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Positive solutions of boundary value problems for nth order ordinary differential equations [1]

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Abstract

In this paper, we investigate the problem of existence and nonexistence of positive solutions for the nonlinear boundary value problem:

$$u^{(n)}(t) + \lambda a(t) f(u(t)) = 0, \;\; 0 < t < 1,$$

satisfying three kinds of different boundary value conditions. Our analysis relies on Krasnoselskii's fixed point theorem of cone. An example is also given to illustrate the main results.

1. Introduction

There is currently a great deal of interest in positive solutions for several types of boundary value problems. A large part of the literature on positive solutions to boundary value problems seems to be traced back to Krasnoselskii's work on nonlinear operator equations [6], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [3] applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter λ for which there is at least one positive solution. In 1995, Eloe and Henderson [1] obtained the solutions that are positive to a cone for the boundary value problem

$$u^{(n)}(t) + a(t) f(u) = 0, \qquad 0 < t < 1,$$

$$u^{(i)}(0) = u^{(n-2)}(1), \qquad 0 \le i \le n-2.$$

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Since this pioneering works, a lot research has been done in this area [2, 3, 5, 7, 8, 9]. The purpose of this paper is to establish the existence of positive solutions to nonlinear nth order boundary value problems:

$$u^{(n)}(t) + \lambda\ a(t)f(u(t)) = 0,\ \ 0 < t < 1, \tag{1}$$

$$u(0) = u''(0) = u'''(0) = ... = u^{(n-1)} = 0,\ u'(1) = 0, \tag{2}$$

$$u(0) = u'(0) = u''(0) = ... = u^{(n-2)} = 0,\ u'(1) = 0, \tag{3}$$

$$u(0) = u'(0) = u''(0) = ... = u^{(n-2)} = 0,\ u''(1) = 0, \tag{4}$$

where λ is a positive parameter. Throughout the paper, we assume that

C1: $f : [0, \infty) \to [0, \infty)$ is continuous

C2: $a :\ (0, 1) \to [0, \infty)$ is continuous function such that $\int_0^1 a(t)\, dt > 0$.

2. Preliminaries

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof our main results.

Definition 1. *Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called* ***cone*** *of E if it satisfies the following conditions:*

1. *$x \in K$, $\sigma \geq 0$ implies $\sigma x \in K$;*
2. *$x \in K$, $-x \in K$ implies $x = 0$.*

Definition 2. *An operator is called* ***completely continuous*** *if it is continuous and maps bounded sets into precompact sets.*

Lemma 1. *Let E be a Banach space and $K \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be completely continuous operator. In addition suppose either:*

H1 : $\|Tu\| \leq \|u\|,\ \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|,\ \forall u \in K \cap \partial\Omega_2$ *or*

H2 : $\|Tu\| \leq \|u\|,\ \forall u \in K \cap \partial\Omega_2$ *and* $\|Tu\| \geq \|u\|,\ \forall u \in K \cap \partial\Omega_1$.

holds. Then T has a fixed pint in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Green functions and their properties

Lemma 2. *Let $y \in C[0,1]$, then the boundary value problem*

$$u_2^{(n)}(t) + y(t) = 0, \;\; 0 < t < 1, \tag{5}$$

$$u_2(0) = u_2''(0) = u_2'''(0) = ...u_2^{(n-1)}(0) = 0, \; u_2'(1) = 0, \tag{6}$$

has a unique solution

$$u_2(t) = \int_0^1 G_2(t,s)y(s)ds,$$

where

$$G_2(t,s) = \begin{cases} \frac{t(1-s)^{n-2}}{(n-2)!} - \frac{(t-s)^{n-1}}{(n-1)!} & \text{if} \quad 0 \le s \le t \le 1; \\ \frac{t(1-s)^{n-2}}{(n-2)!} & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

Proof. Applying the Laplace transform to Eq(5) we get

$$s^n u_2(s) - s^{n-2}u_2'(0) = -y(s), \tag{7}$$

where $u_2(s)$and $y(s)$ is the Laplace transform of $u_2(t)$ and $y(t)$ respectively. The Laplace inversion of Eq (7) gives the final solution as:

$$u_2(t) = \int_0^1 \frac{t(1-s)^{n-2}}{(n-2)!}y(s)ds - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds. \tag{8}$$

The proof is complete. □

It is obvious that

$$G_2(t,s) \ge 0 \text{ and } G_2(1,s) \ge G_2(t,s), \;\; 0 \le t, s \le 1. \tag{9}$$

Lemma 3. *$G_2(t,s) \ge q_2(t)G_2(1,s)$ for $0 \le t, s \le 1$, where $q_2(t) = \frac{(n-2)t}{n-1}$.*

Proof. If $t \ge s$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{(n-1)t(1-s)^{n-2} - (t-s)^{n-1}}{(n-1)(1-s)^{n-2} - (1-s)^{n-1}}$$

$$= \frac{(n-1)t(1-s)^{n-2} - (t-s)(t-s)^{n-2}}{(1-s)^{n-2}(n-2+s)} \ge \frac{(n-1)t}{n-1}.$$

If $t \le s$, then $\frac{G_2(t,s)}{G_2(1,s)} = t \ge \frac{(n-1)t}{n-1}$. The proof is complete. □

Lemma 4. *Let $y \in C[0,1]$, then the boundary value problem*

$$u_3^{(n)}(t) + y(t) = 0, \ \ 0 < t < 1, \tag{10}$$

$$u_3(0) = u_3'(0) = u_3''(0) = ... = u_3^{n-2}(0) = 0, \, u_3'(1) = 0, \tag{11}$$

has a unique solution

$$u_3(t) = \int_0^1 G_3(t,s)y(s)ds,$$

where

$$G_3(t,s) = \begin{cases} \frac{t^{n-1}(1-s)^{n-2}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!} & \text{if} \quad 0 \le s \le t \le 1; \\ \frac{t^{n-1}(1-s)^{n-2}}{(n-1)!} & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

Proof. Applying the Laplace transform to Eq(10) we get

$$s^n u_2(s) - u_2^{n-1}(0) = -y(s), \tag{12}$$

where $u_3(s)$ is the Laplace transform of $u_3(t)$. The Laplace inversion of Eq (12) gives the final solution as:

$$u_3(t) = \int_0^1 \frac{t^{n-1}(1-s)^{n-2}}{(n-1)!} y(s)ds - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds. \tag{13}$$

The proof is complete. □

It is obvious that

$$G_3(t,s) \ge 0 \text{ and } G_3(1,s) \ge G_3(t,s), \ \ 0 \le t, s \le 1. \tag{14}$$

Lemma 5. $G_3(t,s) \ge q_3(t)G_3(1,s)$ *for* $0 \le t, s \le 1$, *where* $q_3(t) = t^{n-1}$.

Proof. If $t \ge s$, then

$$\frac{G_3(t,s)}{G_3(1,s)} = \frac{t^{n-1}(1-s)^{n-2} - (t-s)^{n-1}}{(1-s)^{n-2} - (1-s)^{n-1}}$$

$$= \frac{t(t-ts)^{n-2} - (t-s)(t-s)^{n-2}}{s(1-s)^{n-2}} \ge t^{n-2} \ge t^{n-1}.$$

If $t \le s$, then $\frac{G_3(t,s)}{G_3(1,s)} = t^{n-1}$. The proof is complete. □

Lemma 6. *Let $y \in C[0,1]$, then the boundary value problem*

$$u_4^{(n)}(t) + y(t) = 0, \;\; 0 < t < 1, \tag{15}$$

$$u_4(0) = u_4'(0) = u_4''(0) = ... = u_4^{n-2}(0) = 0, \; u_4''(1) = 0, \tag{16}$$

has a unique solution

$$u_4(t) = \int_0^1 G_4(t,s)y(s)ds,$$

where

$$G_4(t,s) = \begin{cases} \frac{t^{n-1}(1-s)^{n-3}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!} & \text{if} \quad 0 \le s \le t \le 1; \\ \frac{t^{n-1}(1-s)^{n-3}}{(n-1)!} & \text{if} \quad 0 \le t \le s \le 1.. \end{cases}$$

The proof of Lemma 6 is very similar to that of Lemma 4 and therefore omitted. It is obvious that

$$G_4(t,s) \ge 0 \text{ and } G_4(1,s) \ge G_4(t,s), \;\; 0 \le t, s \le 1. \tag{17}$$

Lemma 7. $G_4(t,s) \ge q_4(t)G_4(1,s)$ *for* $0 \le t, s \le 1$, *where* $q_4(t) = \frac{t^{n-1}}{2}$.

Proof. If $t \ge s$, then

$$\begin{aligned} \frac{G_4(t,s)}{G_4(1,s)} &= \frac{t^{n-1}(1-s)^{n-3} - (t-s)^{n-1}}{(1-s)^{n-3} - (1-s)^{n-1}} \\ &\ge \frac{t^2(t-ts)^{n-3} - (t-s)^2(t-ts)^{n-3}}{(1-s)^{n-3}(2s-s^2)} \ge \frac{t^{n-1}}{2}. \end{aligned}$$

If $t \le s$, then $\frac{G_4(t,s)}{G_4(1,s)} = t^{n-1} \ge \frac{t^{n-1}}{2}$. The proof is complete. □

3. Main results

In this section, we will apply Krasnoselskii's fixed point theorem to the eigenvalue problem (1), (i) (i=2,3,4). We note that $u_i(t)$ is a solution of (1),(i) if and only if

$$u_i(t) = \lambda \int_0^1 G_i(t,s)\, a(s) f(u_i(s))\, ds, \quad\;\; 0 \le t \le 1.$$

For our constructions, we shall consider the Banach space $X = C[0,1]$ equipped with standard norm $\|u_i\| = \max_{0 \le t \le 1} |u_i(t)|$, $u_i \in X$. We define a cone P by

$$P = \{u_i \in X : u_i(t) \ge q(t)\, \|u_i\|, t \in [0,1]\},$$

It is easy to see that if $u_i \in P$, then $\|u_i\| = u_i(1)$. Define an integral operator by:

$$Tu_i(t) = \lambda \int_0^1 G_i(t,s)\, a(s) f(u_i(s))\, ds, \qquad 0 \le t \le 1,\ u_i \in P. \qquad (18)$$

Lemma 8. $T(P) \subset P$.

Proof. Notice from (9), (12) and (15) that, for $u_i \in P$, $Tu_i(t) \ge 0$ on $[0,1]$ and

$$\begin{aligned} Tu_i(t) &= \lambda \int_0^1 G_i(t,s) a(s) f(u_i(s))\, ds \\ &\ge \lambda q_i(t) \int_0^1 G_i(1,s) a(s) f(u_i(s))\, ds \\ &\ge \lambda q_i(t) \max_{0 \le t \le 1} \int_0^1 G_i(t,s)\, a(s) f(u_i(s))\, ds \\ &= q(t) \|Tu_i(t)\|, \quad \text{for all } t,s \in [0,1]. \end{aligned}$$

Thus, $T(P) \subset P$. □

By standard argument, it is easy to see that $T : P \longrightarrow P$ is a completely continuous operator. Following Sun and Wen [8] , we define some important constants:

$$A = \int_0^1 G_i(1,s)\, a(s) q_i(s)\, ds, \qquad B = \int_0^1 G_i(1,s)\, a(s)\, ds,$$

$$F_0 = \lim_{u_i \to 0^+} \sup \frac{f(u_i)}{u_i}, \qquad f_0 = \lim_{u_i \to 0^+} \inf \frac{f(u_i)}{u_i},$$

$$F_\infty = \lim_{u_i \to +\infty} \sup \frac{f(u_i)}{u_i}, \qquad f_\infty = \lim_{u_i \to +\infty} \inf \frac{f(u_i)}{u_i}.$$

Here we assume that $\frac{1}{Af_\infty} = 0$ if $f_\infty \to \infty$ and $\frac{1}{BF_0} = \infty$ if $F_0 \to 0$ and $\frac{1}{Af_0} = 0$ if $f_0 \to \infty$ and $\frac{1}{BF_\infty} = \infty$ if $F_\infty \to 0$.

Theorem 1. *Suppose that $Af_\infty > BF_0$, then for each $\lambda \in \left(\frac{1}{Af_\infty}, \frac{1}{BF_0}\right)$, the problem (1) , (i)(i=2,3,4) has at least one positive solution.*

Proof. We choose $\epsilon > 0$ sufficiently small such that $(F_0 + \epsilon)\lambda B \leq 1$. By definition of F_0, we can see that there exists an $l_1 > 0$, such that $f(u_i) \leq (F_0 + \epsilon)u_i$ for $0 < u_i \leq l_1$. If $u_i \in P$ with $\|u_i\| = l_1$, we have

$$\begin{aligned}\|Tu_i(t)\| = Tu_i(1) &= \lambda \int_0^1 G_i(1,s)\, a(s) f(u_i(s))\, ds \\ &\leq \lambda \int_0^1 G_i(1,s) a(s)(F_0+\epsilon)u_i(s)\, ds \\ &\leq \lambda\, (F_0+\epsilon)\|u_i\| \int_0^1 G_i(1,s)a(s)\, ds \\ &\leq \lambda B\, (F_0+\epsilon)\|u_i\| \leq \|u_i\|.\end{aligned}$$

Then we have $\|Tu_i\| \leq \|u_i\|$. Thus if we let $\Omega_1 = \{u_i \in X : \|u_i\| < l_1\}$, then $\|Tu_i\| \leq \|u_i\|$ for $u_i \in P \cap \partial\Omega_1$. Following Yang [9], we choose $\delta > 0$ and $c \in (0, \frac{1}{4})$, such that

$$\lambda \left((f_\infty - \delta) \int_c^1 G_i(1,s)a(s)q(s)\, ds \right) \geq 1.$$

There exists $l_3 > 0$, such that $f(u_i) \geq (f_\infty - \delta)\, u_i$ for $u_i > l_3$. Let $l_2 = \max\{\frac{1}{q_i(c)}, 2l_1\}$. If $u_i \in P$ with $\|u_i\| = l_2$, then we have

$$u_i(t) \geq q_i(t) l_2 \geq q_i(c) l_2 \geq l_3.$$

Therefore, for each $u_i \in P$ with $\|u_i\| = l_2$,we have

$$\begin{aligned}\|Tu_i(t)\| = (Tu_i)(1) &= \lambda \int_0^1 G_i(1,s)\, a(s) f(u_i(s))\, ds \\ &\geq \lambda \int_c^1 G_i(1,s)a(s)(f_\infty - \epsilon)u_i(s)\, ds \\ &\geq \lambda\, (f_\infty - \epsilon)\|u_i\| \int_c^1 G_i(1,s)a(s)\, q_i(s)\, ds \geq \|u_i\|.\end{aligned}$$

Thus if we let $\Omega_2 = \{u_i \in E : \|u_i\| < l_2\}$, then $\Omega_1 \subset \overline{\Omega}_2$ and $\|Tu_i\| \geq \|u_i\|$ for $u_i \in P \cap \partial\Omega_2$. Condition (H1) of Krasnoselskii's fixed point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof. $\square$

Theorem 2. *Suppose that $A\, f_0 > B\, F_\infty$, then for each $\lambda \in \left(\frac{1}{Af_0}, \frac{1}{BF_\infty}\right)$ the problem (1), (i) $(i = 2, 3, 4)$ has at least one positive solution.*

The proof of Theorem 2 is very similar to that of Theorem 1 and therefore omitted.

Theorem 3. *Suppose that $\lambda B f(u_i) < u_i$ for $u_i \in (0, \infty)$. Then the problem (1), (i)(i=2,3,4) has no positive solution.*

Proof. Following Sun and Wen [8], assume to the contrary that u_i is a positive solution of (1),(i). Then

$$\begin{aligned} u_i(1) &= \lambda \int_0^1 G_i(1,s)\, a(s) f(u_i(s))\, ds < \frac{1}{B} \int_0^1 G_i(1,s) a(s) u_i(s)\, ds \\ &\leq \frac{1}{B}\, u_i(1) \int_0^1 G_i(1,s) a(s)\, ds = u_i(1). \end{aligned}$$

This is a contradiction and completes the proof. □

Theorem 4. *Suppose that $\lambda A f(u_i) > u_i$ for $u_i \in (0, \infty)$. Then the problem (1), (i)(i=2,3,4) has no positive solution.*

The proof of Theorem 4 is very similar to that of Theorem 3 and therefore omitted.

Example 1. Consider the equation

$$u_1^5(t) + \lambda(5t+2)\, \frac{7u_1^2 + u_1}{u_1 + 1}\, (8 + \sin u_1) = 0, \quad 0 \leq t \leq 1, \tag{19}$$

$$u_1(0) = u_1''(0) = u_1'''(0) = u_1''''(0) = 0,\ u_1'(1) = 0, \tag{20}$$

Then $F_0 = f_0 = 5$, $F_\infty = 45$, $f_\infty = 27$ and $5u_1 < f(u_1) < 45u_1$. By direct calculations, we obtain that $A = 0.0193452$ and $B = 0.101389$. From theorem 1 we see that if $\lambda \in (1.05495, 1.23288)$, then the problem (19)-(20) has a positive solution. From theorem 3 we have that if $\lambda < 0.156556$, then the problem (19)-(20) has a positive solution. By theorem 4 we have that if $\lambda > 6.46154$, then the problem (19)-(20) has a positive solution.

References

[1] **P. W. Eloe and J. Henderson**, Positive solutions for a fourth order boundary value problem, *Electronic Journal of Differential Equations*, 3 (1995) 1-8.

[2] **P. W. Eloe and B. Ahmed**, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Applied Mathematics Letters*, 18 (2005) 521-527.

[3] **L. H. Erbe and H. Wang**, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc*, **120**. (1994) 743-748.

[4] **Guo, D and Lakshmikantham, V**, 1988, Nonlinear Problems in Abstract Cones, Academic Press, San Diego.

[5] **J. Henderson and S. K. Ntouyas**, Positive Solutions for Systems of nth Order Three-point Nonlocal Boundary Value Problems, *Electronic Journal of Qualitative Theory of Differential Equations*, 18 (2007) 1-12.

[6] **M. A. Krasnoselskii**, Positive Solutions of Operators Equations, Noordhoff, Groningen, 1964.

[7] **Li, S**, Positive solutions of nonlinear singular third-order two-point boundary value problem, *Journal of Mathematical Analysis and Applications*, 323 (2006) 413-425.

[8] **Sun, H. and Wen, W**, On the Number of positive solutions for a nonlinear third order boundary value problem, *International Journal of Difference Equations*, 1 (2006) 165-176.

[9] **Yang, B**, Positive solutions for a fourth order boundary value problem, *Electronic Journal of Qualitative Theory of differential Equations*, 3 (2005) 1-17.

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