots, u_n, \cdots) : u_n \to 0\}$ with $\|u\| = \sup_n |u_n|$. Thus, (14) can be regarded as BVP of the form (1) in E. In this case, $k(t, s) = (1+st)^{-1}$, $h(t, s) = e^{-2s}\cos(t-s)$, $u = (u_1, u_2, \cdots, u_n, \cdots)$, $f = (f_1, f_2, \cdots, f_n, \cdots)$, in which

$$\begin{aligned} f_n(t, u, u', u'', Tu, Su) &= \frac{e^{-2t}}{20n}[1 + u_{n+1} + \sin(u'_n + u''_{n+2})] \\ &\quad - \frac{te^{-2t}}{6n^2}\left(1 - \int_0^t \frac{u_n(s)ds}{1+ts}\right)^{\frac{1}{5}} + \frac{e^{-3t}}{10\sqrt{n}}\int_0^\infty e^{-2s}\cos(t-s)u_{2n}(s)ds, \end{aligned}$$

and

$$\begin{aligned} I_{0kn}(u, u', u'') &= \frac{1}{n4^{2k}}u_{n+1} - \frac{1}{(n+1)^{k^2}}, \\ I_{1kn}(u, u', u'') &= \frac{1}{\sqrt{n}5^{2k}}[u_n - u''_{2n}] \\ I_{2kn}(u, u', u'') &= \frac{1}{n^2 5^{2k}}u'_{n+2}, \end{aligned}$$

where, $t_k = k$ $(k = 1, 2, \cdots)$. It is easy to see that all conditions of Theorem 1 are fulfilled, so our claim is true by Theorem 1.

Example 2. Let L and $E = C_0$ be given in Example 1. For fixed $t_0 \in J'$ and any $y \in E$, there exists obviously $x \in DPC[J, E]$ such that $x(t_0) = y$. Let us denote by $DPC[t_0, E]$ the set $\{x(t_0) : x \in DPC[J, E]\}$. Then $DPC[t_0, E] = E$. Define the function $F : DPC[J, E] \to E$ by

$$F(x(t)) = x(t_0) =: (x_1(t_0), x_2(t_0), \cdots, x_n(t_0), \cdots)$$

for $x(t) = (x_1(t), x_2(t), \cdots, x_n(t), \cdots), x \in DPC[J, E]$ and $t \in J$. F is clearly continuous but cannot be completely continuous since $F(x^n(t)) = e^n$ for $n = 1, 2, \cdots$, where $x^n(t) \equiv e^n$ and the sequence $\{e^n\}$, defined by $e^n = (e^n_1, e^n_2, \cdots, e^n_k, \cdots)$ with $e^n_k = \begin{cases} 0, & ifn \neq k, \\ 1, & ifn = k \end{cases}$, stands for a standard basis in E. Define the function $\varphi : J \times DPC[J, E] \to E$ by

$$\varphi(t, x(t)) = e^{-2t}F(x(t)).$$

Now let

$$\psi(t, x, x', x'', Tx, Sx) = \varphi(t, x) + f(t, x, x', x'', Tx, Sx),$$

where $f = (f_1, f_2 \cdots, f_n, \cdots)$ with $f_n(t, x, x', x'', Tx, Sx)$ given in Example 1. Consider equation (14) for which the corresponding function is ψ instead of f. We can prove that the operator A defined as in (4) is not compact. However, the hypotheses (h1)-(h3) are satisfied which implies that (14) has a solution under the inequality (3) holding.

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(Received April 9, 2008)