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Stability of a monotonic solution of a non-autonomous multidimensional delay differential equation of arbitrary (fractional) order

A. M. A. El-Sayed
Faculty of Science, Alexandria University, Alexandria, Egypt
e.mail: amasayed@hotmail.com
E. M. El-Maghrabi
Faculty of Science, Benha University, Benha 13518, Egypt
e.mail: esam_mh@yahoo.com

Abstract

We are concerned here with the existence of monotonic and uniformly asymptotically stable solution of an initial-value problem of non-autonomous delay differential equations of arbitrary (fractional) orders.

Key words: Fractional calculus, delay differential equations, monotonic solutions, asymptotically stable solution.

1 Introduction

In a number of papers [1, 2, 3, 7] stability and existence of solution of some equations of fractional order has been investigated.
In [2] the authors proved the stability (and some other properties concerning the existence and uniqueness) of solution of the problem of non-autonomous system

$$D_{t_0}^{\alpha} X(t) = A(t) X(t) + f(t), \quad \alpha \in (0,1], \quad X(t_0) = X^0 ,$$

where the coefficients of the matrix functions $A(t)$ and $f(t)$ are absolutely continuous functions. And the problem

$$\frac{d}{dt}X(t) \;=\; A(t)\,\frac{d}{dt}\,I_{t_0}^{\alpha}\,X(t)+f(t), \qquad \alpha \;\in\; (0,1], \quad X(t_0) \;=\; X^0$$

where the coefficients of the matrix $A(t)$ are bounded and measurable and the function $f(t)$ is integrable.

Let $\alpha_i \in (0,1]$, $\beta_j \in [0,1)$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, and r_i, σ_j are positive constants. Consider the initial-value problem

$$\begin{aligned}\frac{d}{dt}x(t) - f(t,x(t),&D_{r_1}^{\alpha_1}x(t-r_1),\ldots,D_{r_n}^{\alpha_n}x(t-r_n))\\ &= g(t,x(t),D_{\sigma_1}^{\beta_1}x(t-\sigma_1),\ldots,D_{\sigma_m}^{\beta_m}x(t-\sigma_m)), \qquad t \;>\; 0,\end{aligned} \tag{1.1}$$

$$x(t) = x_0, \qquad t \;\leq\; 0, \tag{1.2}$$

We prove here the existence of uniformly asymptotically stable and monotonic increasing solution $x \in C[0,T]$ of (1.1) with $\frac{dx}{dt} \in C[0,T]$.
As an application the special (linear) case

$$\begin{aligned}\frac{d}{dt}\;x(t) - a(t)\;x(t) &= \sum_{i=1}^{n} a_i(t)D_{r_i}^{\alpha_i}x(t-r_i) + \sum_{j=1}^{m} b_j(t)D_{\sigma_j}^{\beta_j}x(t-\sigma_j)\\ x(t) &= x_0 \qquad t \leq 0\end{aligned} \tag{1.3}$$

and the initial value problem of the multidimensional neutral delay differential equation

$$\begin{aligned}\frac{d}{dt}\;\left(x(t) - \sum_{i=0}^{n} a_i(t)\;x(t-r_i)\right) &= \sum_{j=1}^{m} b_j(t)\;x(t-\sigma_j)\\ x(t) &= 0 \qquad t \leq 0\end{aligned} \tag{1.4}$$

will be studied

2 Preliminaries

Let $L^1[a,b]$ denote the space of all Lebesgue integrable functions on the interval $[a,b]$, $0 \leq a < b < \infty$.

Definition 2.1. The fractional (arbitrary) order integral of the function $f \in L^1[a,b]$ of order $\beta \in R^+$ is defined by (see [4] - [6])

$$I_a^\beta\ f(t) \ = \ \int_a^t \frac{(t \ - \ s)^{\beta \ - \ 1}}{\Gamma(\beta)}\ f(s)\ ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2. The (Caputo) fractional-order derivative D^α of order $\alpha \in (0,1]$ of the function $g(t)$ is defined as (see [4] - [6])

$$D_a^\alpha\ g(t) \ = \ I_a^{1 \ - \ \alpha}\ \frac{d}{dt}\ g(t), \quad t \ \in \ [a,b].$$

The following properties are some of the main ones of the fractional derivatives and integrals. Let $\beta, \gamma \in R^+$ and $\alpha \ \in (0,1)$. Then we have

(i) $I_a^\beta \ : \ L^1 \to L^1$, and if $f(x) \in L^1$, then $I_a^\gamma\ I_a^\beta\ f(x) \ = \ I_a^{\gamma+\beta}\ f(x)$.

(ii) $\lim_{\beta \to n}\ I_a^\beta\ f(x) \ = \ I_a^n\ f(x)$ uniformly on [a,b], $n \ = \ 1,\ 2,\ 3, \cdots$,
where $I_a^1\ f(x) \ = \ \int_a^x\ f(s)\ ds$.

(iii) $\lim_{\beta \to 0}\ I_a^\beta\ f(x) \ = \ f(x)$ weakly.

(iv) If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1}\ D_a^\alpha\ f(x) \ = \ \frac{df(x)}{dx}$.

(v) If $f(x) \ = \ k \neq 0$, k is a constant, then $D_a^\alpha\ k \ = \ 0$.

3 Existence of solution

Let $C[0,T]$ be the class of continuous functions. For $x \in C[0,T]$ we use the norm $\|x\|_1 = \sup_t e^{-Nt}|x(t)|\ , N > 0$.

Let $f : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}_+,\ \ g : [0,T] \times \mathbb{R}^{m+1} \to \mathbb{R}_+$ and $f,\ g$ satisfy the Lipschitz condition

$$\begin{aligned} |f(t,x_0,x_1,\ldots,x_n) - f(t,y_0,y_1,\ldots,y_n)| &\leq k_1 \sum_{i=0}^{n} |x_i - y_i|\,, \quad k_1 > 0 \\ |g(t,x_0,x_1,\ldots,x_m) - g(t,y_0,y_1,\ldots,y_m)| &\leq k_2 \sum_{j=0}^{m} |x_j - y_j|\,, \quad k_2 > 0 \end{aligned} \tag{3.1}$$

Theorem 3.1. *If f and g, satisfy the Lipschitz condition (3.1), then the Cauchy problem (1.1) - (1.2) has a monotonic increasing solution $x \in C[0,T]$ and $\frac{dx}{dt} \in C[0,T]$.*

Proof. Let $y(t) = \frac{d}{dt}\ x(t)$ then $x(t) = x_0 + I\ y(t)$, which implies

$$x(t - r_i) \ = \ x_0 \ + \ \int_{r_i}^{t} y(\theta - r_i) d\theta, \quad i = 0, 1, 2, \ldots, n$$

i.e.

$$x(t - r_i) = x_0 + I_{r_i}\ y(t - r_i),$$

from which we obtain

$$I_{r_i}^{1-\alpha_i}\ \frac{d}{dt}\ x(t - r_i) = I_{r_i}^{1-\alpha_i}\ y(t - r_i), \quad i = 0, 1, \ldots, n\ .$$

Similarly we obtain

$$I_{\sigma_j}^{1-\beta_j}\ \frac{d}{dt}\ x(t - \sigma_j) = I_{\sigma_j}^{1-\beta_j}\ y(t - \sigma_j), \quad j =, 0, 1, \ldots, m\ .$$

Now equation (1.1) can be written as

$$\begin{aligned} y(t) = f(t, x_0 + Iy(t), I_{r_1}^{1-\alpha_1}\ y(t - r_1), \ldots, I_{r_n}^{1-\alpha_n}\ y(t - r_n)) \\ + g(t, x_0 + I\ y(t), I_{\sigma_1}^{1-\beta_1}\ y(t - \sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ y(t - \sigma_n)), \end{aligned} \tag{3.2}$$

Define the operator $F\ :\ C[0, T]\ \rightarrow\ C[0, T]$ by

$$\begin{aligned} Fy(t) = f(t, x_0 + Iy(t), I_{r_1}^{1-\alpha_1}\ y(t - r_1), \ldots, I_{r_n}^{1-\alpha_n}\ y(t - r_n)) \\ + g(t, x_0 + I\ y(t), I_{\sigma_1}^{1-\beta_1}\ y(t - \sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ y(t - \sigma_n)), \end{aligned}$$

then

$$\begin{aligned} Fy(t) - Fz(t) \ \ = \ \ & f\left(t, x_0 + Iy(t), I_{r_1}^{1-\alpha_1}\ y(t - r_1), \ldots, I_{r_n}^{1-\alpha_n}\ y(t - r_n)\right) \\ & - f\left(t, x_0 + Iz(t), I_{r_1}^{1-\alpha_1}\ z(t - r_1), \ldots, I_{r_n}^{1-\alpha_n}\ z(t - r_n)\right) \\ & + g\left(t, x_0 + I\ y(t), I_{\sigma_1}^{1-\beta_1}\ y(t - \sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ y(t - \sigma_n)\right) \\ & - g\left(t, x_0 + I\ z(t), I_{\sigma_1}^{1-\beta_1}\ z(t - \sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ z(t - \sigma_n)\right), \end{aligned}$$

and

$$\begin{aligned}
e^{-Nt}\,|Fy(t)-Fz(t)| &= e^{-Nt}\left|f\left(t,x_0+Iy(t),I_{r_1}^{1-\alpha_1}\;y(t-r_1),\ldots,I_{r_n}^{1-\alpha_n}\;y(t-r_n)\right)\right.\\
&\quad\left.-\,f\left(t,x_0+Iz(t),I_{r_1}^{1-\alpha_1}\;z(t-r_1),\ldots,I_{r_n}^{1-\alpha_n}\;z(t-r_n)\right)\right|\\
&\quad+\,e^{-Nt}\left|g\left(t,x_0+I\;y(t),I_{\sigma_1}^{1-\beta_1}\;y(t-\sigma_1),\ldots,I_{\sigma_n}^{1-\beta_n}\;y(t-\sigma_n)\right)\right.\\
&\quad\left.-\,g\left(t,x_0+I\;z(t),I_{\sigma_1}^{1-\beta_1}\;z(t-\sigma_1),\ldots,I_{\sigma_n}^{1-\beta_n}\;z(t-\sigma_n)\right)\right|,\\
&\leq (k_1\;+\;k_2)\int_0^t e^{-N(t)}\;|y(s)-z(s)|ds\\
&\quad+\,k_1\sum_{i=1}^{n}\int_{r_i}^{t}e^{-N(t)}\frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)}\;|y(s-r_i)-z(s-r_i)|ds\\
&\quad+\,k_2\sum_{j=1}^{m}\int_{\sigma_j}^{t}e^{-N(t)}\frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)}\;|y(s-\sigma_j)-z(s-\sigma_j)|ds
\end{aligned}$$

which implies that

$$\begin{aligned}
\|Fy-Fz\|_1 &\leq (k_1+k_2)\,\|y-z\|_1\int_0^t e^{-N(t-\theta)}\;d\theta\\
&\quad+\,k_1\,\|y-z\|_1\sum_{i=1}^{n}e^{-Nr_i}\int_0^{t-r_i}e^{-N(t-r_i-\theta)}\frac{(t-r_i-\theta)^{-\alpha_i}}{\Gamma(1-\alpha_i)}\;d\theta\\
&\quad+\,k_2\,\|y-z\|_1\sum_{j=1}^{m}e^{-N\sigma_j}\int_0^{t-\sigma_j}e^{-N(t-\sigma_j-\theta)}\frac{(t-\sigma_j-\theta)^{-\beta_j}}{\Gamma(1-\beta_j)}\;d\theta\\
&\leq \frac{k_1\;+\;k_2}{N}\;\|y-z\|_1\\
&\quad+\,k_1\,\|y-z\|_1\sum_{i=1}^{n}\int_0^{N(t-r_i)}\frac{e^{-u}\;u^{-\alpha_i}}{N^{-\alpha_i}\;\Gamma(1-\alpha_i)}\;\frac{du}{N}\\
&\quad+\,k_2\,\|y-z\|_1\sum_{j=1}^{m}\int_0^{N(t-\sigma_j)}\frac{e^{-u}\;u^{-\beta_j}}{N^{-\beta_j}\;\Gamma(1-\beta_j)}\;\frac{du}{N}\\
&\leq \left(\frac{k_1\;+\;k_2}{N}+k_1\sum_{i=1}^{n}\frac{1}{N^{1-\alpha_i}}+k_2\sum_{j=1}^{m}\frac{1}{N^{1-\beta_j}}\right)\|y-z\|_1<\tilde{K}\,\|y-z\|_1\,,
\end{aligned}$$

where

$$\tilde{K}=\left(\frac{k_1\;+\;k_2}{N}+k_1\sum_{i=1}^{n}\frac{1}{N^{1-\alpha_i}}+k_2\sum_{j=1}^{n}\frac{1}{N^{1-\beta_j}}\right)$$

Now we choose N large enough such that $\tilde{K}<1$, then

$$\|Fy-Fz\|_1\;<\;\|y-z\|_1\,,$$

and the map F is contraction and has a unique positive fixed point $y \in C[0,T]$ which proves the existence of a unique solution $x \in C[0,T]$ of the Cauchy problem (1.1) - (1.2). Now the solution of (1.1) - (1.2) can be written as $x(t) = x_0 + I\ y(t)$, this implies that $\frac{dx}{dt} = y > 0$ which prove that $\frac{dx}{dt} \in C[0,T]$ and x is monotonic increasing. $\square$

4 Stability

In this section we study the stability of the solution of the initial value problem (1.1) - (1.2).

Theorem 4.1. *If the solution of the initial value problem (1.1) - (1.2) exists, then this solution is uniformly asymptotically stable.*

Proof. Let $y(t)$ be a solution of problem (1.1) - (1.2) and $y^*(t)$ be a solution of problem (1.1) with the initial value $x^*(t) = x_0^* \quad t \le 0$ then

$$\begin{aligned}
e^{-Nt}|y(t) - y^*(t)| \le\ & e^{-Nt}\left|f(t, x_0 + Iy(t), I_{r_1}^{1-\alpha_1}\ y(t-r_1), \ldots, I_{r_n}^{1-\alpha_n}\ y(t-r_n))\right. \\
& \left. -\ f(t, x_0^* + Iy^*(t), I_{r_1}^{1-\alpha_1}\ y^*(t-r_1), \ldots, I_{r_n}^{1-\alpha_n}\ y^*(t-r_n))\right| \\
& +\ e^{-Nt}\left|g(t, x_0 + I\ y(t), I_{\sigma_1}^{1-\beta_1}\ y(t-\sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ y(t-\sigma_n))\right., \\
& \left. -\ g(t, x_0^* + I\ y^*(t), I_{\sigma_1}^{1-\beta_1}\ y^*(t-\sigma_1), \ldots, I_{\sigma_n}^{1-\beta_n}\ y^*(t-\sigma_n))\right|, \\
\le\ & (k_1 + k_2)\left\{e^{-Nt}|x_0 - x_0^*| + \int_0^t e^{-N(t-s)}\ e^{-Ns}\ |y(s) - y^*(s)|ds\right\} \\
& + k_1 e^{-Nt}\sum_{i=1}^{n}\int_{r_i}^{t}\frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)}\,|y(s-r_i) - y^*(s-r_i)|\,ds \\
& + k_2 e^{-Nt}\sum_{j=1}^{n}\int_{\sigma_j}^{t}\frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)}\,|y(s-\sigma_i) - y^*(s-\sigma_i)|\,ds \\
\le\ & (k_1 + k_2)\left\{e^{-Nt}|x_0 - x_0^*| + \int_0^t e^{-N(t-s)}\ e^{-Ns}\ |y(s) - y^*(s)|ds\right\} \\
& +\ k_1\sum_{i=1}^{n}\int_0^{t-r_i} e^{-N(t-\theta)}\frac{(t-r_i-\theta)^{-\alpha_i}}{\Gamma(1-\alpha_i)}\ e^{-N\theta}\ |y(\theta) - y^*(\theta)|d\theta \\
& +\ k_2\sum_{j=1}^{n}\int_0^{t-\sigma_j} e^{-N(t-\theta)}\frac{(t-\sigma_j-\theta)^{-\beta_j}}{\Gamma(1-\beta_j)}\ e^{-N\theta}\ |y(\theta) - y^*(\theta)|d\theta
\end{aligned}$$

and

$$
\begin{aligned}
\|y-y^*\|_1 &\leq (k_1+k_2)\ e^{-Nt}\ \ \|x_0-x_0^*\|_1+\frac{(k_1+k_2)}{N}\|y-y^*\|_1 \\
&\quad + k_1\,\|y-y^*\|_1\sum_{i=1}^{n}\int_0^{N(t-r_i)}\frac{e^{-u}\ u^{-\alpha_i}}{N^{-\alpha_i}\ \Gamma(1-\alpha_i)}\ \frac{du}{N} \\
&\quad + k_2\,\|y-y^*\|_1\sum_{j=1}^{n}\int_0^{N(t-\sigma_j)}\frac{e^{-u}\ u^{-\beta_j}}{N^{-\beta_j}\ \Gamma(1-\beta_j)}\ \frac{du}{N} \\
&\leq (k_1+k_2)\ e^{-Nt}\ \ \|x_0-x_0^*\|_1 \\
&\quad + \left(\frac{k_1\ +\ k_2}{N}+k_1\sum_{i=1}^{n}\frac{1}{N^{1-\alpha_i}}+k_2\sum_{j=1}^{n}\frac{1}{N^{1-\beta_j}}\right)\|y-y^*\|_1
\end{aligned}
$$

this implies that

$$
\|y-y^*\|_1 \leq \left(1-\tilde{K}\right)^{-1}(k_1+k_2)\ e^{-Nt}\ \ \|x_0-x_0^*\|_1
$$

where

$$
\tilde{K}=\left(\frac{k_1\ +\ k_2}{N}+k_1\sum_{i=1}^{n}\frac{1}{N^{1-\alpha_i}}+k_2\sum_{i=1}^{n}\frac{1}{N^{1-\beta_i}}\right)<1.
$$

Now

$$
x(t)-x^*(t) \ = \ x_0-x_0^*+\int_0^t (y(s)-y^*(s))ds
$$

this implies that

$$
e^{-Nt}|x(t)-x^*(t)| \ \leq \ e^{-Nt}|x_0-x_0^*|+\int_0^t e^{-N(t-s)}\ e^{-Ns}|(y(s)-y^*(s))|ds
$$

then

$$
\begin{aligned}
\|x-x^*\|_1 &\leq e^{-Nt}\ \ \|x_0-x_0^*\|_1+\frac{1}{N}\|y-y^*\|_1 \\
&\leq e^{-Nt}\ \ \|x_0-x_0^*\|_1+\left(1-\tilde{K}\right)^{-1}\left(\frac{k_1+k_2}{N}\right)\ e^{-Nt}\ \ \|x_0-x_0^*\|_1 \\
&\leq \left(1+\frac{k_1+k_2}{N(1-\tilde{K})}\right)\ e^{-Nt}\ \ \|x_0-x_0^*\|_1\,,
\end{aligned}
$$

therefore $\lim_{t\to\infty}\|x\ -\ x^*\|_1=0$, then the solution of the system (1.1) is uniformly asymptotically stable. □

5 Application

(1) Consider now the initial value problem (1.3). Letting

$$f(t, x(t), D_{r_1}^{\alpha_1} x(t-r_1), \ldots, D_{r_n}^{\alpha_n} x(t-r_n)) = a(t)\ x(t) + \sum_{i=1}^{n} a_i(t) D_{r_i}^{\alpha_i} x(t-r_i)$$

and

$$g(t, x(t), D_{\sigma_1}^{\beta_1} x(t-\sigma_1), \ldots, D_{\sigma_m}^{\beta_m} x(t-\sigma_m)) = \sum_{j=1}^{m} b_j(t) D_{\sigma_j}^{\beta} x(t-\sigma_j)$$

where a_i and b_j, $i, j = 0, 1, 2 \cdots$ are positive continuous functions on $[0, T]$ with $|a_i(t)| \le a$ and $|b_j(t)| \le b$. Then our results here can be applied to the initial value problem (1.3) and prove the existence of positive monotonic and uniformly asymptotically stable solution for the initial value problem (1.3).

(2) Consider now the initial value problem (1.4). Letting $\alpha_i \to 1$ and $\beta \to 0$ in (1.3) and use the proberties of the fractional derivative ([1]) we obtain (1.4).

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