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Two positive solutions for a nonlinear four-point boundary value problem with a p-Laplacian operator *

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Abstract: In this paper, we study the existence of positive solutions for a nonlinear four-point boundary value problem with a p-Laplacian operator. By using a three functionals fixed point theorem in a cone, the existence of double positive solutions for the nonlinear four-point boundary value problem with a p-Laplacian operator is obtained. This is different than previous results.

Key words: p-Laplacian operator; Positive solution; Fixed point theorem; Four-point boundary value problem

1. Introduction

In this paper we are interested in the existence of positive solutions for the following nonlinear four-point boundary value problem with a p-Laplacian operator:

$$(\phi_p(u'))' + e(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$\mu\phi_p(u(0)) - \omega\phi_p(u'(\xi)) = 0, \quad \rho\phi_p(u(1)) + \tau\phi_p(u'(\eta)) = 0. \tag{1.2}$$

where $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{q} + \frac{1}{p} = 1, \mu > 0, \omega \geq 0, \rho > 0, \tau \geq 0, \xi, \eta \in (0, 1)$ is prescribed and $\xi < \eta, e : (0, 1) \to [0, \infty), f : [0, +\infty) \to [0, +\infty)$.

In recent years, because of the wide mathematical and physical background [1,2,12], the existence of positive solutions for nonlinear boundary value problems with p-Laplacian has received wide attention. There exists a very large number of papers devoted to the existence of solutions of the p-Laplacian operators with two or three-point boundary conditions, for example,

$$u(0) = 0, \quad u(1) = 0,$$

$$u(0) - B_0(u'(0)) = 0, \quad u(1) + B_1(u'(1)) = 0,$$

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$$u(0)-B_0(u'(0))=0,\quad u'(1)=0,$$
$$u'(0)=0,\quad u(1)+B_1(u'(1))=0,$$

and

$$u(0)=0,\quad u(1)=u(\eta),$$
$$u(0)-B_0(u'(\eta))=0,\quad u(1)+B_1(u'(1))=0,$$
$$u(0)-B_0(u'(0))=0,\quad u(1)+B_1(u'(\eta))=0,$$
$$au(r)-bp(r)u'(r)=0,\quad cu(R)+dp(R)u'(R)=0.$$

For further knowledge, see [3-11,13]. The methods and techniques employed in these papers involve the use of Leray-Shauder degree theory [4], the upper and lower solution method [5], fixed point theorem in a cone [3,6-8,10,11,13], and the quadrature method [9]. However, there are several papers dealing with the existence of positive solutions for four-point boundary value problem [13-15,18].

Motivated by results in [14], this paper is concerned with the existence of two positive solutions of the boundary value problem (1.1)-(1.2). Our tool in this paper will be a new double fixed point theorem in a cone [11,16,17,19] . The result obtained in this paper is essentially different from the previous results in [14].

In the rest of the paper, we make the following assumptions:

(H1) $f\in C([0,+\infty),[0,+\infty))$;

(H2) $e(t)\in C((0,1),[0,+\infty))$, and $0<\int_0^1 e(t)dt<\infty$. Moreover, $e(t)$ does not vanish identically on any subinterval of $(0,1)$.

Define

$$f_0=\lim_{u\to 0^+}\frac{f(u)}{u^{p-1}},\quad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u^{p-1}}.$$

2. Some background definitions

In this section we provide some background material from the theory of cones in Banach space, and we state a two fixed point theorem due to Avery and Henderson [19].

If $P\subset E$ is a cone, we denote the order induced by P on E by $\le$. That is

$$x\le y \quad \text{if and only if} \quad y-x\in P.$$

Definition 2.1 Given a cone P in a real Banach spaces E, a functional $\psi:P\to R$ is said to be increasing on P, provided $\psi(x)\le\psi(y)$, for all $x,y\in P$ with $x\le y$.

Definition 2.2 Given a nonnegative continuous functional γ on a cone P of a real Banach space E(i.e., $\gamma:P\to[0,+\infty)$ continuous), we define, for each $d>0$, the set

$$P(\gamma,d)=\{x\in P|\gamma(x)<d\}.$$

In order to obtain multiple positive solutions of (1.1)-(1.2), the following fixed point theorem of Avery and Henderson will be fundamental.

Theorem 2.1 [19] *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $\Phi : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad for \quad 0 \leq \lambda \leq 1 \quad and \quad x \in \partial P(\theta, b),$$

and

(i) $\gamma(\Phi x) < c$, *for all* $x \in \partial P(\gamma, c)$,
(ii) $\theta(\Phi x) > b$ *for all* $x \in \partial P(\theta, b)$,
(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(\Phi x) < a$, *for* $x \in \partial P(\alpha, a)$.

Then Φ has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \quad with \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2) \quad with \quad \gamma(x_2) < c.$$

3. Existence of two positive solutions of (1.1)-(1.2)

In this section, by defining an appropriate Banach space and cones, we impose growth conditions on f which allow us to apply the above two fixed point theorem in establishing the existence of double positive solutions of (1.1)-(1.2). Firstly, we mention without proof several fundamental results.

Lemma 3.1 [Lemma 2.1, 14]. *If condition* (H2) *holds, then there exists a constant $\delta \in (0, \frac{1}{2})$ that satisfies*

$$0 < \int_{\delta}^{1-\delta} e(t)dt < \infty.$$

Furthermore, the function:

$$y_1(t) = \int_{\delta}^{t} \phi_q\Big(\int_{s}^{t} e(r)dr\Big)ds + \int_{t}^{1-\delta} \phi_q\Big(\int_{t}^{s} e(r)dr\Big)ds, \quad t \in [\delta, 1-\delta],$$

is a positive continuous function on $[\delta, 1-\delta]$. Therefore $y_1(t)$ has a minimum on $[\delta, 1-\delta]$, so it follows that there exists $L_1 > 0$ such that

$$\min_{t\in[\delta,1-\delta]} y_1(t) = L_1.$$

If $E = C[0, 1]$, then E is a Banach space with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. We note that, from the nonnegativity of e and f, a solution of (1.1)-(1.2) is nonnegative and concave on $[0, 1]$. Define

$$P = \{u \in E : u(t) \geq 0, u(t) \text{ is concave function}, \ t \in [0, 1]\}.$$

Lemma 3.2 [Lemma 2.2, 14]. *Let $u \in P$ and δ be as Lemma 3.1, then*

$$u(t) \geq \delta\|u\|, \quad t \in [\delta, 1-\delta].$$

Lemma 3.3 [Lemma 2.3, 14]. *Suppose that conditions* (H1), (H2) *hold. Then $u(t) \in E \cap C^2(0,1)$ is a solution of boundary value problem* (1.1)-(1.2) *if and only if $u(t) \in E$ is a solution of the following integral equation:*

$$u(t) = \begin{cases} \phi_q\Big(\frac{\omega}{\mu}\displaystyle\int_\xi^\sigma e(r)f(u(r))dr\Big) + \displaystyle\int_0^t \phi_q\Big(\int_s^\sigma e(r)f(u(r))dr\Big)ds, & 0 \leq t \leq \sigma, \\ \phi_q\Big(\frac{\tau}{\rho}\displaystyle\int_\sigma^\eta e(r)f(u(r))dr\Big) + \displaystyle\int_t^1 \phi_q\Big(\int_\sigma^s e(r)f(u(r))dr\Big)ds, & \sigma \leq t \leq 1, \end{cases}$$

where $\sigma \in [\xi, \eta] \subset (0,1)$ and $u'(\sigma) = 0$.

By means of the well known Guo-Krasnoselskii fixed point theorem in a cone, Su et al. [14] established the existence of at least one positive solution for (1.1)-(1.2) under some superlinear and sublinear assumptions imposed on the nonlinearity of f, which can be listed as
(i) $f_0 = 0$ and $f_\infty = +\infty$ (superlinear), or
(ii) $f_0 = +\infty$ and $f_\infty = 0$ (sublinear).

Using the same theorem, the authors also proved the existence of two positive solutions of (1.1)-(1.2) when f satisfies
(iii) $f_0 = f_\infty = 0$, or
(iv) $f_0 = f_\infty = +\infty$.

When $f_0, f_\infty \notin \{0, +\infty\}$, set

$$\theta^* = \frac{2}{L_1}, \quad \theta_* = \frac{1}{\Big(1 + \phi_q(\frac{\omega}{\mu})\Big)\phi_q\Big(\displaystyle\int_0^1 e(r)dr\Big)},$$

and in the following, always assume δ be as in Lemma 3.1, the existence of double positive solutions of boundary value problem (1.1)-(1.2) can be list as follows:

Theorem 3.1 [Theorem 4.3, 14]. *Suppose that conditions* (H1),(H2) *hold. Also assume that f satisfies*

(A1) $f_0 = \lambda_1 \in \Big[0, \Big(\frac{\theta_*}{4}\Big)^{p-1}\Big)$;
(A2) $f_\infty = \lambda_2 \in \Big[0, \Big(\frac{\theta_*}{4}\Big)^{p-1}\Big)$;
(A3) $f(u) \leq (MR)^{p-1}, 0 \leq u \leq R$,

where $M \in (0, \theta_*)$. Then the boundary value problem (1.1)-(1.2) *has at least two positive solutions u_1, u_2 such that*

$$0 < \|u_1\| < R < \|u_2\|.$$

Theorem 3.2 [Theorem 4.4, 14]. *Suppose that conditions* (H1),(H2) *hold. Also assume that f satisfies*

(A4) $f_0 = \lambda_1 \in \Big[\Big(\frac{2\theta^*}{\delta}\Big)^{p-1}, \infty\Big)$;

(A5) $f_\infty = \lambda_2 \in \Big[\Big(\frac{2\theta^*}{\delta}\Big)^{p-1}, \infty\Big)$;
(A6) $f(u) \geq (mr)^{p-1}, \delta r \leq u \leq r$,

where $m \in (\theta^*, \infty)$. *Then the boundary value problem* (1.1)-(1.2) *has at least two positive solutions* u_1, u_2 *such that*

$$0 < \|u_1\| < r < \|u_2\|.$$

When we see such a fact, we cannot but ask *"Whether or not we can obtain a similar conclusion if neither* $f_0 \in [(\frac{2\theta^*}{\delta})^{p-1}, \infty)$ *nor* $f_0 \in [0, (\frac{\theta_*}{4})^{p-1})$*."* Motivated by the above mentioned results, in this paper, we attempt to establish simple criteria for the existence of at least two positive solutions of (1.1)-(1.2). Our result is based on Theorem 2.1 and gives a positive answer to the question stated above.

Set

$$y_2(t) := \phi_q\Big(\int_\delta^t e(r)dr\Big) + \phi_q\Big(\int_t^{1-\delta} e(r)dr\Big), \quad \delta \leq t \leq 1-\delta.$$

For notational convenience, we introduce the following constants:

$$L_2 = \min_{\delta \leq t \leq 1-\delta} y_2(t),$$

and

$$L_3 = \delta\phi_q\Big(\int_0^1 e(r)dr\Big) + \max\Big\{\phi_q\Big(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big), \phi_q\Big(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big)\Big\},$$

$$Q = \phi_q\Big(\int_0^1 e(r)dr\Big) + \max\Big\{\phi_q\Big(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big), \phi_q\Big(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big)\Big\}.$$

Finally, we define the nonnegative, increasing continuous functions γ, θ and α by

$$\gamma(u) = \min_{t \in [\delta, 1-\delta]} u(t),$$

$$\theta(u) = \frac{1}{2}[u(\delta) + u(1-\delta)], \quad \alpha(u) = \max_{0 \leq t \leq 1} u(t).$$

We observe here that, for every $u \in P$,

$$\gamma(u) \leq \theta(u) \leq \alpha(u).$$

It follows from Lemma 3.2 that, for each $u \in P$, one has $\gamma(u) \geq \delta\|u\|$, so $\|u\| \leq \frac{1}{\delta}\gamma(u)$, for all $u \in P$. We also note that $\theta(\lambda u) = \lambda\theta(u)$, for $0 \leq \lambda \leq 1$, and $u \in \partial P(\theta, b)$.

The main result of this paper is as follows:

Theorem 3.3 *Assume that* (H_1) *and* (H_2) *hold, and suppose that there exist positive constants* $0 < a < b < c$ *such that* $0 < a < \delta b < \frac{\delta^2 L_2}{2L_3}c$, *and* f *satisfies the following conditions*

(D1) $f(v) < \phi_p(\frac{a}{Q})$, *if* $0 \leq v \leq a$;
(D2) $f(v) > \phi_p(\frac{2b}{\delta L_2})$, *if* $\delta b \leq v \leq \frac{b}{\delta}$;
(D3) $f(v) < \phi_p(\frac{c}{L_3})$, *if* $0 \leq v \leq \frac{c}{\delta}$;

Then, the boundary value problem (1.1) *and* (1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$a < \max_{t \in [0,1]} u_1(t), \quad \textit{with} \quad \frac{1}{2}[u_1(\delta) + u_1(1-\delta)] < b;$$

and

$$b < \frac{1}{2}[u_2(\delta) + u_2(1-\delta)], \quad \textit{with} \quad \min_{t\in[\delta,1-\delta]} u_2(t) < c.$$

Proof. We define the operator: $\Phi : P \to P$,

$$(\Phi u)(t) := \begin{cases} \phi_q\Big(\frac{\omega}{\mu}\displaystyle\int_\xi^\sigma e(r)f(u(r))dr\Big) + \int_0^t \phi_q\Big(\int_s^\sigma e(r)f(u(r))dr\Big)ds, & 0 \le t \le \sigma, \\ \phi_q\Big(\frac{\tau}{\rho}\displaystyle\int_\sigma^\eta e(r)f(u(r))dr\Big) + \int_t^1 \phi_q\Big(\int_\sigma^s e(r)f(u(r))dr\Big)ds, & \sigma \le t \le 1, \end{cases}$$

for each $u \in P$, where $\sigma \in [\xi, \eta] \subset (0,1)$. It is shown in Lemma 3.3 that the operator $\Phi : P \to P$ is well defined with $\|\Phi u\| = \Phi u(\sigma)$. In particular, if $u \in P(\gamma, c)$, we also have $\Phi u \in P$, moreover, a standard argument shows that $\Phi : P \to P$ is completely continuous (see [Lemma 2.4, 14]) and each fixed point of Φ in P is a solution of (1.1)-(1.2).

We now show that the conditions of Theorem 2.1 are satisfied.

To fulfill property (i) of Theorem 2.2, we choose $u \in \partial P(\gamma, c)$, thus $\gamma(u) = \min_{t\in[\delta,1-\delta]} u(t) = c$. Recalling that $\|u\| \le \frac{1}{\delta}\gamma(u) = \frac{c}{\delta}$, we have

$$0 \le u(t) \le \|u\| \le \frac{1}{\delta}\gamma(u) = \frac{c}{\delta}, \quad 0 \le t \le 1.$$

Then assumption (D3) of Theorem 3.2 implies

$$f(u(t)) < \phi_p(\frac{c}{L_3}), \quad 0 \le t \le 1.$$

(i) If $\sigma \in (0, \delta)$, we have

$$\begin{aligned} \gamma(\Phi u) &= \min_{t\in[\delta,1-\delta]} (\Phi u)(t) = (\Phi u)(1-\delta) \\ &= \phi_q\Big(\tfrac{\tau}{\rho}\int_\sigma^\eta e(r)f(u(r))dr\Big) + \int_{1-\delta}^1 \phi_q\Big(\int_\sigma^s e(r)f(u(r))dr\Big)ds \\ &\le \phi_q\Big(\tfrac{\tau}{\rho}\int_\xi^\eta e(r)f(u(r))dr\Big) + \int_{1-\delta}^1 \phi_q\Big(\int_0^1 e(r)f(u(r))dr\Big)ds \\ &\le \Big[\phi_q\Big(\tfrac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big) + \delta\phi_q\Big(\int_0^1 e(r)dr\Big)\Big] \cdot \frac{c}{L_3} < c. \end{aligned}$$

(ii) If $\sigma \in [\delta, 1-\delta]$, we have

$$\begin{aligned} &\gamma(\Phi u) = \min\nolimits_{t\in[\delta,1-\delta]}(\Phi u)(t) = \min\{(\Phi u)(\delta), (\Phi u)(1-\delta)\} \\ &= \min\Big\{\phi_q\Big(\tfrac{\omega}{\mu}\int_\xi^\sigma e(r)f(u(r))dr\Big) + \int_0^\delta \phi_q\Big(\int_s^\sigma e(r)f(u(r))dr\Big)ds, \\ &\qquad \phi_q\Big(\tfrac{\tau}{\rho}\int_\sigma^\eta e(r)f(u(r))dr\Big) + \int_{1-\delta}^1 \phi_q\Big(\int_\sigma^s e(r)f(u(r))dr\Big)ds\Big\} \\ &\le \max\Big\{\phi_q\Big(\tfrac{\omega}{\mu}\int_\xi^\eta e(r)f(u(r))dr\Big) + \int_0^\delta \phi_q\Big(\int_0^1 e(r)f(u(r))dr\Big)ds, \\ &\qquad\quad \phi_q\Big(\tfrac{\tau}{\rho}\int_\xi^\eta e(r)f(u(r))dr\Big) + \int_{1-\delta}^1 \phi_q\Big(\int_0^1 e(r)f(u(r))dr\Big)ds\Big\} \\ &< \Big[\max\Big\{\phi_q\Big(\tfrac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big), \phi_q\Big(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big)\Big\} + \delta\phi_q\Big(\int_0^1 e(r)dr\Big)\Big] \cdot \frac{c}{L_3} \\ &= c. \end{aligned}$$

(iii) If $\sigma \in (1-\delta, 1)$, we have

$$
\begin{aligned}
&\gamma(\Phi u) = \min_{t\in[\delta,1-\delta]}(\Phi u)(t) = \Phi u(\delta) \\
&= \phi_q\Big(\tfrac{\omega}{\mu}\int_{\xi}^{\sigma} e(r)f(u(r))dr\Big) + \int_0^{\delta}\phi_q\Big(\int_s^{\sigma} e(r)f(u(r))dr\Big)ds \\
&\le \phi_q\Big(\tfrac{\omega}{\mu}\int_{\xi}^{\eta} e(r)f(u(r))dr\Big) + \int_0^{\delta}\phi_q\Big(\int_0^{1} e(r)f(u(r))dr\Big)ds \\
&\le \Big[\phi_q\Big(\tfrac{\omega}{\mu}\int_{\xi}^{\eta} e(r)dr\Big) + \delta\phi_q\Big(\int_0^{1} e(r)dr\Big)\Big]\cdot\frac{c}{L_3} \\
&< c.
\end{aligned}
$$

Therefore, condition (i) of Theorem 2.2 is satisfied.

We next address (ii) of Theorem 2.2. For this, we choose $u \in \partial P(\theta, b)$ so that $\theta(u) = \frac{1}{2}[u(\delta) + u(1-\delta)] = b$. Noting that

$$\|u\| \le (1/\delta)\gamma(u) \le (1/\delta)\theta(u) = b/\delta,$$

we have

$$\delta b < \delta\|u\| \le u(t) \le \frac{b}{\delta}, \quad \text{for } \ t \in [\delta, 1-\delta].$$

Then (D2) yields

$$f(u(t)) > \phi_p(\frac{2b}{\delta L_2}), \quad \text{for } \ t \in [\delta, 1-\delta].$$

As $\Phi u \in P$:

(i) If $\sigma \in (0, \delta)$, we have

$$
\begin{aligned}
&\theta(\Phi u) = \tfrac{1}{2}(\Phi u(\delta) + \Phi u(1-\delta)) \ge \Phi u(1-\delta) \\
&= \phi_q\Big(\tfrac{\tau}{\rho}\int_{\sigma}^{\eta} e(r)f(u(r))dr\Big) + \int_{1-\delta}^{1}\phi_q\Big(\int_{\sigma}^{s} e(r)f(u(r))dr\Big)ds \\
&\ge \int_{1-\delta}^{1}\phi_q\Big(\int_{\sigma}^{1-\delta} e(r)f(u(r))dr\Big)ds \\
&\ge \int_{1-\delta}^{1}\phi_q\Big(\int_{\delta}^{1-\delta} e(r)f(u(r))dr\Big)ds \\
&= \delta\phi_q\Big(\int_{\delta}^{1-\delta} e(r)f(u(r))dr\Big) \\
&\ge \delta\phi_q\Big(\int_{\delta}^{1-\delta} e(r)dr\Big)\cdot\frac{2b}{\delta L_2} \ge 2b > b.
\end{aligned}
$$

(ii) If $\sigma \in [\delta, 1-\delta]$, we have

$$
\begin{aligned}
&2\theta(\Phi u) = [\Phi u(\delta) + \Phi u(1-\delta)] \\
&\ge \int_0^{\delta}\phi_q\Big(\int_s^{\sigma} e(r)f(u(r))dr\Big)ds + \int_{1-\delta}^{1}\phi_q\Big(\int_{\sigma}^{s} e(r)f(u(r))dr\Big)ds \\
&\ge \int_0^{\delta}\phi_q\Big(\int_{\delta}^{\sigma} e(r)f(u(r))dr\Big)ds + \int_{1-\delta}^{1}\phi_q\Big(\int_{\sigma}^{1-\delta} e(r)f(u(r))dr\Big)ds \\
&= \delta\Big[\phi_q\Big(\int_{\delta}^{\sigma} e(r)f(u(r))dr\Big) + \phi_q\Big(\int_{\sigma}^{1-\delta} e(r)f(u(r))dr\Big)\Big] \\
&\ge \delta\Big[\phi_q\Big(\int_{\delta}^{\sigma} e(r)dr\Big) + \phi_q\Big(\int_{\sigma}^{1-\delta} e(r)dr\Big)\Big]\cdot\frac{2b}{\delta L_2} \\
&\ge 2b.
\end{aligned}
$$

(iii) If $\sigma \in (1-\delta, 1)$, we have

$$
\begin{aligned}
\theta(\Phi u) &= \tfrac{1}{2}(\Phi u(\delta) + \Phi u(1-\delta)) \geq \Phi u(\delta) \\
&= \phi_q\Big(\tfrac{\omega}{\mu}\int_\xi^\sigma e(r)f(u(r))dr\Big) + \int_0^\delta \phi_q\Big(\int_s^\sigma e(r)f(u(r))dr\Big)ds \\
&\geq \int_0^\delta \phi_q\Big(\int_\delta^{1-\delta} e(r)f(u(r))dr\Big)ds \\
&> \delta\phi_q\Big(\int_\delta^{1-\delta} e(r)dr\Big)\cdot\frac{2b}{\delta L_2} \geq 2b > b.
\end{aligned}
$$

Hence, condition (ii) of Theorem 2.2 holds.

To fulfill property (iii) of Theorem 2.2, we note $u_*(t) \equiv a/2, 0 \leq t \leq 1$, is a member of $P(\alpha, a)$ and $\alpha(u_*) = a/2$, so $P(\alpha, a) \neq 0$. Now, choose $u \in \partial P(\alpha, a)$, so that $\alpha(u) = \max_{t\in[0,1]} u(t) = a$ and implies $0 \leq u(t) \leq a, 0 \leq t \leq 1$. It follows from assumption (D1), $f(u(t)) \leq \phi_p(a/Q), t \in [0,1]$. As before we obtain

$$
\begin{aligned}
\alpha(\Phi u) &= \|\Phi u\| = \Phi u(\sigma) \\
&= \phi_q\Big(\tfrac{\omega}{\mu}\int_\xi^\sigma e(r)f(u(r))dr\Big) + \int_0^\sigma \phi_q\Big(\int_s^\sigma e(r)f(u(r))dr\Big)ds \\
&= \phi_q\Big(\tfrac{\tau}{\rho}\int_\sigma^\eta e(r)f(u(r))dr\Big) + \int_\sigma^1 \phi_q\Big(\int_\sigma^s e(r)f(u(r))dr\Big)ds \\
&\leq \max\Big\{\phi_q\Big(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big) + \phi_q\Big(\int_0^1 e(r)dr\Big), \\
&\qquad \phi_q\Big(\tfrac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big) + \phi_q\Big(\int_0^1 e(r)dr\Big)\Big\}\cdot\frac{a}{Q} \\
&\leq a.
\end{aligned}
$$

Thus, condition (iii) of Theorem 2.1 is also satisfied. Consequently, an application of Theorem 2.1 completes the proof. □

Finally, we present an example to explain our result.

Example. Consider the boundary value problem (1.1)-(1.2) with

$$
p = \frac{3}{2}, \mu = 2, \rho = \omega = 1, \xi = \frac{1}{4}, \eta = \frac{1}{2}, \tau = 1, \delta = \frac{1}{4}, e(t) = t^{-\frac{1}{2}},
$$

and

$$
f(u) = \begin{cases} \dfrac{6\sqrt{2u}}{u+1}, & 0 \leq u \leq 200, \\ \dfrac{40}{67} + \dfrac{1202}{335}(u-200), & 200 \leq u \leq 250, \\ 180, & 250 < u, \end{cases}
$$

Then (1.1)-(1.2) has at least two positive solutions.

Proof. In this example we have

$$
L_1 = \min_{1/4\leq x\leq 3/4}\Big\{\int_{1/4}^x \phi_q(\int_s^x t^{-1/2}dt)ds + \int_x^{3/4}\phi_q(\int_x^s t^{-1/2}dt)ds\Big\} = \frac{3\sqrt{3}-5}{9},
$$

$$
L_2 = \min_{1/4\leq x\leq 3/4}\Big(\phi_q(\int_{1/4}^x t^{-1/2}dt) + \phi_q(\int_x^{3/4} t^{-1/2}dt)\Big) = 2-\sqrt{3},
$$

$$
L_3 = \delta\phi_q\Big(\int_0^1 e(r)dr\Big) + \max\Big\{\phi_q\Big(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big), \phi_q\Big(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big)\Big\} = 4-2\sqrt{2},
$$

$$Q=\phi_q\Big(\int_0^1 e(r)dr\Big)+\max\Big\{\phi_q\Big(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\Big),\phi_q\Big(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\Big)\Big\}=7-2\sqrt{2}.$$

Let $a=80, b=1000, c=40000$. Then we have

$$f(u)=\frac{6\sqrt{2u}}{u+1}<\phi_p(a/Q),\quad \text{for}\quad 0\le u\le 80,$$

$$f(u)=180>\phi_p((2b)/(\delta L_2)),\quad \text{for}\ \ 250\le u\le 4000,$$

$$f(u)=180<\phi_p(c/L_3),\quad \text{for}\ \ 0\le u\le 160000.$$

Therefore, by Theorem 3.3 we deduce that (1.1)-(1.2) has at least two positive solutions u_1 and u_2 satisfying

$$80<\max_{t\in[0,1]}u_1(t),\quad with\ \ \frac{1}{2}[u_1(\delta)+u_1(1-\delta)]<1000;$$

and

$$1000<\frac{1}{2}[u_2(\delta)+u_2(1-\delta)],\quad with\quad \min_{t\in[\delta,1-\delta]}u_2(t)<40000.$$

Remark. We notice that in the above example, $f_0=6\sqrt{2}\approx 8.48528$, $(\frac{\theta_*}{4})^{p-1}=\frac{\sqrt{5}}{10}\approx 0.223607$ and $(\frac{2\theta^*}{\delta})^{p-1}=6\sqrt{10+6\sqrt{3}}\approx 27.0947$. Therefore, Theorem 3.1 and Theorem 3.2 are not applicable to this example since conditions (A1) and (A4) fail.

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