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EXISTENCE OF POSITIVE SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS

BASANT KARNA
BONITA LAWRENCE

DEPARTMENT OF MATHEMATICS
MARSHALL UNIVERSITY
HUNTINGTON, WV 25755
KARNA@MARSHALL.EDU
LAWRENCE@MARSHALL.EDU

Abstract. The existence of positive solutions are established for the multi-point boundary value problems

$$\begin{cases} (-1)^n u^{(2n)}(x) = \lambda p(x) f(u(x)), \quad 0 < x < 1 \\ u^{(2i)}(0) = \sum_{j=1}^m a_j u^{(2i)}(\eta_j), \quad u^{(2i+1)}(1) = \sum_{j=1}^m b_j u^{(2i+1)}(\eta_j), \quad i = 0, 1, \ldots, n-1 \end{cases}$$

where $a_j, b_j \in [0, \infty)$, $j = 1, 2, \ldots, m$, with $0 < \sum_{j=1}^m a_j < 1$, $0 < \sum_{j=1}^m b_j < 1$, and $\eta_j \in (0,1)$ with $0 < \eta_1 < \eta_2 < \ldots < \eta_m < 1$, under certain conditions on f and p using the Krasnosel'skii fixed point theorem for certain values of λ. We use the positivity of the Green's function and cone theory to prove our results.

1. Introduction

In this paper, we are concerned with determining eigenvalues, λ, for which there exist positive solutions of the $2n$th order boundary value problem,

$$(-1)^n u^{(2n)} = \lambda p(x) f(u(x)), \quad 0 < x < 1 \tag{1}$$

$$u^{(2i)}(0) = \sum_{j=1}^m a_j u^{(2i)}(\eta_j), \quad u^{(2i+1)}(1) = \sum_{j=1}^m b_j u^{(2i+1)}(\eta_j), \quad i = 0, 1, \ldots, n-1, \tag{2}$$

where $a_j, b_j \in [0, \infty)$, $j = 1, 2, \ldots, m$, with $0 < \sum_{j=1}^m a_j < 1$, $0 < \sum_{j=1}^m b_j < 1$, and $\eta_j \in (0,1)$ with $0 < \eta_1 < \eta_2 < \ldots < \eta_m < 1$, and

(A) the function $f : [0, \infty) \to [0, \infty)$ is continuous,
(B) $p : [0, 1] \to [0, \infty)$ is continuous and is not zero on any compact subinterval of $[0, 1]$,
(C) $f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}$, $f_\infty = \lim_{x \to +\infty} \frac{f(x)}{x}$ exist in the extended reals.

Our methods will involve those utilized in dealing with elliptic problems in annular regions. These methods have been effectively adapted for the cases when f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$) and when f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$).

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A large part of the literature on multiple solutions to boundary value problems seems to be traced back to Kransnosel'skii's work on nonlinear operator equations [9], especially the part dealing with the theory of cones in Banach Spaces. In 1994, Erbe and Wang [6] applied Krasnoselskii's work to eigenvalue problems such as the one above to establish intervals of the parameter λ for which there is at least one positive solution. Many authors have used this approach or a variation thereof to obtain eigenvalue intervals. For a small sample of such work, we refer the reader to works by Davis, Henderson, Prasad and Yin [2], Eloe and Henderson [4], Eloe, Henderson and Wong [5], Erbe and Wang [6], and references therein.

The paper is organized as follow. In section 2, we are going to define the appropriate Green's function which is, later, used to define the operator of which the fixed points are the solutions of our boundary value problem (1) and (2). The Krasnosel'skii fixed point theorem stated in that section 2 will be applied in section 3 to yield positive solutions for certain intervals of eigenvalues.

2. Preliminary Results

In this section, we construct the Green's function and state it's properties. We also state the fixed point theorem. The following lemma helps us in constructing the Green's function for the problem (1) and (2).

Lemma 1. *Let $a_j, b_j \in [0,\infty)$ with $(1-\sum_{j=1}^m a_j)(1-\sum_{j=1}^m b_j) \neq 0$. Then*

$$\begin{cases} -u''(x) = f(x), \quad 0<x<1 \\ u(0) = \sum_{j=1}^m a_j u(\eta_j), \quad u'(1) = \sum_{j=1}^m b_j u'(\eta_j), \end{cases}$$

where f is a continuous function and $0 < \eta_1 < \eta_2, \ldots < \eta_m < 1$, has a unique solution given by

$$u(x) = \int_0^1 G(x,s) f(s) ds, \quad 0 \le x \le 1.$$

The function $G(x,s)$ is defined by

$$G(x,s) = D(x,s) + \sum_{j=1}^m A_j D(\eta_j, s) + (\sum_{j=1}^m A_j \eta_j + x)(\sum_{j=1}^m B_j H(s-\eta_j))$$

where

$$D(x,s) = \min(x,s), \ A_j = \frac{a_j}{A}, \ B_j = \frac{b_j}{B}, \ A = 1 - \sum_{j=1}^m a_j, \ B = 1 - \sum_{j=1}^m b_j,$$

and $H(t)$ is the Heaviside function, i.e., $H(t) = 1$ for $t \ge 0$ and $H(t) = 0$ for $t < 0$.

Proof. Let

$$u(x) := \int_0^1 D(x,s) f(s) ds + Cx + D$$

where C and D are arbitrary constants and $D(x,s)$ is defined above.

Then $u'(x) = \int_x^1 f(s)ds + C = \int_0^1 H(s-x) f(s) ds + C$, and $u''(x) = -f(x)$.

So, $u(x)$ is the general solution of $u''(x) = -f(x)$.

Using the condition $u'(1) = \sum_{j=1}^{m} b_j u'(\eta_j)$, we get

$$C = \sum_{j=1}^{m} b_j \left\{ \int_0^1 H(s-\eta_j) f(s) ds + C \right\}$$

$$\Longrightarrow C = \int_0^1 \sum_{j=1}^{m} B_j H(s-\eta_j) f(s) ds$$

and using the condition, $u(0) = \sum_{j=1}^{m} a_j u(\eta_j)$, we get

$$D = \sum_{j=1}^{m} a_j \left(\int_0^1 D(\eta_j, s) f(s) ds + C\eta_j + D \right)$$

$$\Longrightarrow D = \int_0^1 \sum_{j=1}^{m} A_j D(\eta_j, s) f(s) ds + \int_0^1 \left(\sum_{j=1}^{m} A_j \eta_j \right) \left(\sum_{j=1}^{m} B_j H(s-\eta_j) \right) f(s) ds$$

i.e.,

$$D = \int_0^1 \left\{ \sum_{j=1}^{m} A_j D(\eta_j, s) + \left(\sum_{j=1}^{m} A_j \eta_j \right) \left(\sum_{j=1}^{m} B_j H(s-\eta_j) \right) \right\} f(s) ds$$

and so,

$$u(x) = \int_0^1 \left[D(x,s) + \sum_{j=1}^{m} A_j D(\eta_j, s) + \left(\sum_{j=1}^{m} A_j \eta_j + x \right) \left(\sum_{j=1}^{m} B_j H(s-\eta_j) \right) \right] f(s) ds$$

i.e.,

$$u(x) = \int_0^1 G(x,s) f(s) ds.$$

□

Note that the Green's function $G(x,s)$ satisfies $G(x,s) > 0$ on $(0,1] \times (0,1]$, and $\frac{\partial}{\partial x} G(x,s) \geq 0$ a.e. on $[0,1] \times [0,1]$.

Now, if we let $G_1(x,s) := G(x,s)$ and for $i = 2, 3, \ldots, n$,

$$G_i(x,s) = \int_0^1 G_1(x,t) G_{i-1}(t,s) dt,$$

then it turns out that $G_n(x,s)$ is the Green's function for

$$(-1)^n u^{(2n)} = 0, \quad 0 < x < 1$$

satisfying the boundary conditions

$$u^{(2i)}(0) = \sum_{j=1}^{m} a_j u^{(2j)}(\eta_j), \quad u^{(2i+1)}(1) = \sum_{j=1}^{m} b_j u^{(2i+1)}(\eta_j), \quad i = 0, 1, \ldots, n-1.$$

So, $u \in C[0,1]$ is a solution of (1) and (2) if and only if $u(x) = \int_0^1 G_n(x,s) \lambda p(s) f(u(s)) ds$.

It is straight forward from the properties of $G_1(x,s)$ to see that $G_n(x,s) > 0$ on $(0,1] \times (0,1]$, and $\frac{\partial}{\partial x} G_n(x,s) \geq 0$ a.e. on $[0,1] \times [0,1]$.

Lemma 2. *Let* $\delta := \inf \left\{ \frac{G(0,s)}{G(1,s)} : 0 < s \leq 1 \right\}$. *Then,* $\delta > 0$ *and for all* $0 \leq x \leq 1$ *and* $0 < s \leq 1$, $\delta^n G_n(1,s) \leq G_n(0,s) \leq G_n(x,s) \leq G_n(1,s)$.

Proof. Note that $G_1(0,s)$ and $G_1(1,s)$ have the following bounds.

$$\begin{aligned} G_1(0,s) &= \sum_{j=1}^m A_j D(\eta_j, s) + \left(\sum_{j=1}^m A_j \eta_j \right) \left(\sum_{j=1}^m B_j H(s - \eta_j) \right) \\ &\geq s \sum_{j=1}^m A_j \eta_j \quad \text{[Note: equality holds for } s = 0] \end{aligned}$$

and

$$\begin{aligned} G_1(1,s) &= s + \sum_{j=1}^m A_j D(\eta_j, s) + \left(\sum_{j=1}^m A_j \eta_j + 1 \right) \left(\sum_{j=1}^m B_j H(s - \eta_j) \right) \\ &\leq s \left\{ 1 + \sum_{j=1}^m A_j + \left(\sum_{j=1}^m A_j \eta_j + 1 \right) \left(\sum_{j=1}^m \frac{B_j}{\eta_j} \right) \right\} \quad (0 < \eta_j < 1) \end{aligned}$$

So, for $0 < s \leq 1$, we have

$$\begin{aligned} \frac{G_1(0,s)}{G_1(1,s)} &\geq \frac{\sum_{j=1}^m A_j \eta_j s}{s \left\{ 1 + \sum_{j=1}^m A_j + \left(\sum_{j=1}^m A_j \eta_j + 1 \right) \left(\sum_{j=1}^m \frac{B_j}{\eta_j} \right) \right\}} \\ &= \left(\sum_{j=1}^m A_j \eta_j \right) \left\{ 1 + \sum_{j=1}^m A_j + \left(\sum_{j=1}^m A_j \eta_j + 1 \right) \left(\sum_{j=1}^m \frac{B_j}{\eta_j} \right) \right\}^{-1} > 0 \end{aligned}$$

and so, $\delta = \inf \left\{ \frac{G_1(0,s)}{G_1(1,s)} : 0 < s \leq 1 \right\} > 0$.

Note that, for $0 < s \leq 1$,

$$\frac{G_1(0,s)}{G_1(1,s)} \geq \delta$$

i.e., $\delta G_1(1,s) \leq G_1(0,s)$.

Also, $G_1(0,s) \leq G_1(x,s) \leq G_1(1,s)$ for $0 \leq x \leq 1$ as $G_1(x,s)$ is increasing in x.

Therefore, $\delta G_1(1,s) \leq G_1(0,s) \leq G_1(x,s) \leq G_1(1,s)$.

Now, since $G_2(x,s) = \int_0^1 G_1(x,t) G_1(t,s) dt$ and $\delta G_1(1,s) \leq G_1(0,s)$ [$\delta < 1$ as $G_1(0,s) \leq G_1(1,s)$, $0 < s \leq 1$], we have

$$
\begin{aligned}
\delta^2 G_2(1,s) &= \delta \int_0^1 (\delta G_1(1,t))\, G_1(t,s)dt \le \delta \int_0^1 G_1(0,t)G_1(t,s)dt \\
&\le \int_0^1 G_1(0,t)\,(\delta G_1(1,s))\, dt \le \int_0^1 G_1(0,t)G_1(0,s)dt \\
&\le G_2(0,s) \quad \text{for } 0 < \delta \le 1.
\end{aligned}
$$

If we continue this way, we would get that for $0 < s \le 1$, $\delta^n G_n(1,s) \le G_n(0,s)$ which implies that

$$\delta^n G_n(1,s) \le G_n(0,s) \le G_n(x,s) \le G_n(1,s).$$

□

Now we are going to define the Banach Space and the cone where we will produce the solution of our boundary value problems.

Consider the Banach space $\mathcal{B} = C[0,1]$, with norm $\|u\| = \sup_{[0,1]} |u(x)|$ and the cone $\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \ge \delta^n \|u\| \text{ on } [0,1]\}$.

The following straightforward results will be used in proving our main theorems.

(D1) When $f_0 = 0$, there exist an $\eta > 0$ and $H_1 > 0$ such that $f(x) \le \eta x$ for $0 < x < H_1$, and $\lambda\eta \int_0^1 G_n(1,s)p(s)ds \le 1$ for any fixed $\lambda \in (0,\infty)$.
(D2) When $f_\infty = \infty$, there exist a $\mu > 0$ and $\hat{H}_2 > 0$, such that $f(x) \ge \mu x$ for $x \ge \hat{H}_2$, and $\lambda\mu\delta^{2n} \int_0^1 G_n(1,s)p(s)ds \ge 1$ for any fixed $\lambda \in (0,\infty)$.
(D3) When $f_0 = \infty$, there exist $\bar{\eta} > 0$ and $J_1 > 0$, such that $f(x) \ge \bar{\eta}x$ for $0 < x < J_1$, and $\lambda\delta^{2n}\bar{\eta} \int_0^1 G_n(1,s)p(s)ds \ge 1$ for any fixed $\lambda \in (0,\infty)$.
(D4) When $f_\infty = 0$, there exist a $\bar{\mu} > 0$, and $\hat{J}_2 > 0$, such that $f(x) \le \bar{\mu}x$ for $x \ge \hat{J}_2$, and $\lambda\bar{\mu} \int_0^1 G_n(1,s)p(s)ds \le 1$ for any fixed $\lambda \in (0,\infty)$.

We seek a fixed point of the integral operator $T : \mathcal{P} \to \mathcal{B}$ defined by

$$Tu(x) = \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds, \; u \in \mathcal{P}$$

which is a solution of the equation (1) satisfying the boundary conditions (2).

Note that the operator T preserves $\mathcal{P}$; i.e., $T : \mathcal{P} \to \mathcal{P}$ and is completely continuous.

To see this, let $0 < \lambda < \infty$ be given and $u \in \mathcal{P}$. Then, $u(x) \ge \delta^n \|u\|$, on $[0,1]$.

Since, $(Tu)(x) = \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds$,
we have

$$
\begin{aligned}
(Tu)'(x) &= \lambda \int_0^1 \frac{\partial}{\partial x} G_n(x,s)p(s)f(u(s)ds \\
&\ge 0,
\end{aligned}
$$

which implies that (Tu) is increasing on $[0,1]$ and so, $\|Tu\| = Tu(1)$.

Also,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s)ds \\ &\geq \lambda \int_0^1 \delta^n G_1(1,s)p(s)f(u(s))ds \\ &= \delta^n Tu(1) = \delta^n \|Tu\|. \end{aligned}$$

Hence, $Tu \in \mathcal{P}$, that is, $T : \mathcal{P} \to \mathcal{P}$. Moreover, the standard argument shows that T is completely continuous.

The fixed point theorem by Krasnosel'skii which can be found in [9] and used to obtain our results in next section is as follows.

Krasnosel'skii Fixed Point Theorem *Let X be a Banach space and let K be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \delta\Omega_1$, and $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \delta\Omega_2$; or
(ii) $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \delta\Omega_1$, and $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \delta\Omega_2$.

Then, Φ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

Now we are going to prove the existence of at least one positive solution of the above mentioned boundary value problems.

Theorem 3. *Let $0 < \lambda < \infty$, and the function f satisfies superlinearity conditions, that is, $f_0 = 0$, $f_\infty = \infty$ and the conditions (A), (B), and (C) hold. Then the boundary value problem* (1) *and* (2) *has at least one solution belonging to the cone $\mathcal{P}$.*

Proof. Let $0 < \lambda < \infty$. Let $u \in \mathcal{P}$ with $\|u\| = H_1$,
then for $0 \leq x \leq 1$ (Note: $f(u(s)) \leq \eta u(s)$)

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\leq \lambda \int_0^1 G_n(1,s)p(s)\eta u(s)ds \\ &\leq \lambda\eta\|u\| \int_0^1 G_n(1,s)p(s)ds \leq \|u\| \text{ by D1.} \end{aligned}$$

So, $\|Tu\| \leq \|u\|$.

Now, let $\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < H_1\}$, then $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Next, let $H_2 = \max\left\{2H_1, \delta^{-n}\hat{H}_2\right\}$, and $\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}$.

Then, for $u \in \mathcal{P}$, $\|u\| = H_2$, we have $u(s) \geq \delta^n \|u\| = \delta^n H_2 \geq \hat{H}_2,\ \ 0 \leq s \leq 1$, and

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\geq \lambda \int_0^1 \delta^n G_n(1,s)p(s)\mu u(s)ds \\ &\geq \lambda\mu \int_0^1 \delta^n G_n(1,s)p(s)\delta^n\|u\|ds \\ &= \|u\|\lambda\mu\delta^{2n} \int_0^1 G_n(1,s)p(s)ds \\ &\geq \|u\| \ \text{ by D2 for } \mu \ \text{ large.} \end{aligned}$$

This implies $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$.

So, by part (i) of Krasnoselskii's Fixed Point Theorem, there is a fixed point of the operator T, that belongs to $\mathcal{P} \cap (\overline{\Omega}_2 - \Omega_1)$. The fixed point $u(x)$ is the desired solution of (1) and (2) for the given λ.

□

The next result proves the existence of at least one positive solution when f satisfies sublinearity conditions.

Theorem 4. *Let $0 < \lambda < \infty$, and the function f satisfies sublinearity conditions, that is, $f_0 = \infty$, $f_\infty = 0$ and the conditions (A), (B), and (C) hold. Then the boundary value problem (1) and (2) has at least one solution belonging to the cone $\mathcal{P}$.*

Proof. Let $u \in \mathcal{P}$ with $\|u\| = J_1$. Then,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\geq \lambda\delta^n \int_0^1 G_n(1,s)p(s)\bar{\eta}u(s)ds \\ &\geq \lambda\delta^n\bar{\eta} \int_0^1 G_n(1,s)p(s)\delta^n\|u\|ds \\ &\geq \|u\|\lambda\delta^{2n}\bar{\eta} \int_0^1 G_n(1,s)p(s)ds \geq \|u\| \end{aligned}$$

for $\bar{\eta}$ large, J_1 small by D3. i.e., $\|Tu\| \geq \|u\|$.
Let $\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < J_1\}$, then $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$. since $f_0 = \infty$, the conditoin (D4) holds.

Case(a) f is bounded. Let $f(x) \leq M$, $M > 0$.
Let $J_2 = \max\left\{2J_1, M\lambda \int_0^1 G_n(1,s)p(s)ds\right\}$.

Then, for $u \in \mathcal{P}$ with $\|u\| = J_2$, and $0 \le x \le 1$, we have

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\le \lambda \int_0^1 G_n(1,s)p(s)f(u(s))ds \\ &\le \lambda M \int_0^1 G_n(1,s)p(s)ds \le J_2 = \|u\|. \end{aligned}$$

Thus, $\|Tu\| \le \|u\|$.
So, if $\Omega_2 = \{u \in \mathcal{B} \mid \|u\|, J_2\}$, then $\|Tu\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$.

Case(b) f is unbounded. Then there exists
$J_2 > \max\left\{2J_1, \hat{J}_2\right\}$ such that $f(x) \le f(J_2)$ for $0 < x \le J_2$. Let $u \in \mathcal{P}$ with $\|u\| = J_2$. Then,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\le \lambda \int_0^1 G_n(1,s)p(s)f(J_2)ds \\ &\le \lambda \int_0^1 G_n(1,s)p(s)\overline{\mu}J_2 ds \\ &\le \|u\|\lambda\overline{\mu} \int_0^1 G_n(1,s)p(s)ds \le \|u\| \text{ by (D4)}. \end{aligned}$$

Thus, $\|Tu\| \le \|u\|$. For this case, if $\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < J_2\}$, then $\|Tu\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$.

Therefore, by part (ii) of Krasnoselskii's Fixed Point Theorem, there is a fixed point of T belonging to $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ and is a solution of the problem (1) and (2) for given λ. $\square$

The following two results give us existence of at least one positive solution when f_0, f_∞ exist as positive real numbers and λ satisfies certain inequality.

Theorem 5. *Let f_0, f_∞ exist as positive real numbers. Assume (A), (B) hold. Then, for each λ satisfying,*

$$\frac{1}{\delta^{2n} f_\infty \int_0^1 G_n(1,s)p(s)ds} < \lambda < \frac{1}{f_0 \int_0^1 G_n(1,s)p(s)ds},$$

there is at least one solution of (1) *and* (2) *belonging to $\mathcal{P}$.*

Proof. Let $\varepsilon > 0$ be given such that

$$0 < \frac{1}{\delta^{2n}(f_\infty - \varepsilon) \int_0^1 G_n(1,s)p(s)ds} \le \lambda \le \frac{1}{(f_0 + \varepsilon) \int_0^1 G_n(1,s)p(s)ds}.$$

Since $f_0 = \lim_{x\to 0} \frac{f(x)}{x}$ is a finite positive real number, there exists an $H_1 > 0$ s.t. $f(x) \le (f_0+\varepsilon)x$ for $0 < x \le H_1$.
Let $u \in \mathcal{P}$ with $\|u\| = H_1$, then for $0 \le x \le 1$,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\le \lambda \int_0^1 G_n(x,s)p(s)(f_0+\varepsilon)u(s)ds \\ &\le \|u\|\lambda(f_0+\varepsilon)\int_0^1 G_n(1,s)p(s)ds \le \|u\|. \end{aligned}$$

Let $\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < H_1\}$, then $\|Tu\| \le \|u\|$ for $u \in \mathcal{P}\cap\partial\Omega_1$.

Since $f_\infty = \lim_{x\Rightarrow\infty} \frac{f(x)}{x}$ is a finite positive real number, there exists an $\hat{H}_2 > 0$ such that $f(x) \ge (f_\infty - \varepsilon)x$ for $x \ge \hat{H}_2$.

Let $H_2 = \max\left\{2H_1, \delta^{-n}\hat{H}_2\right\}$ and $\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}$.

Let $u \in \mathcal{P}$ with $\|u\| = H_2$. Then, we have $u(x) \ge \delta^n\|u\| = \delta^n H_2 \ge \hat{H}_2$. Thus,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\ge \lambda\delta^n \int_0^1 G_n(1,s)p(s)f(u(s))ds \\ &\ge \lambda\delta^n \int_0^1 G_n(1,s)p(s)(f_\infty-\varepsilon)u(s)ds \\ &\ge \lambda\delta^n(f_\infty-\varepsilon)\int_0^1 G_n(1,s)p(s)\delta^n\|u\|ds \\ &= \|u\|\lambda\delta^{2n}(f_\infty-\varepsilon)\int_0^1 G_n(1,s)p(s)ds \\ &\ge \|u\|. \end{aligned}$$

i.e., $\|Tu\| \ge \|u\|$ for $u \in \mathcal{P}\cap\partial\Omega_2$.

So, by part (i) of Krasnoselskii's Fixed Point Theorem, there is a fixed point of the operator T, that belongs to $\mathcal{P}\cap(\overline{\Omega}_2 - \Omega_1)$. The fixed point $u(x)$ is the desired solution of (1) and (2) for the given λ.

$\square$

Theorem 6. *Let f_0, f_∞ exist as positive real numbers. Assume (A), (B) hold. Then, for each λ satisfying,*

$$\frac{1}{\delta^{2n} f_0 \int_0^1 G_n(1,s)p(s)ds} < \lambda < \frac{1}{f_\infty \int_0^1 G_n(1,s)p(s)ds},$$

there is at least one solution of (1) *and* (2) *belonging to* $\mathcal{P}$.

Proof. Let $\varepsilon > 0$ be given such that

$$0 < \frac{1}{\delta^{2n}(f_0 - \varepsilon)\int_0^1 G_n(1,s)p(s)ds} \le \lambda \le \frac{1}{(f_\infty + \varepsilon)\int_0^1 G_n(1,s)p(s)ds}.$$

Since $f_0 = \lim_{x\to 0}\frac{f(x)}{x}$ is a finite positive real number, there exists a $J_1 > 0$ such that $f(x) \ge (f_0 - \varepsilon)x$ for $0 < x \le J_1$. Choose $u \in \mathcal{P}$ with $\|u\| = J_1$, then for $0 \le x \le 1$,

$$\begin{aligned}
Tu(x) &= \lambda\int_0^1 G_n(x,s)p(s)f(u(s))ds \\
&\ge \lambda\int_0^1 G_n(x,s)p(s)(f_0-\varepsilon)u(s)ds \\
&\ge \lambda(f_0-\varepsilon)\int_0^1 \delta^n G_n(1,s)p(s)u(s)ds \\
&\ge \lambda(f_0-\varepsilon)\delta^n\int_0^1 G_n(1,s)p(s)\delta^n\|u\|ds \\
&\ge \|u\|\lambda(f_0-\varepsilon)\delta^{2n}\int_0^1 G_n(1,s)p(s)ds \ge \|u\|.
\end{aligned}$$

So, if $\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < J_1\}$, then $\|Tu\| \ge \|u\|$ for $u \in \mathcal{P}\cap\partial\Omega_1$.

Since $f_\infty = \lim_{x\to\infty}\frac{f(x)}{x}$ is finite positive real number, there exists an $\hat{J}_2 > 0$ such that $f(x) \le (f_\infty + \varepsilon)x$ for $x \ge \hat{J}_2$.

Case(a) The function f is bounded.
Let $N > 0$ such that $f(x) \le N$ for all $x \ge 0$ and $J_2 = \max\left\{2J_1, N\lambda\int_0^1 G_n(1,s)p(s)ds\right\}$. Then, for $u \in \mathcal{P}$ with $\|u\| = J_2$,

$$\begin{aligned}
Tu(x) &= \lambda\int_0^1 G_n(x,s)p(s)f(u(s))ds \\
&\le \lambda\int_0^1 G_n(1,s)p(s)f(u(s))ds \\
&\le \lambda\int_0^1 G_n(1,s)p(s)Nds \\
&= N\lambda\int_0^1 G_n(1,s)p(s)ds \le J_2 = \|u\|.
\end{aligned}$$

So, if $\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < J_2\}$, then $\|Tu\| \le \|u\|$ for $u \in \mathcal{P}\cap\partial\Omega_2$.

Case(b) The function f is unbounded.
Let $J_2 > \max\left\{2J_1, \hat{J}_2\right\}$ be such that $f(x) \le f(J_2)$ for $0 < x \le J_2$. Let $u \in \mathcal{P}$ with

$\|u\| = J_2$. Then,

$$\begin{aligned} Tu(x) &= \lambda \int_0^1 G_n(x,s)p(s)f(u(s))ds \\ &\leq \lambda \int_0^1 G_n(1,s)p(s)f(J_2)ds \\ &\leq \lambda \int_0^1 G_n(1,s)p(s)(f_\infty + \varepsilon)J_2 ds \\ &= \|u\|\lambda(f_\infty + \varepsilon)\int_0^1 G_n(1,s)p(s)ds \leq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$ where $\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < J_2\}$.

So, by part (ii) of Krasnoselskii's Fixed Point Theorem, there is a fixed point of the operator T, that belongs to $\mathcal{P} \cap (\overline{\Omega}_2 - \Omega_1)$, say $u(x)$ which is the desired solution of (1) and (2) for the given λ. □

References

[1] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, vol. 220, Springer-Verlag, New York and Berlin, 1971.

[2] J. M. Davis, J. Henderson, K. R. Prasad, and W. Yin, *Eigenvalue intervals for nonlinear right focal problems*, Appl. Anal., **74** (2000), 215-231.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[4] P. W. Eloe and J. Henderson, *Positive solutions for higher order ordinary differential equations*, Electronic J. Diff. Eqns., **3** (1995), 1-8.

[5] P. W. Eloe, J. Henderson, and P. Wong, *Positive solutions for two-point boundary value problems*, Proc. of Dyn. Sys. Appl., **2** (1995), 135-144.

[6] L. H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **120** (1994), 743-748.

[7] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[8] C. P. Gupta, *A generalized multi-point boundary value problems for second order ordinary differential equations*, Appl. Math. Comput., **89** (1998), 133-146.

[9] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[10] Ruyun Ma and N. Casteneda, *Existence of solutions of nonlinear m-point boundary value problems*, J. Math. Anal. Appl., **256** (2001), 556-567.

[11] A. Peterson, *On the sign of Green's functions*, J. Diff. Eqns., **21** (1976), 167-178.

[12] Z. Zhang and J. Wang, *On existence and multiplicity of positive solutions to singular mulit-point boundary value problems*, J. Math. Anal. Appl., **295** (2004), 502-512.

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