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Multiplicity of solutions for a class of quasilinear elliptic equations with concave and convex terms in $\mathbb{R}$ *

Uberlandio B. Severo †

Departamento de Matemática

Universidade Federal da Paraíba

58051-900, João Pessoa - PB - Brazil

Abstract

In this paper the Fountain theorem is employed to establish infinitely many solutions for the class of quasilinear Schrödinger equations $-L_p u + V(x)|u|^{p-2}u = \lambda|u|^{q-2}u + \mu|u|^{r-2}u$ in $\mathbb{R}$, where $L_p u = (|u'|^{p-2}u')' + (|(u^2)'|^{p-2}(u^2)')'u$, λ, μ are real parameters, $1 < p < \infty$, $1 < q < p$, $r > 2p$ and the potential $V(x)$ is nonnegative and satisfies a suitable integrability condition.

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1 Introduction

In this paper we establish multiple solutions for quasilinear elliptic equations of the form

$$-L_p u + V(x)|u|^{p-2}u = \lambda|u|^{q-2}u + \mu|u|^{r-2}u, \tag{1.1}$$

$u \in W^{1,p}(\mathbb{R})$, where

$$L_p u := (|u'|^{p-2}u')' + (|(u^2)'|^{p-2}(u^2)')'u,$$

$\lambda, \mu \in \mathbb{R}$, $1 < p < \infty$, $1 < q < p$ and $r > 2p$. We assume that the potential $V(x)$ is nonnegative, locally bounded and satisfies the conditions

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†Tel.:55 83 3216 7434; Fax: 55 83 3216 7277; E-mail Address: uberlandio@mat.ufpb.br

(V_1) For some $R_0 \in (0, \frac{1}{2p})$, there holds $V(x) \geq \alpha > 0$ for all $x \in \mathbb{R}$ such that $|x| > R_0$.

(V_2) $\displaystyle\int_{|x|\geq R_0} V(x)^{-1/(p-1)}\mathrm{d}x < \infty$.

We work with the space of functions

$$X = \left\{u \in W^{1,p}(\mathbb{R}) : \int_{\mathbb{R}} V(x)|u|^p \ \mathrm{d}x < \infty\right\}$$

which is a separable and reflexive Banach Space when endowed with the norm

$$\|u\|^p = \int_{\mathbb{R}} |u'|^p \ \mathrm{d}x + \int_{\mathbb{R}} V(x)|u|^p \ \mathrm{d}x.$$

Notice that (1.1) is the Euler-Lagrange equation of the functional

$$\mathcal{F}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \frac{2^{p-1}}{p}\int_{\mathbb{R}} |u'|^p|u|^p \ \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}} |u|^q \ \mathrm{d}x - \frac{\mu}{r}\int_{\mathbb{R}} |u|^r \ \mathrm{d}x.$$

The functional $\mathcal{F}_{\lambda,\mu}$ is well defined and of class C^1 on the space X (see Lemma 2.2) and we study the existence of solutions of (1.1) understood as critical points of $\mathcal{F}_{\lambda,\mu}$. The next theorem contains our main result:

Theorem 1.1 *Under the assumptions* $(V_1) - (V_2)$ *and supposing* $1 < p < \infty$, $1 < q < p$ *and* $r > 2p$ *we have*

(*a*) *for every* $\mu > 0$, $\lambda \in \mathbb{R}$, *equation (1.1) has a sequence of solutions* (u_k) *such that* $\mathcal{F}_{\lambda,\mu}(u_k) \to \infty$ *as* $k \to \infty$.

(*b*) *for every* $\lambda > 0$, $\mu \in \mathbb{R}$, *equation (1.1) has a sequence of solutions* (v_k) *such that* $\mathcal{F}_{\lambda,\mu}(v_k) < 0$ *and* $\mathcal{F}_{\lambda,\mu}(v_k) \to 0$ *as* $k \to \infty$.

After the well-known results of Ambrosetti-Brezis-Cerami [4], problems involving elliptic equations with concave and convex type nonlinearities have been studied by several authors, see for example [2], [6], [9], [11] for semilinear problems and [3], [8], [18] for quasilinear problems.

The study of problem (1.1) was in part motivated by the works of Bartsch-Willem [6], Poppenberg-Schmitt-Wang [16] and Ambrosetti-Wang [5]. In [6], Bartsch and Willem proved similar results to the Theorem 1.1 for the semilinear problem $-\Delta u = \lambda|u|^{q-2}u + \mu|u|^{r-2}u$ in an open bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions and $1 < q < 2 < r < 2^*$. Poppenberg, Schmitt and Wang [16] studied the existence of a positive ground state solution for the quasilinear elliptic equation $-u'' + V(x)u - (u^2)''u = \theta|u|^{p-1}u$ in $\mathbb{R}$, in the superlinear case by using a constrained minimization argument. Ambrosetti-Wang [5] consider the quasilinear problem $-u'' + [1+\varepsilon a(x)]u - k[1+\varepsilon b(x)](u^2)''u = [1+\varepsilon c(x)]u^q$ where $q > 1$, k, ε are real numbers and a, b, c are real-valued functions belonging to a certain class S. The authors use a variational method, together with a perturbation technique to prove that there exists $k_0 > 0$ such that for $k > -k_0$ and a, b, c belonging to S, the equation has a positive solution $u \in H^1(\mathbb{R})$, provided that $|\varepsilon|$ is sufficiently small. Equations of type (1.1) were also studied in [1].

The special features of this class of problems in the present paper is that it is defined in $\mathbb{R}$, involves the p-Laplacian operator, the nonlinear term $(|(u^2)'|^{p-2}(u^2)')'u$ and concave and convex type nonlinearities. Here, we adapt an argument developed by Poppenberg-Schmitt-Wang [16]. Our main results complement and improve some of their results, in sense that we are considering a more general class of operators and nonlinearities, and we have allowed that the potential V vanishes in a bounded part of the domain, so that our results are new even in the semilinear case because we deal with a more general class of potential. To obtain multiplicity of solutions for (1.1), we will apply the Bartsch-Willem Fountain theorem as well as the Dual Fountain theorem, see [6], [19].

Equations like (1.1) in the case $p = 2$ model several physical phenomena. More explicitly, solutions of the equation

$$-\Delta u + V(x)u - k\Delta(u^2)u = g(u) \quad \text{in} \quad \mathbb{R}^N \tag{1.2}$$

are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form $i\partial_t z = -\Delta z + V(x)z - h(|z|^2)z - k\Delta f(|z|^2)f'(|z|^2)z$ where $V = V(x), \quad x \in \mathbb{R}^N$, is a given potencial, k is a real constant and f, h are real functions. For more details on physical motivations and applications, we refer to [5], [16] and references therein.

There are several recent works on elliptic problems involving versions of (1.2), for instance, see [5], [10], [12], [13], [14], [15]. In [13], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.1) by the mountain-pass theorem. The same method of change of variables was used recently also in [10], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space and they studied a different class of nonlinearities. In [14], the existence of both one-sign and nodal ground states of soliton type solutions were established by the Nehari method. But in these papers, the authors do not deal with the concave and convex case.

The organization of this work is as follows: *Section 2* contains some preliminary results and we state the abstract theorems which we apply in this work. In *Section 3*, we establish the Palais-Smale condition and *Section 4* presents the proof of the main theorem.

Notation. In this paper we make use of the following notation:

- $\int_{\mathbb{R}}$ denotes the integral on the line real and C, C_0, C_1, C_2, ... denote positive (possibly different) constants.

- For $R > 0$, I_R denotes the open interval $(-R, R)$.

- For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ denotes the usual Lebesgue space with norms

$$\begin{aligned} \|u\|_p &= \left(\int_{\mathbb{R}} |u|^p \mathrm{d}x \right)^{1/p}, \ 1 \leq p < \infty; \\ \|u\|_\infty &= \inf \left\{ C > 0 : |u(x)| \leq C \text{ almost everywhere in } \mathbb{R} \right\}. \end{aligned}$$

- For $1 \leq p < \infty$, $W^{1,p}(\mathbb{R})$ denotes the Sobolev space modeled on $L^p(\mathbb{R})$ with norm

$$\|u\|_{1,p} = \left[\int_{\mathbb{R}} (|u'|^p + |u|^p) \mathrm{d}x \right]^{1/p}.$$

- By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X and its dual X^*;

- We denote weak convergence in X by " $\rightharpoonup$ " and strong convergence by " $\to$ ".

2 Abstract framework

In this section we establish some properties of the space X and functional $\mathcal{F}_{\lambda,\mu}$. By the condition (V_1), it follows that the embedding $X \hookrightarrow W^{1,p}(\mathbb{R})$ is continuous. Indeed,

$$\begin{aligned} \|u\|_{1,p}^p &\leq \int_{\mathbb{R}} |u'|^p \ \mathrm{d}x + \int_{-R_0}^{R_0} |u|^p \ \mathrm{d}x + \alpha^{-1} \int_{|x|>R_0} V(x)|u|^p \ \mathrm{d}x \\ &\leq \max\{1, \alpha^{-1}\} \|u\|^p + 2R_0 \|u\|_\infty^p. \end{aligned}$$

Since $\|u\|_\infty^p \leq p\|u\|_{1,p}^p$, see for instance Brezis [7], we conclude that

$$\|u\|_{1,p} \leq \left(\frac{\max\{1, \alpha^{-1}\}}{1 - 2pR_0} \right)^{1/p} \|u\|.$$

Furthermore, we have the following proposition:

Proposition 2.1 *Under the conditions $(V_1)-(V_2)$, the embedding from X into $L^s(\mathbb{R})$ is continuous for $1 \leq s \leq \infty$ and compact for $1 \leq s < \infty$.*

Proof. Using Hölder's inequality, we get

$$\begin{aligned}\int_{|x|\geq R_0}|u| \leq &\left(\int_{|x|\geq R_0} V(x)|u|^p\,\mathrm{d}x\right)^{1/p}\left(\int_{|x|\geq R_0} V(x)^{-1/(p-1)}\mathrm{d}x\right)^{p/(p-1)}\\ \leq &C\|u\|.\end{aligned}$$

Hence

$$\|u\|_1 = \int_{-R_0}^{R_0}|u|\mathrm{d}x + \int_{|x|>R_0}|u|\mathrm{d}x \leq 2R_0\|u\|_\infty + C\|u\| \leq C_1\|u\|.$$

Since X is immersed continuously in $L^\infty(\mathbb{R})$, we can conclude by interpolation that the embedding from X into $L^s(\mathbb{R})$ is continuous for $1 \leq s \leq \infty$. Now, we prove the compactness. Let (u_n) be a sequence in X satisfying $\|u_n\| \leq C$. Thus, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in} \quad X.$$

Without lost of generality, we can assume that $u_0 = 0$. Let us show that $u_n \to 0$ in $L^1(\mathbb{R})$. Given $\epsilon > 0$, for $R > 0$ large enough we obtain

$$\int_{|x|\geq R} V(x)^{-1/(p-1)}\mathrm{d}x < \left(\frac{\epsilon}{2C}\right)^{(p-1)/p}.$$

Hence

$$\begin{aligned}\int_{|x|\geq R}|u_n|\mathrm{d}x \leq &\left(\int_{|x|\geq R} V(x)|u_n|^p\,\mathrm{d}x\right)^{1/p}\left(\int_{|x|\geq R} V(x)^{-1/(p-1)}\mathrm{d}x\right)^{p/(p-1)}\\ < &\frac{\epsilon}{2C}\|u_n\| \leq \frac{\epsilon}{2}.\end{aligned}$$

On other hand, as the embedding $W^{1,p}(I_R) \hookrightarrow L^1(I_R)$ is compact there exists n_0 such that for all $n \geq n_0$

$$\int_{-R}^{R}|u_n|\mathrm{d}x < \frac{\epsilon}{2}.$$

Thus, for all $n \geq n_0$

$$\|u_n\|_1 = \int_{-R}^{R}|u_n|\mathrm{d}x + \int_{|x|\geq R}|u_n|\mathrm{d}x < \epsilon$$

which implies that $u_n \to 0$ in $L^1(\mathbb{R})$. From this convergence, if $1 < s < \infty$ we obtain

$$\|u_n\|_s^s = \int_{\mathbb{R}}|u_n|^{s-1}|u_n|\mathrm{d}x \leq \|u_n\|_\infty^{s-1}\|u_n\|_1 \leq C\|u_n\|_1 \to 0 \quad \text{as} \quad n \to \infty,$$

and consequently

$$u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}) \quad \text{for} \quad 1 \leq s < \infty$$

and the proof is complete. ∎

Lemma 2.2 *The functional $\mathcal{F}_{\lambda,\mu} : X \to \mathbb{R}$ is of class C^1 in X and for $u, v \in X$*

$$\begin{aligned}\langle \mathcal{F}'_{\lambda,\mu}(u), v\rangle = &\int_{\mathbb{R}} |u'|^{p-2}u'v'\mathrm{d}x + \int_{\mathbb{R}} V(x)|u|^{p-2}uv\mathrm{d}x + 2^{p-1}\int_{\mathbb{R}} |u|^p|u'|^{p-2}u'v'\mathrm{d}x \\ &+ 2^{p-1}\int_{\mathbb{R}} |u'|^p|u|^{p-2}uv\ \mathrm{d}x - \lambda\int_{\mathbb{R}} |u|^{q-2}uv\ \mathrm{d}x - \mu\int_{\mathbb{R}} |u|^{r-2}uv\mathrm{d}x.\end{aligned}$$

Proof. It will be sufficient to show that $\Phi(u) := \int_{\mathbb{R}} |u'|^p|u|^p\ \mathrm{d}x$ is of class C^1 in X, because for the other summands the proof is standard.
Existence of Gateaux derivative: Let $u, v \in X$ and $0 \neq t \in \mathbb{R}$. We have

$$\begin{aligned}\frac{\Phi(u+tv) - \Phi(u)}{t} = \frac{1}{t}\int_{\mathbb{R}} \big[&|u' + tv'|^p(|u+tv|^p - |u|^p) \\ &+ (|u'+tv'|^p - |u'|^p)|u|^p\big]\mathrm{d}x.\end{aligned} \tag{2.1}$$

Now, given $x \in \mathbb{R}$ and $0 < |t| < 1$, by the mean value theorem, there exist $\lambda, \theta \in (0, 1)$ such that

$$\begin{aligned}\frac{||u(x)+tv(x)|^p - |u(x)|^p|}{|t|} &= p|u(x) + \lambda tv(x)|^{p-1}|v(x)| \\ &\leq p\left(|u(x)| + |v(x)|\right)^{p-1}|v(x)|\end{aligned}$$

and

$$\begin{aligned}\frac{||u'(x)+tv'(x)|^p - |u'(x)|^p|}{|t|} &= p|u'(x) + \theta tv'(x)|^{p-1}|v'(x)| \\ &\leq p\left(|u'(x)| + |v'(x)|\right)^{p-1}|v'(x)|.\end{aligned}$$

As $\left(|u| + |v|\right)^{p-1}|v|, |u|^p \in L^\infty(\mathbb{R})$ and $(|u'| + |v'|)^p, \left(|u'| + |v'|\right)^{p-1}|v'| \in L^1(\mathbb{R})$, it follows, from (2.1) and Lebesgue's dominated convergence theorem, that

$$\langle \Phi'(u), v\rangle = p\int_{\mathbb{R}} |u|^p|u'|^{p-2}u'v'\mathrm{d}x + p\int_{\mathbb{R}} |u'|^p|u|^{p-2}uv\ \mathrm{d}x.$$

Continuity of the Gateaux derivative: We choose a sequence $u_n \to u$ in X and $v \in X$ with $\|v\| \leq 1$. By the Hölder's inequality we have

$$\begin{aligned}&|\langle \Phi'(u_n), v\rangle - \langle \Phi'(u), v\rangle| \\ &\leq p\int_{\mathbb{R}} \left||u_n|^p|u_n'|^{p-2}u_n' - |u|^p|u'|^{p-2}u'\right||v'|\mathrm{d}x \\ &\quad + p\int_{\mathbb{R}} \left||u_n'|^p|u_n|^{p-2}u_n - |u'|^p|u|^{p-2}u\right||v|\mathrm{d}x \\ &\leq p\left(\int_{\mathbb{R}} \left||u_n|^p|u_n'|^{p-2}u_n' - |u|^p|u'|^{p-2}u'\right|^{p/(p-1)}\mathrm{d}x\right)^{(p-1)/p} \\ &\quad + pC\left(\int_{\mathbb{R}} \left||u_n'|^p|u_n|^{p-2}u_n - |u'|^p|u|^{p-2}u\right|^{p/(p-1)}\mathrm{d}x\right)^{(p-1)/p}.\end{aligned}$$

According to the Lebesgue's dominated convergence theorem, the right hand side tends to 0 uniformly for $\|v\| \leq 1$. Thus $\Phi' : X \to X^*$ is continuous. ■

Let us consider the condition below:

(A_1) the compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{j=0}^{\infty} X_j}$, the spaces X_j are invariant and there exists a finite dimensional space V such that, for every $j \in \mathbb{N}$, $X_j \simeq V$ and the action of G on V is admissible, that is, every continuous equivariant map $\partial U \to V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in $V^k, k \geq 2$, has a zero.

From now on, we follow the notations:

$$Y_k := \bigoplus_{j=0}^{k} X_j, \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j},$$

To prove the item (a) of the Theorem 1.1, we shall use the Fountain theorem of T. Bartsch [6] as given in [19, Theorem 3.6]:

Lemma 2.3 (Fountain Theorem) *Under the assumption (A_1), let $I \in C^1(X, \mathbb{R})$ be an invariant functional. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A_2) $a_k := \max\limits_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0;$

(A_3) $b_k := \min\limits_{u \in Z_k, \|u\| = r_k} I(u) \to \infty, k \to \infty;$

(A_4) *I satisfies the $(PS)_c$ condition for every $c > 0$,*

then I has an unbounded sequence of critical values.

For the item (b), we shall apply a dual version of the Fountain Theorem, see [6, Theorem 2] or [19, theorem 3.18].

Lemma 2.4 (Dual Fountain Theorem) *Under the assumption (A_1), let $I \in C^1(X, \mathbb{R})$ be an invariant functional. Moreover, suppose that I satisfies the following conditions:*

(B_1) *for every $k \geq k_0$, there exists $R_k > 0$ such that $I(u) \geq 0$ for every $u \in Z_k$ with $\|u\| = R_k$;*

(B_2) $b_k := \inf\limits_{u \in Z_k, \|u\| \leq R_k} I(u) \to 0$ *as* $k \to \infty;$

(B_3) *for every $k \geq 1$ there exists $r_k \in (0, R_k)$ and $d_k < 0$ such that $I(u) \leq d_k$ for every $u \in Y_k$ with $\|u\| = r_k$;*

(B_4) *every sequence* $(u_n) \subset Y_n$ *with* $I(u_n) < 0$ *bounded and* $(I|_{Y_n})'(u_n) \to 0$ *as* $n \to \infty$ *has a subsequence which converges to a critical point of* I.

Then for each $k \geq k_0$, I *has a critical value* $c_k \in [b_k, d_k]$ *and* $c_k \to 0$ *as* $k \to \infty$.

Observe that (B_2) and (B_3) imply $b_k \leq d_k < 0$.

3 The Palais-Smale condition

We begin this section by proving that any Palais-Smale sequence of the functional $\mathcal{F}_{\lambda,\mu}$ is bounded.

Lemma 3.1 *Any* $(PS)_c$ *sequence in* X, *that is, satisfying* $\mathcal{F}_{\lambda,\mu}(u_n) \to c$ *and* $\mathcal{F}'_{\lambda,\mu}(u_n) \to 0$ *is bounded.*

Proof. We have for n large

$$\mathcal{F}_{\lambda,\mu}(u_n) - \frac{1}{r}\langle \mathcal{F}'_{\lambda,\mu}(u_n), u_n\rangle \leq \|u_n\| + c + 1.$$

On the other hand, by Lemma 2.2,

$$\begin{aligned}\mathcal{F}_{\lambda,\mu}(u_n) - \frac{1}{r}\langle \mathcal{F}'_{\lambda,\mu}(u_n), u_n\rangle = &\left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|^p + 2^{p-1}\left(\frac{1}{p} - \frac{2}{r}\right)\int_{\mathbb{R}} |u_n'|^p |u_n|^p \,\mathrm{d}x \\ &+ \lambda\left(\frac{1}{r} - \frac{1}{q}\right)\int_{\mathbb{R}} |u_n|^q \,\mathrm{d}x.\end{aligned}$$

Thus,

$$\|u_n\| + c + 1 \geq \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|^p - C\|u_n\|^q,$$

from which it follows that (u_n) is bounded since $r > 2p$ and $p > q$. ■

Lemma 3.2 (The Palais-Smale condition) *The functional* $\mathcal{F}_{\lambda,\mu}$ *satisfies the* $(PS)_c$ *condition for all* $c \in \mathbb{R}$.

Proof. Let (u_n) be in X satisfying $\mathcal{F}_{\lambda,\mu}(u_n) \to c$ and $\mathcal{F}'_{\lambda,\mu}(u_n) \to 0$. We will show that (u_n) has a convergent subsequence. By Lemma 3.1, (u_n) is bounded. Thus, up to a subsequence, $u_n \rightharpoonup u$ in X and using Proposition 2.1 $u_n \to u$ in $L^q(\mathbb{R})$ and in $L^s(\mathbb{R})$. Therefore, the Hölder's inequality implies that

$$\begin{aligned}&\int_{\mathbb{R}} |u_n|^{q-2} u_n (u_n - u) \,\mathrm{d}x \to 0 \quad \text{and} \\ &\int_{\mathbb{R}} |u_n|^{r-2} u_n (u_n - u) \,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{3.1}$$

We have

$$\begin{aligned}
o_n(1) =& \langle \mathcal{F}'_{\lambda,\mu}(u_n), u_n - u\rangle + \lambda \int_{\mathbb{R}} |u_n|^{q-2}u_n(u_n-u)\mathrm{d}x \\
&+ \mu \int_{\mathbb{R}} |u_n|^{r-2}u_n(u_n-u)\mathrm{d}x \\
=& \int_{\mathbb{R}} |u_n'|^{p-2}u_n'(u_n'-u')\mathrm{d}x + \int_{\mathbb{R}} V(x)|u_n|^{p-2}u_n(u_n-u)\mathrm{d}x \\
&+ 2^{p-1}\Big[\int_{\mathbb{R}} |u_n|^p|u_n'|^{p-2}u_n'(u_n'-u')\mathrm{d}x + \int_{\mathbb{R}} |u_n'|^p|u_n|^{p-2}u_n(u_n-u)\mathrm{d}x\Big]
\end{aligned}$$

and since $\int_{\mathbb{R}} |u'|^{p-2}u'(u_n'-u')\mathrm{d}x = o_n(1)$ and $\int_{\mathbb{R}} V(x)|u|^{p-2}u(u_n-u)\mathrm{d}x = o_n(1)$ as $n \to \infty$, the equality above can be rewritten as

$$\begin{aligned}
o_n(1) =& \int_{\mathbb{R}} (|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n'-u')\mathrm{d}x \\
&+ \int_{\mathbb{R}} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\mathrm{d}x \\
&+ 2^{p-1}\Big[\underbrace{\int_{-R}^{R} |u_n|^p|u_n'|^{p-2}u_n'(u_n'-u')\mathrm{d}x}_{I_1} + \underbrace{\int_{|x|>R} |u_n|^p|u_n'|^{p-2}u_n'(u_n'-u')\mathrm{d}x}_{I_2}\Big] \\
&+ 2^{p-1}\Big[\underbrace{\int_{-R}^{R} |u_n'|^p|u_n|^{p-2}u_n(u_n-u)\mathrm{d}x}_{I_3} + \underbrace{\int_{|x|>R} |u_n'|^p|u_n|^{p-2}u_n(u_n-u)\mathrm{d}x}_{I_4}\Big].
\end{aligned}$$

As $|u|^p, |u'|^p \in L^1(\mathbb{R})$, given $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|>R} |u|^p\ \mathrm{d}x < \epsilon^p \quad \text{and} \quad \int_{|x|>R} |u'|^p\ \mathrm{d}x < \epsilon^p. \tag{3.2}$$

By the convergence $\int_{\mathbb{R}} |u|^p|u'|^{p-2}u'(u_n'-u') \to 0$ as $n \to \infty$ we obtain

$$\begin{aligned}
I_1 =& \int_{-R}^{R} (|u_n|^p|u_n'|^{p-2}u_n' - |u|^p|u'|^{p-2}u')(u_n'-u')\mathrm{d}x \\
=& \int_{-R}^{R} (|u_n|^p - |u|^p)|u'|^{p-2}u'(u_n'-u')\mathrm{d}x \\
&+ \int_{-R}^{R} |u_n|^p(|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n'-u')\mathrm{d}x.
\end{aligned}$$

Using the inequality

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p, & \text{if} \quad p \geq 2 \\ C_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if} \quad 1 < p < 2 \end{cases} \tag{3.3}$$

where $x, y \in \mathbb{R}^N$, $C_p > 0$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^N$ (see Simon [17]), the second summand in I_1 is nonnegative and applying the mean value theorem to the integrand of the first integral, we get

$$I_1 \geq \int_{-R}^{R} |\xi_n|^{p-2}\xi_n(u_n - u)|u'|^{p-2}u'(u_n' - u')\mathrm{d}x + o_n(1),$$

where

$$\min\{u_n, u\} \leq \xi_n \leq \max\{u_n, u\}.$$

Notice that for all n and for some $C > 0$, $|\xi_n(x)| \leq C$ almost everywhere in $\mathbb{R}$ because (u_n) is bounded in $L^\infty(\mathbb{R})$. Observe that

$$\begin{aligned}
&\left| \int_{-R}^{R} |\xi_n|^{p-2}\xi_n(u_n - u)|u'|^{p-2}u'(u_n' - u')\mathrm{d}x \right| \\
&\leq \int_{-R}^{R} |\xi_n|^{p-1}|u_n - u||u'|^{p-1}|u_n' - u'|\mathrm{d}x \\
&\leq C\|u_n - u\|_{L^\infty(I_R)} \int_{-R}^{R} |u'|^{p-1}|u_n' - u'|\mathrm{d}x \\
&\leq C\|u_n - u\|_{L^\infty(I_R)} \left(\int_{-R}^{R} |u'|^p \ \mathrm{d}x \right)^{(p-1)/p} \left(\int_{-R}^{R} |u_n' - u'|^p \ \mathrm{d}x \right)^{1/p} \\
&\leq C\|u_n - u\|_{L^\infty(I_R)} \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

since the embedding $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(I_R)$ is compact. Thus $I_1 \geq o_n(1)$. Next, our purpose is to estimate the integral I_2. We have that

$$\begin{aligned}
I_2 &= \int_{|x|>R} |u_n|^p|u_n'|^p \ \mathrm{d}x - \int_{|x|>R} |u_n|^p|u_n'|^{p-2}u_n'u'\mathrm{d}x \\
&\geq - \int_{|x|>R} |u_n|^p|u_n'|^{p-2}u_n'u'\mathrm{d}x.
\end{aligned}$$

By (3.2), we can conclude that

$$\begin{aligned}
\left| \int_{|x|>R} |u_n'|^p|u_n|^{p-2}u_n'u'\mathrm{d}x \right| \leq &\|u_n\|^p \left(\int_{|x|>R} |u_n'|^p \ \mathrm{d}x \right)^{(p-1)/p} \left(\int_{|x|>R} |u'|^p \ \mathrm{d}x \right)^{1/p} \\
\leq &C\epsilon.
\end{aligned}$$

Therefore $I_2 \geq -C\epsilon$. For the integral I_3, we have

$$\begin{aligned}
|I_3| \leq \int_{-R}^{R} |u_n'|^p|u_n|^{p-1}|u_n - u|\mathrm{d}x \leq &\|u_n - u\|_{L^\infty(I_R)}\|u_n\|_\infty^{p-1}\|u_n\| \\
\leq &C\|u_n - u\|_{L^\infty(I_R)} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Writing the integral I_4 in the following way

$$I_4 = \int_{|x|>R} |u_n'|^p|u_n|^p \ \mathrm{d}x - \int_{|x|>R} u_n'|^p|u_n|^{p-2}u_n u \ \mathrm{d}x,$$

similarly to I_2, we conclude that $I_4 \geq -C\epsilon$, where $\epsilon > 0$ was fixed previously. Summarizing, we obtain

$$\begin{aligned} 2^pC\epsilon + o_n(1) \geq &\int_{\mathbb{R}} (|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n' - u')\mathrm{d}x \\ &+ \int_{\mathbb{R}} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x \end{aligned}$$

and by the inequality (3.3) we get

$$2^pC_p\epsilon + o_n(1) \geq \begin{cases} \displaystyle\int_{\mathbb{R}} |u_n' - u'|^p \ \mathrm{d}x + \int_{\mathbb{R}} V(x)|u_n - u|^p \ \mathrm{d}x, \text{ if } \ p \geq 2 \\ \displaystyle\int_{\mathbb{R}} \frac{|u_n' - u'|^2}{(|u_n'| + |u'|)^{2-p}}\mathrm{d}x + \int_{\mathbb{R}} \frac{V(x)|u_n - u|^2}{(|u_n| + |u|)^{2-p}} \ \mathrm{d}x, \text{ if } \ 1 < p < 2. \end{cases}$$

Thus, if $p \geq 2$ we deduce that $\|u_n - u\|^p \to 0$ as $n \to \infty$ since $\epsilon > 0$ was arbitrary. Therefore $u_n \to u$ in X. If $1 < p < 2$ we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{|u_n' - u'|^2}{(|u_n'| + |u'|)^{2-p}}\mathrm{d}x = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}} V(x)\frac{|u_n - u|^2}{(|u_n| + |u|)^{2-p}}\mathrm{d}x = 0. \tag{3.4}$$

Using the Hölder's inequality and the fact that (u_n') is bounded in $L^p(\mathbb{R})$ we get

$$\begin{aligned} \int_{\mathbb{R}} |u_n' - u'|^p \ \mathrm{d}x \leq &\left(\int_{\mathbb{R}} \frac{|u_n' - u'|^2}{(|u_n'| + |u'|)^{2-p}}\mathrm{d}x\right)^{p/2} \left(\int_{\mathbb{R}} (|u_n'| + |u'|)^p \mathrm{d}x\right)^{(2-p)/2} \\ \leq &C \left(\int_{\mathbb{R}} \frac{|u_n' - u'|^2}{(|u_n'| + |u'|)^{2-p}}\mathrm{d}x\right)^{p/2} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

By similar arguments, we also have that $\int_{\mathbb{R}} V(x)|u_n - u|^p\mathrm{d}x \to 0$ which shows that $\|u_n - u\|^p \to 0$ and the proof is complete. ∎

4 Proof of the main result

Now, we are ready to prove the main theorem:

Proof of Theorem 1.1.

(a) Let us verify that in Lemma 2.3 the conditions $(A_1) - (A_4)$ are satisfied. Since X is a reflexive, separable Banach space, let us fix $(e_j)_{j=0}^{\infty}$ a Schauder basis for X.

Thus $X = \overline{\bigoplus_{j=0}^{\infty} X_j}$ where $X_j = \mathbb{R}e_j$ and on X we consider the antipodal action of $\mathbb{Z}_2$ which verifies the condition (A_1). Let us define

$$\beta_k := \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_r}{\|u\|}.$$

By similar arguments as in [19, Lemma 3.8], we have that $\beta_k \to 0$ as $k \to \infty$. On Z_k, we get

$$\mathcal{F}_{\lambda,\mu}(u) \geq \frac{1}{p}\|u\|^p - \frac{|\lambda|}{q}\|u\|_q^q - \frac{\mu}{r}\|u\|_r^r \geq \frac{1}{p}\|u\|^p - \frac{|\lambda|}{q}C\|u\|^q - \frac{\mu}{r}\beta_k^r\|u\|^r.$$

Since $q < p$ we have

$$\frac{|\lambda|}{q}C\|u\|^q \leq \frac{1}{2p}\|u\|^p$$

for $\|u\| \geq R, \quad R > 0$ large. Thus, for $\|u\| \geq R$ we obtain

$$\mathcal{F}_{\lambda,\mu}(u) \geq \frac{1}{2p}\|u\|^p - \frac{\mu}{r}\beta_k^r\|u\|^r.$$

Choosing $r_k := (\mu\beta_k^r)^{1/(p-r)}$, it follows that $r_k \to \infty$. Hence, there exists k_0 such that $r_k \geq R$ for all $k \geq k_0$. Thus, if $u \in Z_k$ and $\|u\| = r_k$ with $k \geq k_0$, we obtain

$$\mathcal{F}_{\lambda,\mu}(u) \geq \left(\frac{1}{2p} - \frac{1}{r}\right)(\mu\beta_k^r)^{p/(p-r)}$$

and since $\beta_k \to 0$ as $k \to \infty$, relation (A_3) is proved. The functional $\mathcal{F}_{\lambda,\mu}$ is even and

$$\begin{aligned}
\mathcal{F}_{\lambda,\mu}(u) &\leq \frac{1}{p}\|u\|^p + \frac{2^{p-1}}{p}C\|u\|_\infty^p\|u\|^p - \frac{\lambda}{q}\|u\|_q^q - \frac{\mu}{r}\|u\|_r^r \\
&\leq \frac{1}{p}\|u\|^p + \frac{2^{p-1}}{p}C\|u\|^{2p} - \frac{\lambda}{q}\|u\|_q^q - \frac{\mu}{r}\|u\|_r^r \\
&\leq \frac{1}{p}\|u\|^p + \frac{2^{p-1}}{p}C\|u\|^{2p} - \frac{\lambda}{q}C\|u\|^q - \frac{\mu}{r}C\|u\|^r
\end{aligned}$$

because on the finite-dimensional space Y_k all norms are equivalent. Since $r > 2p$ the relation (A_2) is satisfied for every $\rho_k > r_k > 0$ large enough.

The condition (A_4) holds by Lemma 3.2. It suffices then to use the Fountain theorem and the item (a) is proved.

(b) In order to see (B_1) of Lemma 2.4 we set

$$\alpha_k := \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_q}{\|u\|}.$$

It follows easily from the Proposition 2.1 that $\alpha_k \to 0$ as $k \to \infty$. We obtain for $u \in Z_k$

$$\mathcal{F}_{\lambda,\mu}(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda}{q}\alpha_k^q\|u\|^q - \frac{|\mu|}{r}C\|u\|^r.$$

Since $r > p$, we have

$$\frac{|\mu|}{r}C\|u\|^r \leq \frac{1}{2p}\|u\|^p$$

for $\|u\| \leq R, \quad R > 0$ small. Thus,

$$\mathcal{F}_{\lambda,\mu}(u) \geq \frac{1}{2p}\|u\|^p - \frac{\lambda}{q}\alpha_k^q\|u\|^q.$$

Setting $R_k := \left(\frac{2p\lambda\alpha_k^q}{q}\right)^{1/(p-q)}$, we get

$$\frac{1}{2p}R_k^p = \frac{\lambda}{q}\alpha_k^q R_k^q.$$

Clearly $R_k \to 0$, so there exists k_0 with $R_k \leq R$ when $k \geq k_0$. Thus, if $u \in Z_k, k \geq k_0$ and satisfies $\|u\| = R_k$ we have

$$\mathcal{F}_{\lambda,\mu}(u) \geq \frac{1}{2p}\|u\|^p - \frac{\lambda}{q}\alpha_k^q\|u\|^q = 0.$$

This proves (B_1). Next, (B_2) follows immediately from $R_k \to 0$. To check (B_3), we observe that on the finite dimensional space Y_k all norms are equivalent. Hence, there exist $C_0, C_1 > 0$ such that

$$\begin{aligned}\mathcal{F}_{\lambda,\mu}(u) \leq& \frac{1}{p}\|u\|^p + C\|u\|^{2p} - \frac{\lambda}{q}C_0\|u\|^q - \frac{\mu}{r}C_1\|u\|^r \\ =&\|u\|^q\left(\frac{1}{p}\|u\|^{p-q} + C\|u\|^{2p-q} - \frac{\lambda}{q}C_0 - \frac{\mu}{r}C_1\|u\|^{r-q}\right)\end{aligned}$$

Since $q < p < r$ and $R_k \to 0$, taking $r_k \in (0, R_k)$ with R_k sufficiently small, (B_3) is satisfied. This is precisely the point where $\lambda > 0$ enters. Finally, the condition (B_4) is showed similarly to Lemma 3.2. ∎

In conclusion, we add some remarks about the behavior of solutions with respect to parameters λ and μ.

Remark 1 *(a) For $\lambda \in \mathbb{R}$ and $\mu \leq 0$ there are no solutions with positive energy. Moreover*

$$\inf\{\|u\| : u \;\; \textit{solves} \;\; (1.1), \;\; \mathcal{F}_{\lambda,\mu}(u) > 0\} \to \infty \quad \textit{as} \quad \mu \to 0^+.$$

(b) For $\mu \in \mathbb{R}$ and $\lambda \leq 0$ there are no solutions with negative energy. Moreover

$$\sup\{\|u\| : u \;\; \textit{solves} \;\; (1.1), \;\; \mathcal{F}_{\lambda,\mu}(u) < 0\} \to 0 \quad \textit{as} \quad \lambda \to 0^+.$$

Proof of (a): Let $\lambda, \mu \in \mathbb{R}$. From $\mathcal{F}'_{\lambda,\mu}(u) = 0$ we obtain

$$\lambda \|u\|_q^q = -\mu \|u\|_r^r + \|u\|^p + 2^p \int_{\mathbb{R}} |u|^p |u'|^p \, \mathrm{d}x.$$

If $\mathcal{F}_{\lambda,\mu}(u) \geq 0$, we have

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|u\|^p + \mu \left(\frac{1}{q} - \frac{1}{r}\right) \|u\|_r^r + 2^p \left(\frac{1}{2p} - \frac{1}{q}\right) \int_{\mathbb{R}} |u|^p |u'|^p \, \mathrm{d}x \geq 0.$$

Since $1 < q < p$ and $r > 2p$, we see immediately that for $\mu \leq 0$ the only solution with nonnegative energy is $u = 0$. Now, if $\mu > 0$, then there are constants $c_1, c_2 > 0$ with

$$-c_1 \|u\|^p + \mu \|u\|^r \geq 0,$$

hence

$$\|u\|^{r-p} \geq \mu^{-1} c_1 / c_2 \to +\infty \quad \text{as} \quad \mu \to 0^+,$$

and the result follows. ■

Proof of (b): Fix $\lambda, \mu \in \mathbb{R}$. Similarly, from $\mathcal{F}'_{\lambda,\mu}(u) = 0$ we obtain

$$\mu \|u\|_r^r = -\lambda \|u\|_q^q + \|u\|^p + 2^p \int_{\mathbb{R}} |u|^p |u'|^p \, \mathrm{d}x.$$

Thus, if $\mathcal{F}_{\lambda,\mu}(u) \leq 0$ we have

$$\left(\frac{1}{p} - \frac{1}{r}\right) \|u\|^p + \lambda \left(\frac{1}{r} - \frac{1}{q}\right) \|u\|_q^q + 2^p \left(\frac{1}{2p} - \frac{1}{r}\right) \int_{\mathbb{R}} |u|^p |u'|^p \, \mathrm{d}x \leq 0.$$

This implies that for $\lambda \leq 0$ the only solution with non-positive energy is $u = 0$. For $\mu > 0$, there are constants $c_3, c_4 > 0$ with

$$c_3 \|u\|^p - \lambda c_4 \|u\|^q \leq 0,$$

hence

$$\|u\|^{p-q} \leq \lambda c_4 / c_3 \to 0 \quad \text{as} \quad \lambda \to 0^+.$$

■

Remark 2 *For $\lambda > 0$ small, it is easy to see that the functional $\mathcal{F}_{\lambda,\mu}$ possesses the mountain-pass geometry. Thus, using the mountain-pass theorem (see [19, Theorem 2.10]) equation (1.1) has a mountain-pass type solution. This solution is nonnegative. Indeed, it is enough to work with the functional*

$$J_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \frac{2^{p-1}}{p} \int_{\mathbb{R}} |u'|^p |u|^p \, \mathrm{d}x - \frac{\lambda}{q} \int_{\mathbb{R}} (u^+)^q \, \mathrm{d}x - \frac{\mu}{r} \int_{\mathbb{R}} (u^+)^r \, \mathrm{d}x$$

and if $\langle J'_{\lambda,\mu}(u), v \rangle = 0$ for all $v \in X$, setting $v = u^-$, where $u^- = \max\{-u, 0\}$, we obtain that $u^- = 0$. Thus $u \geq 0$.

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