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THE FREEZING METHOD FOR VOLTERRA INTEGRAL EQUATIONS IN A BANACH SPACE *

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Abstract

The "freezing" method for ordinary differential equations is extended to the Volterra integral equations in a Banach space of the type

$$x(t) - \int_0^t K(t, t-s)x(s)ds = f(t) \ (t \geq 0),$$

where $K(t, s)$ is an operator valued function "slowly" varying in the first argument. Besides, sharp explicit stability conditions are derived.

Subject Classification: 45M10, 45N05
Key words: Volterra integral equations, Banach space, stability

1 Introduction and statement of the basic lemma

Stability and boundedness of Volterra integral and integrodifferential equations have been extensively considered for a long time (see the well-known books [1, 4], recent papers [5, 8, 15, 16] and papers listed below). The basic method in the theory of stability and boundedness of Volterra integral equations is the direct Liapunov method. But finding the Liapunov functionals is a difficult mathematical problem. The other approach is connected with an interpretation of the Volterra equations as operator equations in appropriate spaces. Such an approach was used in many papers, cf. [3, 6, 7, 12, 14, 16] and references therein. In this paper, for a class of Volterra equations in a Banach space we establish explicit sufficient stability conditions which are also necessary stability conditions when the integral operator is a convolution. Our results improve the well known ones in the case of the considered equations.

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The approach suggested below is based on the extension of the "freezing" method which was introduced by V.M. Alekseev for linear ordinary differential equations cf. [2] (see also [9, Section 3.2]). That method was already extended to difference equations [11].

Let X be a Banach space with a norm $\|.\|$ and the unit operator I, $R_+ := [0,\infty)$, and $C(\omega, X)$ is the space of continuous functions defined on a set $\omega \subset \mathbb{R}$ with values in X and equipped with the sup-norm $|.|_{C(\omega)} = |.|_{C(\omega,X)}$. $L^p(\omega, X)$ $(1 \leq p < \infty)$ is the space of functions defined on ω with values in X and equipped with the

$$|f|_{L^p(\omega)} = [\int_\omega \|f(t)\|^p dt]^{1/p}.$$

Consider in X the equation

$$x(t) - \int_0^t K(t, t-s)x(s)ds = f(t)\ (f \in C(R_+, X),\ t \geq 0), \tag{1.1}$$

where $K(t,s)$ is a functions defined on $[0 \leq s \leq t < \infty]$, whose values are bounded operators in X, and for any fixed $\tau \geq 0$, $K(\tau, .)$ is integrable and bounded on R_+. In addition,

$$\int_0^t \|K(t,s) - K(\tau,s)\|ds \leq q|t-\tau| \ \ (q = const;\ t,\tau \geq 0). \tag{1.2}$$

A solution of Equation (1.1) is a continuous function defined on R_+ and satisfying (1.1) for all finite $t > 0$. The existence of solutions under consideration is checked below.

Note that the approach suggested below enables us to consider also the equation

$$x(t) - \int_0^t K(t-s, s)x(s)ds = f(t)\ (t \geq 0)$$

under condition (1.2). It is clear that under (1.2) the function $K(\tau, s)$, for a fixed τ, admits the Laplace transform

$$\tilde{K}_\tau(z) := \int_0^\infty e^{-zs}K(\tau,s)ds \ \ (Re z \geq c_0 = const).$$

Besides, it is assumed that the operator $W_\tau(z) := I - \tilde{K}_\tau(z)$ is *invertible* for all $z \in C_+ := \{z \in \mathbb{C} : Re\ z \geq 0\}$ and $W_\tau^{-1}(iy) \in L^1(\mathbb{R})$. Introduce the "local Green function"

$$G_\tau(t) := \frac{1}{2\pi}\int_{-\infty}^\infty e^{iyt}W_\tau^{-1}(iy)dy.$$

We will say that Equation (1.1) is stable, if for any $f \in C(R_+, X)$ a solution x of (1.1) satisfies the inequality

$$|x|_{C(R_+)} \leq a_0|f|_{C(R_+)}, \tag{1.3}$$

where the constant a_0 does not depend on f.

Lemma 1.1 *Under condition (1.2), let*

$$q\int_0^\infty s \sup_{\tau\geq 0}\|G_\tau(s)\|ds < 1. \tag{1.4}$$

Then Equation (1.1) is stable. Moreover, constant a_0 in (1.3) is explicitly pointed below.

This lemma is proved in the next section.

2 Proof of Lemma 1.1

Consider the equation

$$x(t) - \int_0^t K(\tau, t-s)x(s)ds = f(t) \ \ (t \geq 0) \tag{2.1}$$

with a fixed $\tau \geq 0$. Applying to (2.1) the Laplace transform, we have

$$\tilde{x}(z) - \tilde{K}_\tau(z)\tilde{x}(z) = \tilde{f}(z),$$

where $\tilde{x}(z)$ and $\tilde{f}(z)$ are the Laplace transforms to $x(t)$ and $f(t)$, respectively, z is the dual variable. Hence,

$$\tilde{x}(z) = W_\tau^{-1}(z)\tilde{f}(z).$$

So

$$x(t) = \int_0^t G_\tau(t-s)f(s)ds. \tag{2.2}$$

Now rewrite (1.1) in the form

$$x(t) - \int_0^t K(\tau, t-s)x(s)ds = f_0(t,\tau) + f(t) \ \ (t \geq 0). \tag{2.3}$$

with

$$f_0(t,\tau) = \int_0^t (K(t, t-s) - K(\tau, t-s))x(s)ds.$$

So according to (2.2),

$$x(t) = \int_0^t G_\tau(t-s)(f(s) + f_0(s,\tau))ds = F(t) + \int_0^t G_\tau(t-s)f_0(s,\tau)ds, \tag{2.4}$$

where

$$F(t) = \int_0^t G_\tau(t-s)f(s)ds.$$

With the notation

$$w(t) := \sup_{\tau \geq 0} \|G_\tau(t)\|$$

we thus get

$$|F|_{C(R_+)} \leq |f|_{C(R_+)} \sup_t \int_0^t w(t-s)ds = |w|_{L^1(R_+)} \, |f|_{C(R_+)}.$$

Due to (1.3)

$$\|f_0(t,\tau)\| \leq \int_0^t \|(K(\tau, t-s) - K(t, t-s))x(s)\|ds \leq |x|_{C(0,t)} q|t-\tau|.$$

Now (2.4) implies

$$\|x(t)\| \leq |w|_{L^1(R_+)} \, |f|_{C(R_+)} + q\int_0^t w(t-s)|x|_{C(0,s)}|s-\tau|ds.$$

Take $t = \tau$. Then

$$\|x(\tau)\| \le |w|_{L^1(R_+)}\ |f|_{C(R_+)} + q\int_0^\tau w(\tau-s)|_X|x|_{C(0,s)}(\tau-s)ds.$$

Hence,

$$\|x(\tau)\| \le |w|_{L^1(R_+)}\ |f|_{C(R_+)} + |x|_{C(0,\tau)}\int_0^\tau (\tau-s)w(\tau-s)ds_1 \le$$
$$|w|_{L^1(R_+)}\ |f|_{C(R_+)} + |x|_{C(0,\tau)}\Theta,$$

where

$$\Theta = q\int_0^\infty sw(s)ds.$$

Therefore, for any $t_0 > 0$,

$$\sup_{\tau\le t_0}\|x(\tau)\| \le |w|_{L^1(R_+)}\ |f|_{C(R_+)} + \sup_{\tau\le t_0}|x|_{C(0,\tau)}\Theta.$$

Now condition (1.4) implies

$$|x|_{C(0,t_0)} \le \frac{|w|_{L^1(R_+)}\ |f|_{C(R_+)}}{1-\Theta}.$$

Since the right hand part does not depend on t_0, inequality (1.3) follows. Besides,

$$a_0 = \frac{|w|_{L^1(R_+)}}{1-\Theta}.$$

The existence of solutions is due to the Neumann series

$$x = \sum_{k=0}^\infty V^k f,$$

where V is the Volterra integral operator defined in (1.1). The lemma is proved. □

3 The main result

First, note that

$$tG_\tau(t) = t\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{zt}W_\tau^{-1}(z)dz = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{zt}T(z)dz,$$

where

$$T_\tau(z) := -\frac{dW_\tau^{-1}(z)}{dz} = W_\tau^{-1}(z)\frac{dW_\tau(z)}{dz}W_\tau^{-1}(z).$$

For a number $b > 0$ and $Re\ z > -b$, let $T_\tau(z)$ be regular and

(3.1) $$\psi_b := \sup_{\tau\ge 0}\frac{1}{2\pi}\int_{-\infty}^\infty \|T_\tau(iy-b)\|dy < \infty.$$

Then

$$\|tG_\tau(t)\| \le e^{-bt}\frac{1}{2\pi}\int_{-\infty}^{\infty}\|T(iy-b)\|dy = e^{-bt}\psi_b.$$

So

$$\int_0^\infty t\sup_\tau \|G_\tau(t)\|dt \le \psi_b\int_0^\infty e^{-bt}dt = \frac{\psi_b}{b}.$$

Now Lemma 1.1 implies our main result.

Theorem 3.1 *Under condition (1.2), for a positive b and all z with $Re\ z > -b$, let $T_\tau(z)$ be regular, and the conditions (3.1) and $q\psi_b < b$ hold. Then Equation (1.1) is stable.*

To illustrate this result, consider in X the equation

$$x(t) - A(t)\int_0^t e^{-(t-s)h}x(s)ds = f(t)\ (h = const > 0,\ t \ge 0), \tag{3.2}$$

where $A(t)$ is a variable bounded operator in X satisfying

$$\|A(t) - A(\tau)\| \le q_1|t-\tau|\ \ (t,\tau \ge 0). \tag{3.3}$$

Take $K(t,s) = A(t)e^{-sh}$. Then

$$\int_0^t \|K(t,s) - K(\tau,s)\|ds \le q_1\|A(t) - A(\tau)\|\int_0^t e^{-sh}ds \le \frac{q_1}{h}|t-\tau|\ (t,\tau \ge 0). \tag{3.4}$$

So (1.2) holds with $q = q_1/h$. We also have

$$\tilde{K}_\tau(z) := A(\tau)\int_0^\infty e^{-zs}e^{-hs}ds = \frac{A(\tau)}{z+h}$$

and

$$W_\tau(z) := I - \frac{A(\tau)}{z+h}.$$

Hence,

$$T_\tau(z) = (I - \frac{A(\tau)}{z+h})^{-2}\frac{A(\tau)}{(z+h)^2} = A(\tau)((z+h)I - A(\tau))^{-2}.$$

So

$$\|T_\tau(z)\| \le \|A(\tau)\|\ \|((z+h)I - A(\tau))^{-1}\|^2\ \ (\tau \ge 0). \tag{3.5}$$

Note that some estimates for resolvents of nonselfadjoint operators can be found in [10]. For instance, take $X = L^2(0,1)$ and

$$A(t)w(y) = a(t,y)\int_0^1 m(y,y_1)w(y_1)dy_1\ \ (y \in [0,1]),$$

where $a(t,.)$ for all $t \ge 0$ is a scalar measurable function satisfying the conditions

$$\sup_{t\ge 0, y\in[0,1]} |a(t,y)| < \infty$$

and

$$(3.6)\qquad |a(t,y)-a(\tau,y)|\le q_0|t-\tau| \ \ (y\in[0,1];\ t,\tau\ge 0).$$

In addition, the scalar function $m(.,.)$ satisfies the condition

$$N_m := [\int_0^1\int_0^1 |m(y,y_1)|^2 dy\, dy_1]^{1/2} < \infty.$$

That is, we consider the equation

$$(3.7)\qquad u(t,y)=f(t,y)+a(t,y)\int_0^t e^{-h(t-s)}\int_0^1 m(y,y_1)u(s,y_1)dy_1 ds \ \ (0\le y\le 1;\ t\ge 0),$$

where $f(t,.)\in L^2(0,1)$. By the Schwarz inequaliy, for any $w\in L^2(0,1)$ we get

$$\|(A(t)-A(\tau))w\|^2=\int_0^1 |(a(t,y)-a(\tau,y))\int_0^1 m(y,y_1)w(y_1)dy_1|^2 dy \le (q_0|t-\tau|N_m)^2\|w\|^2.$$

That is, (3.3) holds with $q_1 = q_0N_m$. So according to (3.4), condition (1.2) is valid with $q=q_0N_m/h$. Furthermore, clearly,

$$\|A(\tau)\|\le c(a,m):=\sup_{\tau,y}|a(\tau,y)|N_m \ \ (\tau\ge 0).$$

Assume that

$$(3.8)\qquad 2c(a,m)<h$$

and take $b=h/2$. Then by (3.5),

$$\|T_\tau(-b+iy)\|\le \frac{c(a,m)}{(\sqrt{y^2+h^2/4}-c(a,m))^2} \ \ (\tau\ge 0).$$

So we have the inequality $\psi_b\le\tilde\psi_h$, where

$$\tilde\psi_h := \frac{c(a,m)}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{(\sqrt{y^2+h^2/4}-c(a,m))^2}<\infty.$$

Thus under conditions (3.6) and (3.8), thanks to Theorem 3.1, Equation (3.7) is stable provided $2q_0N_m\tilde\psi_h<h^2$.

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