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EXISTENCE OF A SOLUTION FOR A CLASS OF NONLINEAR PARABOLIC SYSTEMS

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Abstract. An existence result of a solution for a class of nonlinear parabolic systems is established. The data belong to L^1 and no growth assumption is made on the nonlinearities.

1. Introduction

In the present paper we establish an existence result of a renormalized solution for a class of nonlinear parabolic systems of the type

$$\frac{\partial u}{\partial t} - div\Big(a(x,u,\nabla u) + \Phi(u)\Big) + f_1(x,u,v) = 0 \ \text{ in } (0,T)\times\Omega \ ; \tag{1.1}$$

$$\frac{\partial v}{\partial t} - div\Big(a(x,v,\nabla v) + \Phi(v)\Big) + f_2(x,u,v) = 0 \ \text{ in } (0,T)\times\Omega \ ; \tag{1.2}$$

$$u = v = 0 \ \text{ on } (0,T)\times\partial\Omega \ ; \tag{1.3}$$

$$u(t=0) = u_0 \ \text{ in } \Omega. \tag{1.4}$$

$$v(t=0) = v_0 \ \text{ in } \Omega. \tag{1.5}$$

In Problem (1.1)-(1.5) the framework is the following : Ω is a bounded domain of R^N, $(N \geq 1)$, T is a positive real number while the data u_0 and v_0 in $L^1(\Omega)$. The operator $-div(a(x,u,Du))$ is a Leray-Lions operator which is coercive and which grows like $|Du|^{p-1}$ with respect to Du, but which is not restricted by any growth condition with respect to u (see assumptions (2.1), (2.2), (2.3) and (2.4) of Section 2.). The function Φ, f_1 and f_2 are just assumed to be continuous on $\mathbb{R}$.

When Problem (1.1)-(1.5) is investigated there is difficulty is due to the facts that the data u_0 and v_0 only belong to L^1 and the function $a(x,u,Du)$, $\Phi(u)$, $f_1(x,u,v)$ and $f_2(x,u,v)$ does not belong $(L^1_{loc}((0,T)\times\Omega))^N$ in general, so that proving existence of a weak solution (i.e. in the distribution meaning) seems to be an arduous task. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by Lions and Di Perna [22] for the study of Boltzmann equation (see also P.-L. Lions [17] for a few applications to fluid mechanics models). This notion was then adapted to elliptic vesion of (1.1), (1.2), (1.3) in Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [10], in P.-L. Lions and F. Murat [19] and F. Murat [19], [20]. At the same time the equivalent notion of

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entropy solutions has been developed independently by Bénilan and al. [2] for the study of nonlinear elliptic problems.

As far as the parabolic equation case (1.1)-(1.5), (with, $f_i(x,u,v) = f \in L^1(\Omega \times (0,T))$) is concerned and still in the framework of renormalized solutions, the existence and uniqueness has been proved in D. Blanchard, F. Murat and H. Redwane [5] (see also A. Porretta [21] and H. Redwane [23]) in the case where $f_i(x,u,v)$ is replaced by $f + div(g)$ (where g belong $L^{p'}(Q)^N$). In the case where $a(t,x,s,\xi)$ is independant of s, $\Phi = 0$ and $g = 0$, existence and uniqueness has been established in D. Blanchard [3] ; D. Blanchard and F. Murat [4], and in the case where $a(t,x,s,\xi)$ is independent of s and linear with respect to ξ, existence and uniqueness has been established in D. Blanchard and H. Redwane [7].

In the case where $\Phi = 0$ and where the operator $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$ p-Laplacian replaces a nonlinear term $div(a(x,s,\xi))$, existence of a solution for nonlinear parabolic systems (1.1)-(1.5) is investigated in El Ouardi, A. El Hachimi ([14] [15]), in Marion [18] and in A. Eden and all [1] (see also L. Dung [12]), where an existence result of as (usual) weak solution is proved.

With respect to the previous ones, the originality of the present work lies on the noncontrolled growth of the function $a(x,s,\xi)$ with respect to s, and the function Φ, f_1 and f_2 are just assumed to be continuous on $\mathbb{R}$, and u_0, v_0 are just assumed belong to $L^1(\Omega)$.

The paper is organized as follows : Section 2 is devoted to specify the assumptions on $a(x,s,\xi)$, Φ, f_1, f_2, u_0 and v_0 needed in the present study and gives the definition of a renormalized solution of (1.1)-(1.5). In Section 3 (Theorem 3.0.4) we establish the existence of such a solution.

2. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true : Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$, for $i = 1,\ 2$

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function,} \tag{2.1}$$

$$a(x,s,\xi).\xi \geq \alpha|\xi|^p \tag{2.2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ given real number.

For any $K > 0$, there exists $\beta_K > 0$ and a function C_K in $L^{p'}(\Omega)$ such that

$$|a(x,s,\xi)| \leq C_K(x) + \beta_K|\xi|^{p-1} \tag{2.3}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq K$, and for every $\xi \in \mathbb{R}^N$

$$[a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] \geq 0, \tag{2.4}$$

for any $s \in \mathbb{R}$, for any $(\xi,\xi') \in \mathbb{R}^{2N}$ and for almost every $x \in \Omega$.

$$\Phi \ : \ \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function} \tag{2.5}$$

For $i = 1,\ 2$

$$\text{(2.6)} \qquad f_i\ :\ \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R} \quad \text{is a Carathéodory function,}$$

$$f_1(x,0,s) = f_2(x,s,0) = 0 \qquad\qquad \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R}.$$

For almost every $x \in \Omega$, for every $s_1,\ s_2 \in \mathbb{R}$:

$$\text{(2.7)} \qquad sign(s_1)f_1(x,s_1,s_2) \geq 0 \quad \text{and} \quad sign(s_2)f_2(x,s_1,s_2) \geq 0$$

For any $K > 0$, there exists $\sigma_K > 0$ and a function F_K in $L^1(\Omega)$ such that

$$\text{(2.8)} \qquad |f_1(x,s_1,s_2)| \leq F_K(x) + \sigma_K\ |s_2|$$

for almost every $x \in \Omega$, for every s_1 such that $|s_1| \leq K$, and for every $s_2 \in \mathbb{R}$.
For any $K > 0$, there exists $\lambda_K > 0$ and a function G_K in $L^1(\Omega)$ such that

$$\text{(2.9)} \qquad |f_2(x,s_1,s_2)| \leq G_K(x) + \lambda_K\ |s_1|$$

for almost every $x \in \Omega$, for every s_2 such that $|s_2| \leq K$, and for every $s_1 \in \mathbb{R}$.

$$\text{(2.10)} \qquad (u_0,\ v_0) \in L^1(\Omega)\times L^1(\Omega)$$

Remark 2.0.1. As already mentioned in the introduction Problem (1.1)-(1.5) does not admit a weak solution under assumptions (2.1)-(2.10) (even when $f_1 = f_2 \equiv 0$) since the growths of $a(u,Du)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs $L^p(0,T;W_0^{1,p}(\Omega))$).

Throughout this paper and for any non negative real number K we denote by $T_K(r) = min(K, max(r,-K))$ the truncation function at height K. For any measurable subset E of Q, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q and for any real number s, $\chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}},\ \chi_{\{v>s\}}$) is the characteristic function of the set $\{(x,t)\in Q\ ;\ v(x,t) < s\}$ (respectively, $\{(x,t)\in Q\ ;\ v(x,t) = s\},\ \{(x,t)\in Q\ ;\ v(x,t) > s\}$). The definition of a renormalized solution for Problem (1.1)-(1.5) can be stated as follows.

Definition 2.0.2. A couple of functions (u,v) defined on Q is called a renormalized solution of Problem (1.1)-(1.5) if u and v satisfy :

$$\text{(2.11)} \qquad (T_K(u),\ T_K(v)) \in L^p(0,T;W_0^{1,p}(\Omega))^2 \text{ and } (u,\ v) \in L^\infty(0,T;L^1(\Omega))^2\ ;$$

for any $K \geq 0$.

$$\text{(2.12)} \qquad \int_{\{(t,x)\in Q\ ;\ n\leq|u(x,t)|\leq n+1\}} a(x,u,Du)Du\,dx\,dt \longrightarrow 0 \quad \text{as } n\to+\infty\ ;\ ;$$

$$\text{(2.13)} \qquad \int_{\{(t,x)\in Q\ ;\ n\leq|v(x,t)|\leq n+1\}} a(x,v,Dv)Dv\,dx\,dt \longrightarrow 0 \quad \text{as } n\to+\infty\ ;$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have

$$\text{(2.14)} \qquad \frac{\partial S(u)}{\partial t} - div(S'(u)a(x,u,Du)) + S''(u)a(x,u,Du)Du$$

$$- div(S'(u)\Phi(u)) + S''(u)\Phi(u)Du + f_1(x,u,v)S'(u) = 0 \text{ in } D'(Q)\ ;$$

and

$$\frac{\partial S(v)}{\partial t} - div(S'(v)a(x,v,Dv)) + S''(v)a(x,v,Dv)Dv \tag{2.15}$$
$$- div(S'(v)\Phi(v)) + S''(v)\Phi(v)Dv + f_2(x,u,v)S'(v) = 0 \quad \text{in } D'(Q) \ ;$$

$$S(u)(t=0) = S(u_0) \quad \text{and} \quad S(v)(t=0) = S(v_0) \quad \text{in } \Omega. \tag{2.16}$$

The following remarks are concerned with a few comments on definition 2.0.2.

Remark 2.0.3. Equation (2.14) (and (2.15)) is formally obtained through pointwise multiplication of equation (1.1) by $S'(u)$ (and equation (1.2) by $S'(v)$). Note that in definition 2.0.2, Du is not defined even as a distribution, but that due to (2.11) each term in (2.14) (and (2.15)) has a meaning in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Indeed if K is such that $suppS' \subset [-K,K]$, the following identifications are made in (2.14) (and in (2.15)) :

⋆ $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.

⋆ $S'(u)a(u,Du)$ identifies with $S'(u)a(T_K(u),DT_K(u))$ a.e. in Q. Since indeed $|T_K(u)| \leq K$ a.e. in Q, assumptions (2.1) and (2.3) imply that

$$\Big| a(T_K(u),DT_K(u)) \Big| \leq C_K(t,x) + \beta_K |DT_K(u)|^{p-1} \quad \text{a.e. in } Q.$$

As a consequence of (2.11) and of $S'(u) \in L^\infty(Q)$, it follows that

$$S'(u)a(T_K(u),DT_K(u)) \in (L^{p'}(Q))^N.$$

⋆ $S''(u)a(u,Du)Du$ identifies with $S''(u)a(T_K(u),DT_K(u))DT_K(u)$ and in view of (2.1), (2.3) and (2.11) one has

$$S''(u)a(T_K(u),DT_K(u))DT_K(u) \in L^1(Q).$$

⋆ $S'(u)\Phi(u)$ and $S''(u)\Phi(u)Du$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))DT_K(u)$. Due to the properties of S and (2.5), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (2.11) implies that $S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N$, and $S''(u)\Phi(T_K(u))DT_K(u) \in L^p(Q)$.

⋆ $S'(u)f_1(x,u,v)$ identifies with $S'(u)f_1(x,T_K(u),v)$ a.e. in Q. Since indeed $|T_K(u)| \leq K$ a.e. in Q, assumptions (2.8) imply that

$$\Big| f_1(x,T_K(u),v) \Big| \leq F_K(x) + \sigma_K \ |v| \quad \text{a.e. in } Q.$$

As a consequence of (2.11) and of $S'(u) \in L^\infty(Q)$, it follows that

$$S'(u)f_1(x,T_K(u),v) \in L^1(Q).$$

The above considerations show that equation (2.14) takes place in $D'(Q)$ and that $\frac{\partial S(u)}{\partial t}$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, which in turn implies that $\frac{\partial S(u)}{\partial t}$ belongs to $L^1(0,T;W^{-1,s}(\Omega))$ for all $s < inf(p', \frac{N}{N-1})$. It follows that $S(u)$ belongs to $C^0([0,T];W^{-1,s}(\Omega))$ so that the initial condition (2.16) makes sense. The same holds also for v.

3. Existence result

This section is devoted to establish the following existence theorem.

Theorem 3.0.4. *Under assumptions (2.1)-(2.10) there exists at least a renormalized solution* $(u,\ v)$ *of Problem (1.1)-(1.5).*

Proof. of Theorem 3.0.4. The proof is divided into 9 steps. In Step1, we introduce an approximate problem. Step 2 is devoted to establish a few *a priori* estimates. In Step 3, the limit (u, v) of the approximate solutions $(u^\varepsilon, v^\varepsilon)$ is introduced and is shown of (u, v) belongs to $L^\infty(0, T; L^1(\Omega))^2$ and to satisfy (2.11). In Step 4, we define a time regularization of the field $(T_K(u), T_K(v))$ and we establish Lemma 3.0.5, which a allows us to control the parabolic contribution that arises in the monotonicity method when passing to the limit. Step 5 is devoted to prove that an energy estimate (Lemma 3.0.6) which is a key point for the monotonicity arguments that are developed in Step 6 and Step 7. In Step 8, we prove that u satisfies (2.12) and v satisfies (2.13). At last, Step 9 is devoted to prove that (u, v) satisfies (2.14), (2.15) and (2.16) of definition 2.0.2 □

⋆ **Step 1**. Let us introduce the following regularization of the data :

$$a_\varepsilon(x, s, \xi) = a(x, T_{\frac{1}{\varepsilon}}(s), \xi) \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N\ ; \tag{3.1}$$

$$\Phi_\varepsilon \text{ is a lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N \tag{3.2}$$

such that Φ_ε uniformly converges to Φ on any compact subset of $\mathbb{R}$ as ε tends to 0.

$$f_1^\varepsilon(x, s_1, s_2) = f_1(x, T_{\frac{1}{\varepsilon}}(s_1), s_2) \text{ a.e. in } \Omega,\ \forall s_1,\ s_2 \in \mathbb{R}\ ; \tag{3.3}$$

$$f_2^\varepsilon(x, s_1, s_2) = f_2(x, s_1, T_{\frac{1}{\varepsilon}}(s_2)) \text{ a.e. in } \Omega,\ \forall s_1,\ s_2 \in \mathbb{R}\ ; \tag{3.4}$$

$$u_0^\varepsilon \text{ and } v_0^\varepsilon \text{ are a sequence of } C_0^\infty(\Omega)\text{- functions such that} \tag{3.5}$$

$$u_0^\varepsilon \to u_0 \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad v_0^\varepsilon \to v_0 \quad \text{in } L^1(\Omega)$$

as ε tends to 0.

Let us now consider the following regularized problem.

$$\frac{\partial u^\varepsilon}{\partial t} - div\Big(a_\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)\Big) + f_1^\varepsilon(x, u^\varepsilon, v^\varepsilon) = 0 \text{ in } Q\ ; \tag{3.6}$$

$$\frac{\partial v^\varepsilon}{\partial t} - div\Big(a_\varepsilon(x, v^\varepsilon, \nabla v^\varepsilon) + \Phi_\varepsilon(v^\varepsilon)\Big) + f_2^\varepsilon(x, u^\varepsilon, v^\varepsilon) = 0 \text{ in } Q\ ; \tag{3.7}$$

$$u^\varepsilon = v^\varepsilon = 0 \text{ on } (0, T) \times \partial\Omega\ ; \tag{3.8}$$

$$u^\varepsilon(t = 0) = u_0^\varepsilon \text{ in } \Omega. \tag{3.9}$$

$$v^\varepsilon(t = 0) = v_0^\varepsilon \text{ in } \Omega. \tag{3.10}$$

In view of (2.3), (2.8) and (2.9), a_ε, f_1^ε and f_1^ε satisfiy : there exists $C_\varepsilon \in L^{p'}(\Omega)$, $F_\varepsilon \in L^1(\Omega)$, $G_\varepsilon \in L^1(\Omega)$ and $\beta_\varepsilon > 0$, $\sigma_\varepsilon > 0$, $\lambda_\varepsilon > 0$, such that

$$|a_\varepsilon(x, s, \xi)| \le C_\varepsilon(x) + \beta_\varepsilon |\xi|^{p-1} \text{ a.e. in } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

$$|f_1^\varepsilon(x, s_1, s_2)| \le F_\varepsilon(x) + \sigma_\varepsilon\ |s_2| \quad \text{a.e. in } x \in \Omega,\ \forall s_1,\ s_2 \in \mathbb{R}.$$

and

$$|f_2^\varepsilon(x,s_1,s_2)| \le G_\varepsilon(x) + \lambda_\varepsilon\ |s_1| \quad \text{a.e. in } x\in\Omega,\ \forall s_1,\ s_2\in\mathbb{R}.$$

As a consequence, proving existence of a weak solution $(u^\varepsilon,\ v^\varepsilon) \in \Big(L^p(0,T;W_0^{1,p}(\Omega))\Big)^2$ of (3.6)-(3.10) is an easy task (see e.g. [1], [14] and [15]).

$\star$ **Step 2**. The estimates derived in this step rely on usual techniques for problems of type (3.6)-(3.10) and we just sketch the proof of them (the reader is referred to [3], [4], [7], [9], [5], [6] or to [10], [19], [20] for elliptic versions of (3.6)-(3.10)).

Using $T_K(u^\varepsilon)$ as a test function in (3.6) leads to

$$\int_\Omega \overline{T}_K^\varepsilon(u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds \tag{3.11}$$

$$+\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds + \int_0^t\int_\Omega f_1^\varepsilon(x,u^\varepsilon,v^\varepsilon)T_K(u^\varepsilon)\,dx\,ds = \int_\Omega \overline{T}_K^\varepsilon(u_0^\varepsilon)\,dx$$

for almost every t in $(0,T)$, and where

$$\overline{T}_K^\varepsilon(r) = \int_0^r T_K(s)\,ds = \begin{cases} \frac{r^2}{2} & \text{if } |r|\le K \\ K\ |r| - \frac{K^2}{2} & \text{if } |r|\ge K \end{cases}$$

The Lipschitz character of Φ_ε, Stokes formula together with the boundary condition (3.8) make it possible to obtain

$$\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds = 0, \tag{3.12}$$

for almost any $t\in(0,T)$.

Since a_ε satisfies (2.2), f_1^ε satisfies (2.7) and the properties of $\overline{T}_K^\varepsilon$ and u_0^ε, permit to deduce from (3.11) that

$$T_K(u^\varepsilon) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \tag{3.13}$$

independently of ε for any $K\ge 0$.

Proceeding as in [4], [7] [5] and [6] that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' is compact ($suppS'\subset[-K,K]$)

$$S(u^\varepsilon) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \tag{3.14}$$

and

$$\frac{\partial S(u^\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{3.15}$$

independently of ε, as soon as $\varepsilon < \frac{1}{K}$.

Now for fixed $K>0$: $a_\varepsilon(T_K(u^\varepsilon),DT_K(u^\varepsilon)) = a(T_K(u^\varepsilon),DT_K(u^\varepsilon))$ a.e. in Q as soon as $\varepsilon<\frac{1}{K}$, while assumption (2.3) gives

$$\Big|a_\varepsilon(T_K(u^\varepsilon),DT_K(u^\varepsilon))\Big| \le C_K(x) + \beta_K|DT_K(u^\varepsilon)|^{p-1}$$

where $\beta_K>0$ and $C_K\in L^{p'}(Q)$. In view (3.13), we deduce that,

$$a\Big(T_K(u^\varepsilon),DT_K(u^\varepsilon)\Big) \text{ is bounded in } (L^{p'}(Q))^N. \tag{3.16}$$

independently of ε for $\varepsilon<\frac{1}{K}$.

For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(r) = T_{n+1}(r) - T_n(r)$$

Remark that $\|\theta_n\|_{L^\infty(\mathbb{R})} \leq 1$ for any $n \geq 1$ and that $\theta_n(r) \to 0$ for any r when n tends to infinity.

Using the admissible test function $\theta_n(u^\varepsilon)$ in (3.6) leads to

$$\int_\Omega \overline{\theta_n}(u^\varepsilon)(t)\,dx + \int_0^t \int_\Omega a_\varepsilon(u^\varepsilon, Du^\varepsilon) D\theta_n(u^\varepsilon)\,dx\,ds \tag{3.17}$$

$$+ \int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon) D\theta_n(u^\varepsilon)\,dx\,ds + \int_0^t \int_\Omega f_1^\varepsilon(x, u^\varepsilon, v^\varepsilon)\theta_n(u^\varepsilon)\,dx\,ds = \int_\Omega \overline{\theta_n}(u_0^\varepsilon)\,dx,$$

for almost any t in $(0,T)$ and where $\overline{\theta_n}(r) = \displaystyle\int_0^r \theta_n(s)\,ds$.

The Lipschitz character of Φ_ε, Stokes formula together with boundary condition (3.8) allow to obtain

$$\int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon) D\theta_n(u^\varepsilon)\,dx\,ds = 0. \tag{3.18}$$

Since $\overline{\theta_n}(.) \geq 0,\ f_1^\varepsilon$ satisfies (2.7), we have

$$\int_0^t \int_\Omega a(u^\varepsilon, Du^\varepsilon) D\theta_n(u^\varepsilon)\,dx\,ds \leq \int_\Omega \overline{\theta_n}(u_0^\varepsilon)\,dx, \tag{3.19}$$

for almost $t \in (0,T)$ and for $\varepsilon < \frac{1}{n+1}$.

$\star$ **Step 3**. Arguing again as in [4], [7] [5] and [6] estimates (3.14), (3.15) imply that, for a subsequence still indexed by ε,

$$u^\varepsilon \text{ converges almost every where to } u \text{ in } Q, \tag{3.20}$$

and with the help of (3.13),

$$T_K(u^\varepsilon) \text{ converges weakly to } T_K(u) \text{ in } L^p(0,T; W_0^{1,p}(\Omega)), \tag{3.21}$$

$$\theta_n(u^\varepsilon) \rightharpoonup \theta_n(u) \text{ weakly in } L^p(0,T; W_0^{1,p}(\Omega)) \tag{3.22}$$

$$a_\varepsilon\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big) \rightharpoonup X_K \text{ weakly in } (L^{p'}(Q))^N. \tag{3.23}$$

The same holds for v^ε :

$$v^\varepsilon \text{ converges almost every where to } v \text{ in } Q, \tag{3.24}$$

$$T_K(v^\varepsilon) \text{ converges weakly to } T_K(v) \text{ in } L^p(0,T; W_0^{1,p}(\Omega)), \tag{3.25}$$

$$\theta_n(v^\varepsilon) \rightharpoonup \theta_n(v) \text{ weakly in } L^p(0,T; W_0^{1,p}(\Omega)) \tag{3.26}$$

$$a_\varepsilon\Big(T_K(v^\varepsilon), DT_K(v^\varepsilon)\Big) \rightharpoonup Y_K \text{ weakly in } (L^{p'}(Q))^N \tag{3.27}$$

as ε tends to 0 for any $K > 0$ and any $n \geq 1$ and where for any $K > 0,\ X_K,\ Y_K$ belongs to $(L^{p'}(Q))^N$.

We now establish that u and v belongs to $L^\infty(0,T;L^1(\Omega))$. To this end, recalling (2.7), (3.5), (3.12) and (3.20) allows to pass to the limit-inf in (3.11) as ε tends to 0 and to obtain

$$\int_\Omega \overline{T_K}(u)(t)\,dx \le K\ \|u_0\|_{L^1(\Omega)}.$$

Due to the definition of $\overline{T_K}$, we deduce from the above inequality that

$$K\int_\Omega |u(x,t)|\,dx \le \frac{3K^2}{2} mes(\Omega) + K\ \|u_0\|_{L^1(\Omega)}$$

for almost any $t \in (0,T)$, which shows that u belongs to $L^\infty(0,T;L^1(\Omega))$.

The same holds for v belongs to $L^\infty(0,T;L^1(\Omega))$.

We are now in a position to exploit (3.19). Due to the definition of θ_n we have

$$a(u^\varepsilon, Du^\varepsilon)D\theta_n(u^\varepsilon) = a(u^\varepsilon, Du^\varepsilon)Du^\varepsilon \chi_{\{n\le|u^\varepsilon|\le n+1\}} \ge \alpha\ |D\theta_n(u^\varepsilon)|^p \quad \text{a.e. in } Q$$

Inequality (3.19), the weak convergence (3.22) and the pointwise convergence of u_0^ε to u_0 then imply that

$$\alpha\ \int_Q |D\theta_n(u)|^p\,dx\,dt \le \int_\Omega \overline{\theta}_n(u_0)\,dx.$$

Since θ_n and $\overline{\theta_n}$ both converge to zero everywhere as n goes to zero while

$$|\theta_n(u)| \le 1 \text{ and } |\overline{\theta}_n(u)| \le |u_0| \in L^1(\Omega)$$

the Lebesgue's convergence theorem permits to conclude that

$$\lim_{n\to+\infty} \int_{\{n\le|u|\le n+1\}} |Du|^p\,dx\,dt = 0. \tag{3.28}$$

and

$$\lim_{n\to+\infty} \overline{\lim_{\varepsilon\to 0}} \int_{\{n\le|u^\varepsilon|\le n+1\}} a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon\,dx\,dt = 0. \tag{3.29}$$

$\star$ **Step 4**. This step is devoted to introduce for $K \ge 0$ fixed, a time regularization of the function $T_K(u)$ in order to perform the monotonicity method which will be developed in Step 5 and Step 6. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [16]). More recently, it has been exploited in [8] and [11] to solve a few nonlinear evolution problems with L^1 or measure data.

This specific time regularization of $T_K(u)$ (for fixed $K \ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega) \quad \text{for all } \mu > 0, \tag{3.30}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \le K \quad \forall \mu > 0, \tag{3.31}$$

$$v_0^\mu \to T_K(u_0) \text{ a.e. in } \Omega \text{ and } \frac{1}{\mu}\|Dv_0^\mu\|^p_{L^p(\Omega)} \to 0, \text{ as } \mu \to +\infty. \tag{3.32}$$

Existence of such a subsequence $(v_0^\mu)_\mu$ is easy to establish (see e.g. [13]). For fixed $K \geq 0$ and $\mu > 0$, let us consider the unique solution $T_K(u)_\mu \in L^\infty(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$ of the monotone problem :

$$\frac{\partial T_K(u)_\mu}{\partial t} + \mu\Big(T_K(u)_\mu - T_K(u)\Big) = 0 \text{ in } D'(Q). \tag{3.33}$$

$$T_K(u)_\mu(t=0) = v_0^\mu \text{ in } \Omega. \tag{3.34}$$

Remark that due to (3.33), we have for $\mu > 0$ and $K \geq 0$,

$$\frac{\partial T_K(u)_\mu}{\partial t} \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.35}$$

The behavior of $T_K(u)_\mu$ as $\mu \to +\infty$ is investigated in [16] (see also [11] and [13]) and we just recall here that (3.30)-(3.34) imply that

$$T_K(u)_\mu \to T_K(u) \ \text{ a.e. in } Q\ ; \tag{3.36}$$

and in $L^\infty(Q)$ weak $\star$ and strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$.

$$\|T_K(u)_\mu\|_{L^\infty(Q)} \leq max\Big(\|T_K(u)\|_{L^\infty(Q)}\ ;\ \|v_0^\mu\|_{L^\infty(\Omega)}\Big) \leq K \tag{3.37}$$

for any μ and any $K \geq 0$.

The very definition of the sequence $T_K(u)_\mu$ for $\mu > 0$ (and fixed K) allow to establish the following lemma

Lemma 3.0.5. *Let $K \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \leq K$ and supp(S') is compact. Then*

$$\underline{\lim_{\mu\to+\infty}} \lim_{\varepsilon\to 0} \int_0^T \int_0^s \Big\langle \frac{\partial S(u^\varepsilon)}{\partial t}\ ,\ \Big(T_K(u^\varepsilon) - (T_K(u))_\mu\Big)\Big\rangle\, dt\, ds \geq 0$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof of Lemma 3.0.5 : The Lemma is proved in [5] (see Lemma 1, p.341).

$\star$ **Step 5**. In this step we prove the following lemma which is the key point in the monotonocity arguments that will be developed in Step 6.

Lemma 3.0.6. *The subsequence of u^ε defined is Step 3 satisfies for any $K \geq 0$*

(3.38)
$$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega a(u^\varepsilon, DT_K(u^\varepsilon))DT_K(u^\varepsilon)\, dx\, ds\, dt \leq \int_0^T \int_0^t \int_\Omega X_K DT_K(u)\, dx\, ds\, dt$$

Proof of Lemma 3.0.6 : We first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that, for any $n \geq 1$

$$S_n(r) = r \ \text{ for } |r| \leq n, \tag{3.39}$$

$$supp S_n' \subset [-(n+1), (n+1)], \tag{3.40}$$

$$\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \tag{3.41}$$

Pointwise multiplication of (3.6) by $S_n'(u^\varepsilon)$ (which is licit) leads to

$$(3.42)\qquad \frac{\partial S_n(u^\varepsilon)}{\partial t} - div\Big(S_n(u^\varepsilon)a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)\Big) + S_n''(u^\varepsilon)a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)Du^\varepsilon$$
$$- div\Big(\Phi_\varepsilon(u^\varepsilon)S_n'(u^\varepsilon)\Big) + S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) + f_1^\varepsilon(x,u^\varepsilon,v^\varepsilon)S_n'(u^\varepsilon) = 0 \text{ in } D'(Q).$$

We use the sequence $T_K(u)_\mu$ of approximations of $T_K(u)$ defined by (3.33), (3.34) of Step 4 and plug the test function $T_K(u^\varepsilon) - T_K(u)_\mu$ (for $\varepsilon > 0$ and $\mu > 0$) in (3.42). Through setting, for fixed $K \geq 0$,

$$(3.43)\qquad W_\mu^\varepsilon = T_K(u^\varepsilon) - T_K(u)_\mu$$

we obtain upon integration over $(0,t)$ and then over $(0,T)$:

(3.44)
$$\int_0^T\int_0^t \Big\langle \frac{\partial S_n(u^\varepsilon)}{\partial t}\ ,\ W_\mu^\varepsilon\Big\rangle\,ds\,dt + \int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)DW_\mu^\varepsilon\,dx\,ds\,dt$$
$$+ \int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)W_\mu^\varepsilon a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)Du^\varepsilon\,dx\,ds\,dt$$
$$+ \int_0^T\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)S_n'(u^\varepsilon)DW_\mu^\varepsilon\,dx\,ds\,dt$$
$$+ \int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)W_\mu^\varepsilon\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon\,dx\,ds\,dt$$
$$+ \int_0^T\int_0^t\int_\Omega f_1^\varepsilon(x,u^\varepsilon,v^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt = 0$$

In the following we pass to the limit in (3.44) as ε tends to 0, then μ tends to $+\infty$ and then n tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$:

$$(3.45)\qquad \underline{\lim_{\mu\to+\infty}}\lim_{\varepsilon\to0}\int_0^T\int_0^t \Big\langle \frac{\partial S_n(u^\varepsilon)}{\partial t}\ ,\ W_\mu^\varepsilon\Big\rangle\,ds\,dt \geq 0 \text{ for any } n \geq K,$$

$$(3.46)\qquad \lim_{\mu\to+\infty}\lim_{\varepsilon\to0}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)DW_\mu^\varepsilon\,dx\,ds\,dt = 0 \text{ for any } n \geq 1,$$

$$(3.47)\qquad \lim_{\mu\to+\infty}\lim_{\varepsilon\to0}\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)W_\mu^\varepsilon\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon\,dx\,ds\,dt = 0 \text{ for any } n,$$

$$(3.48)\qquad \lim_{n\to+\infty}\overline{\lim_{\mu\to+\infty}}\,\overline{\lim_{\varepsilon\to0}}\Big|\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)W_\mu^\varepsilon a_\varepsilon(u^\varepsilon,Du^\varepsilon)Du^\varepsilon\,dx\,ds\,dt\Big| = 0,$$

and

$$(3.49)\qquad \lim_{\mu\to+\infty}\lim_{\varepsilon\to0}\int_0^T\int_0^t\int_\Omega f_1^\varepsilon(x,u^\varepsilon,v^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt = 0 \text{ for any } n \geq 1.$$

Proof of (3.45). In view of the definition (3.43) of W_μ^ε, lemma 3.0.5 applies with $S = S_n$ for fixed $n \geq K$. As a consequence (3.45) holds true.

Proof of (3.46). For fixed $n \geq 1$, we have

$$S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)DW_\mu^\varepsilon = S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))DW_\mu^\varepsilon \tag{3.50}$$

a.e. in Q, and for all $\varepsilon \leq \frac{1}{n+1}$, and where $suppS_n' \subset [-(n+1), n+1]$.

Since S_n' is smooth and bounded, (2.5), (3.2) and (3.20) lead to

$$S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon)) \to S_n'(u)\Phi(T_{n+1}(u)) \tag{3.51}$$

a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0.

For fixed $\mu > 0$, we have

$$W_\mu^\varepsilon \rightharpoonup T_K(u) - T_K(u)_\mu \quad \text{weakly } in \ L^p(0,T; W_0^{1,p}(\Omega)) \tag{3.52}$$

and a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0.

As a consequence of (3.50), (3.51) and (3.52) we deduce that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)DW_\mu^\varepsilon \, dx\, ds\, dt \\ &= \int_0^T \int_0^t \int_\Omega S_n'(u)\Phi(u)\Big[DT_K(u) - DT_K(u)_\mu\Big]\, dx\, ds\, dt \end{aligned} \tag{3.53}$$

for any $\mu > 0$.

Appealing now to (3.36) and passing to the limit as $\mu \to +\infty$ in (3.53) allows to conclude that (3.46) holds true.

Proof of (3.47). For fixed $n \geq 1$, and by the same arguments that those that lead to (3.50), we have

$$S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon W_\mu^\varepsilon = S_n''(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))DT_{n+1}(u^\varepsilon)W_\mu^\varepsilon \text{ a.e. in } Q.$$

From (2.5), (3.2) and (3.20), it follows that for any $\mu > 0$

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)W_\mu^\varepsilon \, dx\, ds\, dt \\ &= \int_0^T \int_0^t \int_\Omega S_n''(u)\Phi(u)\Big[DT_K(u) - DT_K(u)_\mu\Big]\, dx\, ds\, dt \end{aligned}$$

with the help of (3.36) passing to the limit, as μ tends to $+\infty$, in the above equality leads to (3.47).

Proof of (3.48). For any $n \geq 1$ fixed, we have $suppS_n'' \subset [-(n+1), -n] \cup [n, n+1]$. As a consequence

$$\begin{aligned} &\Big| \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon W_\mu^\varepsilon \, dx\, ds\, dt \Big| \\ &\leq T\|S_n''\|_{L^\infty(\mathbb{R})}\|W_\mu^\varepsilon\|_{L^\infty(Q)} \int_{\{n\leq |u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon \, dx\, dt, \end{aligned}$$

for any $n \geq 1$, and any $\mu > 0$. The above inequality together with (3.37) and (3.41) make it possible to obtain

$$\overline{\lim_{\mu\to+\infty}}\ \overline{\lim_{\varepsilon\to 0}} \Big| \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon W_\mu^\varepsilon \, dx\, ds\, dt \Big| \tag{3.54}$$

$$\leq C\overline{\lim_{\varepsilon\to 0}}\int_{\{n\leq |u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon\,dx\,dt,$$

for any $n \geq 1$, where C is a constant independent of n.

Appealing now to (3.29) permits to pass to the limit as n tends to $+\infty$ in (3.54) and to establish (3.48).

Proof of (3.49). For fixed $n \geq 1$, we have

$$f_1^\varepsilon(x, u^\varepsilon, v^\varepsilon)S_n'(u^\varepsilon) = f_1(x, T_{n+1}(u^\varepsilon), v^\varepsilon)$$

a.e. in Q, and for all $\varepsilon \leq \frac{1}{n+1}$.

In view (2.8), (3.20) and (3.24), Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega f_1^\varepsilon(x, u^\varepsilon, v^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt$$

$$= \int_0^T\int_0^t\int_\Omega f_1(x,u,v)S_n'(u)\Big(T_K(u) - T_K(u)_\mu\Big)\,dx\,ds\,dt.$$

Now for fixed $n \geq 1$, using (3.36) permits to pass to the limit as μ tends to $+\infty$ in the above equality to obtain (3.49).

We now turn back to the proof of lemma 3.0.6, due to (3.44), (3.45), (3.46), (3.47), (3.48) and (3.49), we are in a position to pass to the lim-sup when ε tends to zero, then to the limit-sup when μ tends to $+\infty$ and then to the limit as n tends to $+\infty$ in (3.44). We obtain using the definition of W_μ^ε that for any $K \geq 0$

$$\lim_{n\to+\infty}\overline{\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)\Big(DT_K(u^\varepsilon) - DT_K(u)_\mu\Big)\,dx\,ds\,dt \leq 0.$$

Since $S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)DT_K(u^\varepsilon) = a(u^\varepsilon, Du^\varepsilon)DT_K(u^\varepsilon)$ for $\varepsilon \leq \frac{1}{K}$ and $K \leq n$.

The above inequality implies that for $K \leq n$

$$\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega a_\varepsilon(u^\varepsilon, Du^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds\,dt \tag{3.55}$$

$$\leq \lim_{n\to+\infty}\overline{\lim_{\mu\to+\infty}}\,\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)DT_K(u)_\mu\,dx\,ds\,dt$$

The right hand side of (3.55) is computed as follows. In view (3.1) and (3.40), we have for $\varepsilon \leq \frac{1}{n+1}$.

$$S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon) = S_n'(u^\varepsilon)a\Big(T_{n+1}(u^\varepsilon), DT_{n+1}(u^\varepsilon)\Big)\text{ a.e. in } Q.$$

Due to (3.23) it follows that for fixed $n \geq 1$

$$S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon) \rightharpoonup S_n'(u)X_{n+1}\text{ weakly in } L^{p'}(Q),$$

when ε tends to 0. The strong convergence of $T_K(u)_\mu$ to $T_K(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ as μ tends to $+\infty$, then allows to conclude that

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)DT_K(u)_\mu\,dx\,ds\,dt \tag{3.56}$$

$$= \int_0^T \int_0^t \int_\Omega S_n'(u) X_{n+1} DT_K(u)\,dx\,ds\,dt = \int_0^T \int_0^t \int_\Omega X_{n+1} DT_K(u)\,dx\,ds\,dt$$

as soon as $K \le n$, since $S_n'(r) = 1$ for $|r| \le n$. Now for $K \le n$ we have

$$a\Big(T_{n+1}(u^\varepsilon), DT_{n+1}(u^\varepsilon)\Big)\chi_{\{|u^\varepsilon|<K\}} = a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)\chi_{\{|u^\varepsilon|<K\}} \text{ a.e. in } Q$$

Passing to the limit as ε tends to 0, we obtain

$$X_{n+1}\chi_{\{|u|<K\}} = X_K \chi_{\{|u|<K\}} \text{ a.e. in } Q - \{|u| = K\} \text{ for } K \le n. \tag{3.57}$$

As a consequence of (3.57) we have for $K \le n$

$$X_{n+1} DT_K(u) = X_K DT_K(u) \text{ a.e. in } Q. \tag{3.58}$$

Recalling (3.55), (3.56) and (3.58) allows to conclude (3.38) holds true and the proof of lemma 3.0.6 is complete.

⋆ **Step 6**. In this step we prove the following monotonicity estimate :

Lemma 3.0.7. *The subsequence of u^ε defined in step 3 satisfies for any $K \ge 0$*

$$\lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega \Big[a(T_K(u^\varepsilon), DT_K(u^\varepsilon)) - a(T_K(u^\varepsilon), DT_K(u))\Big] \Big[DT_K(u^\varepsilon) - DT_K(u)\Big]\,dx\,ds\,dt = 0 \tag{3.59}$$

Proof of Lemma 3.0.7. Let $K \ge 0$ be fixed. The monotone character (2.4) of $a(s,\xi)$ with respect to ξ implies that

$$\int_0^T \int_0^t \int_\Omega \Big[a(T_K(u^\varepsilon), DT_K(u^\varepsilon)) - a(T_K(u^\varepsilon), DT_K(u))\Big] \Big[DT_K(u^\varepsilon) - DT_K(u)\Big]\,dx\,ds\,dt \ge 0, \tag{3.60}$$

To pass to the limit-sup as ε tends to 0 in (3.60), let us remark that (2.1), (2.3) and (3.20) imply that

$$a(T_K(u^\varepsilon), DT_K(u)) \to a(T_K(u), DT_K(u)) \text{ a.e. in } Q,$$

as ε tends to 0, and that

$$\Big|a(T_K(u^\varepsilon), DT_K(u))\Big| \le C_K(t,x) + \beta_K |DT_K(u)|^{p-1}$$

a.e. in Q, uniformly with respect to ε.

It follows that when ε tends to 0

$$a\Big(T_K(u^\varepsilon), DT_K(u)\Big) \to a\Big(T_K(u), DT_K(u)\Big) \text{ strongly in } (L^{p'}(Q))^N. \tag{3.61}$$

Using (3.38) of lemma (3.0.6), (3.21), (3.23) and (3.61) allow to pass to the lim-sup as ε tends to zero in (3.60) and to obtain (3.59) of lemma 3.0.7.

⋆ **Step 7**. In this step we identify the weak limit X_K and we prove the weak L^1 convergence of the "truncated" energy $a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u^\varepsilon)$ as ε tends to 0.

Lemma 3.0.8. *For fixed $K \geq 0$, we have as ε tends to 0*

$$X_K = a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big) \quad \text{a.e. in } Q. \tag{3.62}$$

And as ε tends to 0

(3.63)

$$a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u^\varepsilon) \rightharpoonup a\Big(T_K(u), DT_K(u)\Big)DT_K(u) \text{ weakly in } L^1(Q).$$

Proof of Lemma (3.0.8). The proof is standard once we remark that for any $K \geq 0$, any $0 < \varepsilon < \frac{1}{K}$ and any $\xi \in \mathbb{R}^N$

$$a_\varepsilon(T_K(u^\varepsilon), \xi) = a(T_K(u^\varepsilon), \xi) = a_{\frac{1}{K}}(T_K(u^\varepsilon), \xi) \quad \text{a.e. in } Q$$

which together with (3.21), (3.61) makes it possible to obtain from (3.59) of lemma 3.0.7

$$\lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega a_{\frac{1}{K}}\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u^\varepsilon)\, dx\, ds\, dt \tag{3.64}$$

$$= \int_0^T \int_0^t \int_\Omega \sigma_K DT_K(u)\, dx\, ds\, dt.$$

Since, for fixed $K > 0$, the function $a_{\frac{1}{K}}(s, \xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view (3.21), (3.23), and (3.64). It follows that (3.62) holds true (the case $K = 0$ being trivial). In order to prove (3.63), we observe that the monotone character of a (with respect to ξ) and (3.59) give that for any $K \geq 0$ and any $T' < T$

$$\Big[a(T_K(u^\varepsilon), DT_K(u^\varepsilon)) - a(T_K(u^\varepsilon), DT_K(u))\Big]\Big[DT_K(u^\varepsilon) - DT_K(u)\Big] \to 0 \tag{3.65}$$

strongly in $L^1((0, T') \times \Omega)$ as ε tends to 0.

Moreover (3.21), (3.23), (3.61) and (3.62) imply that

$$a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u) \rightharpoonup a\Big(T_K(u), DT_K(u)\Big)DT_K(u) \text{ weakly in } L^1(Q),$$

$$a\Big(T_K(u^\varepsilon), DT_K(u)\Big)DT_K(u^\varepsilon) \rightharpoonup a\Big(T_K(u), DT_K(u)\Big)DT_K(u) \text{ weakly in } L^1(Q),$$

and

$$a\Big(T_K(u^\varepsilon), DT_K(u)\Big)DT_K(u) \longrightarrow a\Big(T_K(u), DT_K(u)\Big)DT_K(u) \text{ strongly in } L^1(Q),$$

as tends to 0. Using the above convergence results in (3.65) shows that for any $K \geq 0$ and any $T' < T$

$$a\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u^\varepsilon) \rightharpoonup a\Big(T_K(u), DT_K(u)\Big)DT_K(u) \tag{3.66}$$

weakly in $L^1((0, T') \times \Omega)$ as tends to 0.

Remark that for $\overline{T} > T$, we have (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9) hold true with $\overline{T}$ in place of T, we can show that the convergence result (3.66) is still in $L^1(Q)$ weak, namely that (3.63) holds true.

$\star$ **Step 8**. In this step we prove that u satisfies (2.12) (and (2.13)). To this end, remark that for any fixed $n \geq 0$ one has

$$\begin{aligned}&\int_{\{(t,x)/\ n\leq |u^\varepsilon|\leq n+1\}} a(u^\varepsilon, Du^\varepsilon) Du^\varepsilon \, dx\, dt\\ &= \int_Q a_\varepsilon(u^\varepsilon, Du^\varepsilon)\Big[DT_{n+1}(u^\varepsilon) - DT_n(u^\varepsilon)\Big]\, dx\, dt\\ &= \int_Q a_\varepsilon\Big(T_{n+1}(u^\varepsilon), DT_{n+1}(u^\varepsilon)\Big) DT_{n+1}(u^\varepsilon)\, dx\, dt\\ &\quad - \int_Q a_\varepsilon\Big(T_n(u^\varepsilon), DT_n(u^\varepsilon)\Big) DT_n(u^\varepsilon)\, dx\, dt\end{aligned}$$

for $\varepsilon < \frac{1}{n+1}$.

According to (3.63), one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 0$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\{(t,x)/\ n\leq |u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon, Du^\varepsilon) Du^\varepsilon \, dx\, dt\\ &= \int_Q a\Big(T_{n+1}(u), DT_{n+1}(u)\Big) DT_{n+1}(u)\, dx\, dt\\ &\quad - \int_Q a\Big(T_n(u), DT_n(u)\Big) DT_n(u)\, dx\, dt\\ &= \int_{\{(t,x)/\ n\leq |u|\leq n+1\}} a(u, Du) Du\, dx\, dt\end{aligned} \tag{3.67}$$

Taking the limit as n tends to $+\infty$ in (3.68) and using the estimate (3.67) show that u satisfies (2.12), (and v satisfies (2.13)).

$\star$ **Step 9**. In this step, u is shown to satisfies (2.14) and (2.16 for u) (and v is shown to satisfies (2.15) and (2.16) for v). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $suppS' \subset [-K, K]$. Pointwise multiplication of the approximate equation (3.6) by $S'(u^\varepsilon)$ (and (3.7) by $S'(v^\varepsilon)$) leads to

$$\begin{aligned}&\frac{\partial S(u^\varepsilon)}{\partial t} - div\Big(S'(u^\varepsilon) a_\varepsilon(u^\varepsilon, Du^\varepsilon)\Big) + S''(u^\varepsilon) a_\varepsilon(u^\varepsilon, Du^\varepsilon) Du^\varepsilon\\ &- div\Big(S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\Big) + S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) Du^\varepsilon + f_1^\varepsilon(x, u^\varepsilon, v^\varepsilon) S'(u^\varepsilon) = 0 \ \text{ in } D'(Q).\end{aligned} \tag{3.68}$$

In what follows we pass to the limit as ε tends to 0 in each term of (3.68).

$\star$ *Limit of* $\frac{\partial S(u^\varepsilon)}{\partial t}$

Since S is bounded and continuous, and $S(u^\varepsilon)$ converges to $S(u)$ a.e. in Q and in $L^\infty(Q)$ weak $\star$. Then $\frac{\partial S(u^\varepsilon)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $D'(Q)$ as ε tends to 0.

$\star$ *Limit of* $-div\Big(S'(u^\varepsilon) a_\varepsilon(u^\varepsilon, Du^\varepsilon)\Big)$

Since $suppS' \subset [-K, K]$, we have for $\varepsilon < \frac{1}{K}$,

$$S'(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon) = S'(u^\varepsilon)a_\varepsilon\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big) \text{ a.e. in } Q.$$

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S, (3.21) and (3.62) of Lemma (3.0.8) imply that

$$S'(u^\varepsilon)a_\varepsilon\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big) \rightharpoonup S'(u)a\Big(T_K(u), DT_K(u)\Big) \text{ weakly in } L^{p'}(Q),$$

as ε tends to 0, because $S'(u) = 0$ for $|u| \geq K$ a.e. in Q. And the term

$$S'(u)a\Big(T_K(u), DT_K(u)\Big) = S'(u)a(u, Du) \text{ a.e. in } Q.$$

$\star$ *Limit of* $S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon$

Since $suppS'' \subset [-K, K]$, we have for $\varepsilon \leq \frac{1}{K^\star}$

$$S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon = S''(u^\varepsilon)a_\varepsilon\Big(T_K(u^\varepsilon), DT_K(u^\varepsilon)\Big)DT_K(u^\varepsilon) \text{ a.e. in } Q.$$

The pointwise convergence of $S''(u^\varepsilon)$ to $S''(u)$ as ε tends to 0, the bounded character of S'', T_K and (3.63) of lemma (3.0.8) allow to conclude that

$$S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, Du^\varepsilon)Du^\varepsilon \rightharpoonup S''(u)a\Big(T_K(u), DT_K(u)\Big)DT_K(u)$$

weakly in $L^1(Q)$, as ε tends to 0. And

$$S''(u)a\Big(T_K(u), DT_K(u)\Big)DT_K(u) = S''(u)a(u, u)Du \text{ a.e. in } Q.$$

$\star$ *Limit of* $S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)$

Since $suppS' \subset [-K, K]$, we have for $\varepsilon \leq \frac{1}{K^\star}$

$$S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) = S'(u^\varepsilon)\Phi_\varepsilon(T_K(u^\varepsilon)) \text{ a.e. in } Q.$$

As a consequence of (2.5), (3.2) and (3.20), it follows that for any $1 \leq q < +\infty$

$$S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) \to S'(u)\Phi(T_K(u)) \text{ strongly in } L^q(Q),$$

as ε tends to 0. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

$\star$ *Limit of* $S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon$
Since $S' \in W^{1,\infty}(\mathbb{R})$ with $suppS' \subset [-K, K]$, we have

$$S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon = \Phi_\varepsilon(T_K(u^\varepsilon))DS'(u^\varepsilon) \text{ a.e. in } Q,$$

we have, $DS'(u^\varepsilon)$ converges to $DS'(u)$ weakly in $L^p(Q)^N$ as ε tends to 0, while $\Phi_\varepsilon(T_K(u^\varepsilon))$ is uniformly bounded with respect to ε and converges a.e. in Q to $\Phi(T_K(u))$ as ε tends to 0. Therefore

$$S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon \rightharpoonup \Phi_\varepsilon(T_K(u^\varepsilon))DS'(u^\varepsilon) \text{ weakly in } L^p(Q).$$

$\star$ *Limit of* $f^\varepsilon(x, u^\varepsilon, v^\varepsilon)S'(u^\varepsilon)$

Due to (2.6), (2.8), (3.3), (3.20) and (3.24), we have

$$f^\varepsilon(x,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \to f(x,u,v)S'(u) \text{ strongly in } L^1(Q),$$

as ε tends to 0.

As a consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in equation (3.68) and to conclude that u satisfies (2.14), (and v satisfies (2.15))

It remains to show that $S(u)$ (and $S(v)$) satisfies the initial condition (2.16 for u) (and (2.16 for v). To this end, firstly remark that, S being bounded, $S(u^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.68) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (see, e.g, [24], Corollary 4) implies that $S(u^\varepsilon)$ lies in a compact set of $C^0([0,T];W^{-1,s}(\Omega))$ for any $s < inf\Big(p', \frac{N}{N-1}\Big)$. It follows that, on one hand, $S(u^\varepsilon)(t=0) = S(u_0^\varepsilon)$ converges to $S(u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the order hand, (3.5) and the smoothness of S imply that $S(u_0^\varepsilon)$ converges to $S(u)(t=0)$ strongly in $L^q(\Omega)$ for all $q < +\infty$. Then we conclude that

$$S(u)(t=0) = S(u_0) \text{ in } \Omega.$$

The same holds also for v

$$S(v)(t=0) = S(v_0) \text{ in } \Omega.$$

As a conclusion of step 3, step 8 and step 9, the proof of theorem 3.0.4 is complete.

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