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Sufficient condition for existence of solutions for higher-order resonance boundary value problem with one-dimensional p-Laplacian [1]

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Abstract:By using coincidence degree theory of Mawhin, existence results for some higher order resonance multi-point boundary value problems with one dimensional p-Laplacian operator are obtained.

Keywords: boundary value problem; one-dimensional p-Laplacian; resonance; coincidence degree.

1. Introduction

In this paper we consider higher-order multi-point boundary value problem with one-dimensional p-Laplacian

$$(\varphi_p(x^{(i)}(t)))^{(n-i)} = f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) + e(t), t \in (0,1), \tag{1.1}$$

subject to one of the following boundary conditions:

$$x(1) = \sum_{j=1}^{m-2} \alpha_j x(\xi_j), x''(0) = \cdots = x^{(i-1)}(0) = x^{(i+1)}(0) = \cdots = x^{(n-1)}(0) = 0, x^{(i-1)}(1) = x^{(i-1)}(\xi), x^{(i)}(1) = x^{(i)}(\eta), \tag{1.2}$$

where $p > 1$ is a constant; $\varphi_p : R \to R, \varphi_p(u) = |u|^{p-2}u; f : [0,1]\times R^n \to R$ is a continuous function and $1 \le i \le n-1$ is a fixed integer, $e(t) \in L^1[0,1], \alpha_j(1 \le j \le m-2) \in R, \eta, \xi, \xi_j \in (0,1), j = 1, \cdots, m-2, 0 < \xi_1 < \cdots < \xi_{m-2} < 1.$

We notice that the operator $\varphi_p(u) = |u|^{p-2}u$ is called the (one-dimensional) p-Laplacian and it appears in many contexts. For example, it is used extensively in non-Newtonian fluids, in some reaction-diffusion problems, in flow through porous media, in nonlinear elasticity, glaceology and petroleum extraction.

The boundary value problem (1.1), (1.2) is said to be at resonance in the sense that the associate homogeneous problem

$$(\varphi_p(x^{(i)}(t)))^{(n-i)} = 0, 0 \le t \le 1$$

subject to boundary condition (1.2) has nontrival solutions.

The study on multi-point nonlocal boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Since then some existence results have been obtained for general

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nonlinear boundary value problems by several authors. We refer the reader to some recent results, such as [3-7] at non-resonance and [8-12] at resonance. For resonance case, by using Leray-Schauder continuation theorem, nonlinear-alternative of Leray-Schauder and coincidence degree theorem, the main technique of these works is to convert the problem into the abstract form $Lx = Nx$, where L is a non-invertible linear operator. For problem (1.1) with some resonance conditions , if $p = 2$, some existence results are established by [10-12].

But as far as we know, the existence results for high order resonance problems with p-Laplacian operator such as (1.1), (1.2) with $p \neq 2$ have never been studied before. This is mainly due to the facts that in this situation, above methods are not applicable directly since the p-Laplacian operator $(\varphi_p(x^i(t)))^{(n-i)}$ is not linear with respect to x. Inspired by [13,14], the goal of this paper is to fill the gap in this area. By using Mawhin continuation theorem the existence results for above problem are established.

2.Preliminaries

First we recall briefly some notations and an abstract existence results.

Let X,Y be real Banach spaces and let $L : domL \subset X \to Y$ be a Fredholm operator with index zero, here dom L denotes the domain of L. This means that ImL is closed in Y and dim $KerL = dim(Y/ImL) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = KerL \oplus X_1$ and $Y = ImL \oplus Y_1$ and let $P : X \to KerL$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $KerL \cap (domL \cap X_1) = \{0\}$, thus the restrictions $L_p := L|_{domL \cap X_1}$ is invertible. Denote by K the inverse of L_p.

Let Ω be an open bounded subset of X with $domL \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact. We first give the famous Mawhin continuation theorem.

Lemma 2.1(Mawhin [15]). Suppose that X and Y are Banach spaces, and $L : domL \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$.If

(1)$Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap domL, \lambda \in (0, 1)$;

(2)$Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$;

(3)$\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$, where $J : KerL \to ImQ$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap domL$.

3.Existence results for problem (1.1), (1.2)

In order to eliminate the dilemma that L isn't linear for the case $p \neq 2$, we set

$$\begin{cases} x_1(t) = x(t) \\ \\ x_2(t) = \varphi_p(x^{(i)}(t)) \end{cases} \tag{3.1}$$

then problem (1.1), (1.2) is equivalent to system

$$\begin{cases} x_1^{(i)}(t) = \varphi_q(x_2) \\ \\ x_2^{(n-i)}(t) = f(t, x_1, \cdots, x_1^{(i-1)}, \varphi_q(x_2), \cdots, (\varphi_q(x_2))^{(n-i-1)}) + e(t) \end{cases} \tag{3.2}$$

with boundary conditions

$$x_1(1) = \sum_{j=1}^{m-2} \alpha_j x_1(\xi_j), x_1''(0) = \cdots = x_1^{(i-1)}(0) = x_2'(0) = \cdots = x_2^{(n-i-1)}(0) = 0, x_1^{(i-1)}(1) = x_1^{(i-1)}(\xi), x_2(1) = x_2(\eta)$$

where φ_q is the inverse function of $\varphi_p, \varphi_q(u) = |u|^{q-2}u$, where $1/p + 1/q = 1$.Clearly if $x(t) = (x_1(t), x_2(t))$ is a solution for system (3.2), then $x_1(t)$ must be a solution for problem (1.1),(1.2).

Define

$X = \{u(t) = (u_1(t), u_2(t))| \; u_1(t) \in C^i[0,1]u_2(t) \in C^{n-i}[0,1]m\}$ with the norm

$$\|u\| = \max\{|u_1|_\infty, |u_1'|_\infty, \cdots, |u_1^{(i-1)}|_\infty, |\varphi_q(u_2)|_\infty, \cdots, |\varphi_q(u_2)^{(n-i-1)}|_\infty\},$$

$Y = \{v(t) = (v_1(t), v_2(t))| \; v_i(t) \in L^1[0,1], i = 1, 2\}$ with the norm $\|v\| = \max\{|v_1|_1, |\varphi_q(v_2)|_1\}$, where $|u|_\infty = \max_{0 \leq t \leq 1} |u(t)|, |u|_1 = \int_0^1 |u(t)|dt$.Clearly X and Y are Banach spaces. We will use the Sobolev space $W^{(i,n-i)}(0,1)$ defined as

$W^{(i,n-i)}(0,1) = \{u = (u_1, u_2) : (0,1) \to R : u_1, u_2$ are absolutely continuous on [0,1] and $u_1^{(i)}, u_2^{(n-i)} \in L^1[0,1]\}$.

Define $L : dom\ L \subset X \to Y$ by

$$Lx := (x_1^{(i)}(t), x_2^{(n-i)}(t))$$

where $dom\ L = \{x \in W^{(i,n-i)}(0,1) : x_1(1) = \sum_{j=1}^{m-2} \alpha_j x_1(\xi_j),$

$$x_1''(0) = \cdots = x_1^{(i-1)}(0) = x_2'(0) = \cdots = x_2^{(n-i-1)}(0) = 0, x_1^{(i-1)}(1) = x_1^{(i-1)}(\xi), x_2(1) = x_2(\eta)\}$$

and $N : X \to Y$ by

$$Nx := (\varphi_q(x_2), f(t, x_1, \cdots, x_1^{(i-1)}, \varphi_q(x_2), \cdots, (\varphi_q(x_2))^{(n-i-1)}) + e(t)).$$

Then system (3.2) can be written as $Lx = Nx$, here L is a linear operator.

In this section we shall prove existence results for system (3.2) under the case $\sum_{j=1}^{m-2} \alpha_j = 1, \sum_{j=1}^{m-2} \alpha_j \xi_j \neq 1$.

Lemma 3.1 If $\sum\limits_{j=1}^{m-2}\alpha_j=1$, $\sum\limits_{j=1}^{m-2}\alpha_j\xi_j\neq 1$, then

(1) $ImL=\{(y_1,y_2)\in Y:\int_\xi^1 y_1(t)dt=0,\int_\eta^1\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i}=0\}$.

(2) $L:domL\subset X\to Y$ is a Fredholm operator with index zero,

(3) Define projector operator $P:X\to KerL$ as

$$Px=(x_1(0),x_2(0)),$$

then the generalized inverse of operator L, $K_P:ImL\to domL\cap KerP$ can be written as

$K_P(y)=$

$$(-\frac{\sum\limits_{j=1}^{m-2}\alpha_j}{1-\sum\limits_{j=1}^{m-2}\alpha_j\xi_j}\int_{\xi_j}^1\int_0^{s_i}\cdots\int_0^{s_2}y_1(s_1)ds_1\cdots ds_i+\int_0^t\cdots\int_0^{s_2}y_1(s_1)ds_1\cdots ds_i,\int_0^t\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i})$$

satisfying $\|K_P(y(t))\|\leq\triangle\|y\|$, where $\triangle=1+\frac{\sum\limits_{j=1}^{m-2}|\alpha_j|(1-\xi_j)}{|1-\sum\limits_{j=1}^{m-2}\alpha_j\xi_j|}$ is a constant.

Proof: (1):First we show

$$ImL=\{(y_1,y_2)\in Y:\int_\xi^1 y_1(t)dt=0,\int_\eta^1\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i}=0\}.$$

First suppose $y(t)=(y_1(t),y_2(t))\in ImL$, then there exists $x(t)=(x_1(t),x_2(t))\in domL$ such that $Lx=y$. That is

$$x_1(t)=\int_0^t\int_0^{s_i}\cdots\int_0^{s_2}y_1(s_1)ds_1\cdots ds_i+a_{i-1}t^{i-1}+\cdots+a_1t+a_0$$

$$x_2(t)=\int_0^t\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i}+b_{n-i-1}t^{n-i-1}+\cdots+b_1t+b_0$$

Then boundary condition

$$x_1(1)=\sum_{j=1}^{m-2}\alpha_jx_1(\xi_j),x_1''(0)=\cdots=x_1^{(i-1)}(0)=x_2'(0)=\cdots=x_2^{(n-i-1)}(0)=0,x_1^{(i-1)}(1)=x_1^{(i-1)}(\xi),x_2(1)=x_2(\eta)$$

imply that

$$\int_\xi^1 y_1(t)dt=0,\int_\eta^1\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i}=0.$$

Next we suppose $y(t)\in\{(y_1,y_2)\in Y:\int_\xi^1 y_1(t)dt=0,\int_\eta^1\int_0^{s_{n-i}}\cdots\int_0^{s_2}y_2(s_1)ds_1\cdots ds_{n-i}=0\}$.

Let $x(t)=(x_1(t),x_2(t))$, where

$$x_1(t)=-\frac{\sum\limits_{j=1}^{m-2}\alpha_j}{1-\sum\limits_{j=1}^{m-2}\alpha_j\xi_j}\int_{\xi_j}^1\int_0^{s_i}\cdots\int_0^{s_2}y_1(s_1)ds_1\cdots ds_i+\int_0^t\cdots\int_0^{s_2}y_1(s_1)ds_1\cdots ds_i,$$

$$x_2(t) = \int_0^t \int_0^{s_{n-i}} \cdots \int_0^{s_2} y_2(s_1) ds_1 \cdots ds_{n-i}$$

then $Lx = (x_1^{(i)}(t), x_2^{(n-i)}(t)) = (y_1(t), y_2(t))$. Furthermore consider

$$\int_\xi^1 y_1(t)dt = 0, \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} y_2(s_1) ds_1 \cdots ds_{n-i} = 0,$$

by a simple computation,

$$x_1(1) = \sum_{j=1}^{m-2} \alpha_j x_1(\xi_j), x_1''(0) = \cdots = x_1^{(i-1)}(0) = x_2'(0) = \cdots = x_2^{(n-i-1)}(0) = 0, x_1^{(i-1)}(1) = x_1^{(i-1)}(\xi), x_2(1) = x_2(\eta)$$

Then $x(t) \in domL$, thus $y(t) \in ImL$. Sum up all above we obtain that

$$ImL = \{(y_1, y_2) \in Y : \int_\xi^1 y_1(t)dt = 0, \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} y_2(s_1) ds_1 \cdots ds_{n-i} = 0\}.$$

(2):Following we claim that L is a Fredholm operator with index zero. It's easy to see that $KerL = (a, b), a, b \in R$. Suppose $y(t) \in Y$, define the projector operator Q as

$$Q(y) = (Q(y_1(t)), Q(y_2(t))) = (\frac{\int_\xi^1 y_1(t)dt}{1-\xi}, \frac{(n-1)!}{1-\eta^{n-i}} \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} y_2(s_1) ds_1 \cdots ds_{n-i}).$$

Let $y^* = y(t) - Q(y(t)) = (y_1 - Q(y_1), y_2 - Q(y_2))$, it's easy to see that $y^* \in ImL$. Hence $Y = ImL + KerL$, furthermore considering $ImL \cap KerL = \{0\}$, we have $Y = ImL \oplus KerL$. Thus

$$dim\ KerL = co\ dim\ ImL,$$

which means L is a Fredholm operator with index zero.

(3):Define the projector operator $P : X \to KerL$ as

$$Px = (x_1(0), x_2(0)),$$

for $y(t) \in ImL$, we have

$$(LK_P)(y(t)) = y(t),$$

and for $x(t) \in domL \cap KerP$, following facts

$$-\frac{\sum_{j=1}^{m-2} \alpha_j}{1 - \sum_{j=1}^{m-2} \alpha_j \xi_j} \int_{\xi_j}^1 \int_0^{s_i} \cdots \int_0^{s_2} x_1^{(i)}(s_1) ds_1 \cdots ds_i + \int_0^t \cdots \int_0^{s_2} x_1^{(i)}(s_1) ds_1 \cdots ds_i = x_1(t) - x_1(0) = x_1(t),$$

$$\int_0^t \int_0^{s_{n-i}} \cdots \int_0^{s_2} x_2^{(n-i)}(s_1) ds_1 ds_{n-i} = x_2(t) - x_2(0) = x_2(t).$$

show that $K_P = (L_{domL\cap KerP})^{-1}$. Furthermore from the definition of the norms in the X, Y, we have

$$\|K_P(y(t))\| \leq \triangle \|y\|.$$

The above arguments complete the proof of Lemma 3.1.

Theorem 3.1: Let $f : [0,1] \times R^n \to R$ be a continuous function. Assume there exists $m_1 \in 1, 2, \cdots, m-3$ such that $\alpha_j > 0$ for $1 \leq j \leq m_1$ and $\alpha_j < 0$ for $m_1 + 1 \leq j \leq m-2$, furthermore following conditions are satisfied:

(C_1) There exist functions $a_k(t) \in L^1[0,1], k = 1, 2, \cdots, n$ and constant $\theta \in [0,1)$ such that for all $(x_1, x_2, \cdots, x_n) \in R^n, t \in [0,1]$,one of following conditions is satisfied:

$$|f(t, x_1, x_2, \cdots, x_n) + e(t)| \leq (\sum_{k=1}^{n} a_k(t)|x_k| + b(t)|x_n|^{\theta} + r(t))^{p-1}, \tag{3.3}$$

$$|f(t, x_1, x_2, \cdots, x_n) + e(t)| \leq (\sum_{k=1}^{n} a_k(t)|x_k| + b(t)|x_{n-1}|^{\theta} + r(t))^{p-1}, \tag{3.4}$$

$$\cdots \quad \cdots$$

$$|f(t, x_1, x_2, \cdots, x_n) + e(t)| \leq (\sum_{k=1}^{n} a_k(t)|x_k| + b(t)|x_1|^{\theta} + r(t))^{p-1}, \tag{3.5}$$

(C_2) There exists a constant $M > 0$ such that for $x \in dom\ L$, if $|x_1(t)| > M$,

$$\int_{\xi}^{1} \varphi_q(\int_{\sigma}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_2} (f(s_1, x_1, \cdots, x_n) + e(s_1)) ds_1 \cdots ds_{n-i}) dt \neq 0; \tag{3.6}$$

for all $x_2, \cdots, x_n \in R^{n-1}, \sigma \in (0,1), t \in (0,1) \backslash \{\sigma\}$.

(C_3) There exist $M^* > 0$ such that for any $c_1 \in R$, if $|c_1| > M^*$, for all $c_2 \in R$, then either

$$c_2 \times \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_2} (f(s_1, c_1, 0, \cdots, 0, \varphi_q(c_2), 0, \cdots, 0) + e(s_1)) ds_1 \cdots ds_{n-i} < 0 \tag{3.7}$$

or else

$$c_2 \times \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_2} (f(s_1, c_1, 0, \cdots, 0, \varphi_q(c_2), 0, \cdots, 0) + e(s_1)) ds_1 \cdots ds_{n-i} > 0 \tag{3.8}$$

Then for each $e \in L^1[0,1]$, the resonance problem (1.1), (1.2) with $\sum_{j=1}^{m-2} \alpha_j = 1, \sum_{j=1}^{m-2} \alpha_j \xi_j \neq 1$ has at least one solution in $C^{n-1}[0,1]$ provided that

$$\sum_{k=1}^{n} |a_k|_1 < \frac{1}{1 + \triangle}.$$

Proof : We divide the proof into the following steps.

Step 1. Let

$$\Omega_1 = \{x \in dom\ L \setminus KerL : Lx = \lambda Nx\}\ for\ some\ \lambda \in [0,1].$$

Then Ω_1 is bounded.

Suppose that $x \in \Omega_1, Lx = \lambda Nx$, thus $\lambda \neq 0, Nx \in ImL = KerQ$, hence

$$\int_{\xi}^{1} \varphi_q(x_2(t)) dt = 0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_2} (f(s_1, x_1, \cdots, x_1^{(i-1)}, \varphi_q(x_2), \cdots, (\varphi_q(x_2))^{(n-i-1)}) + e(s_1)) ds_1 \cdots ds_{n-i} = 0.$$

For $x_1^{(i-1)}(1)=x_1^{(i-1)}(\xi)$, there exist $\sigma_1\in(\xi,1)$ such that $x_1^{(i)}(\sigma_1)=0$. Integrate both sides of (1.1), we have

$$x_2(t)=\int_{\sigma_1}^{t}\int_0^{s_{n-i}}\cdots\int_0^{s_2}(f(s_1,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)})+e(s_1))ds_1\cdots ds_{n-i}=0, \qquad (3.9)$$

Thus

$$\int_\xi^1\varphi_q(\int_{\sigma_1}^{t}\int_0^{s_{n-i}}\cdots\int_0^{s_2}(f(s_1,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)})+e(s_1))ds_1\cdots ds_{n-i})dt=0. \qquad (3.10)$$

Then (3.10) together with condition (C_2) imply that there exists $t_0\in[0,1]$ such that $|x_1(t_0)|<M$. In view of $x_1(t)=x_1(t_0)+\int_{t_0}^{t}x_1'(s)ds$, we obtain that

$$|x(t)|<M+|x_1'|_1. \qquad (3.11)$$

Furthermore, for $\alpha_j>0,1\le j\le m_1$ and $\alpha_j<0,m_1+1\le j\le m-2$ and $x_1(1)=\sum\limits_{j=1}^{m-2}\alpha_jx_1(\xi_j)$, we have

$$x_1(1)-\sum_{j=m_1+1}^{m-2}\alpha_jx_1(\xi_j)=\sum_{j=1}^{m_1}\alpha_jx_1(\xi_j),$$

then there exists $t_1\in[\xi_{m_1+1},1],t_2\in[0,\xi_{m_1}]$ such that

$$x_1(t_1)=\frac{x_1(1)-\sum\limits_{j=m_1+1}^{m-2}\alpha_jx_1(\xi_j)}{1-\sum\limits_{m_1+1}^{m-2}\alpha_j},x_1(t_2)=\frac{\sum\limits_{j=1}^{m_1}\alpha_jx_1(\xi_j)}{\sum\limits_{j=1}^{m_1}\alpha_j},$$

thus in view of $\sum\limits_{j=1}^{m-2}\alpha_j=1$,we obtain that $x_1(t_1)=x_1(t_2)$, and $t_1\ne t_2$. This implies that there exists $t_3\in(t_1,t_2)$ such that $x_1'(t_3)=0$. Then from $x_1'(t)=x_1'(t_3)+\int_{t_3}^{t}x_1''(s)ds$, we obtain

$$|x_1'|\le|x_1''|_1. \qquad (3.12)$$

Consider the boundary condition
$x_1''(0)=x_1'''(0)=\cdots=x_1^{(i-1)}(0)=x_2'(0)=\cdots=x_2^{(n-i-1)}(0)=0$ together with $x_2(\sigma_1)=0$, it's easy to get

$$|x_1''|_\infty\le|x_1'''|_\infty\le\cdots|x_1^{(i)}|_\infty=|\varphi_q(x_2)|_\infty\cdots\le|(\varphi_q(x_2)^{(n-i-1)})|_\infty. \qquad (3.13)$$

Consider (3.11),(3.12),(3.13) we have

$$\begin{aligned}\|Px\|&\le\max\{|x_1(0)|,|\varphi_q(x_2(0))|\}\\&\le\max\{M+|\varphi_q(x_2)|_1,|\varphi_q(f(t,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)}+e(t))|_1\} \qquad (3.14)\end{aligned}$$

Again for $x\in\Omega_1,x\in domL\setminus KerL$, then $(I-P)x\in domL\cap KerP,LPx=0$, thus from Lemma 3.1, we have

$$\|(I-P)x\|=\|K_PL(I-P)x\|\le\triangle\|L(I-P)x\|_1=\triangle\|Lx\|\le\triangle\|Nx\|$$

$$\leq \triangle \max\{|\varphi_q(x_2)|_1, |\varphi_q(f(t,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)})+e(t))|_1\}. \tag{3.15}$$

From (3.14),(3.15) we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| \leq M + (1+\triangle)|\varphi_q(f(t,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)})+e(t))|_1. \tag{3.16}$$

If assumption (3.3) holds,we obtain that

$$\|x\| \leq M + (1+\triangle)|\varphi_q(f(t,x_1,\cdots,x_1^{(i-1)},\varphi_q(x_2),\cdots,(\varphi_q(x_2))^{(n-i-1)})+e(t))|_1$$

$$\leq (1+\triangle)(|a|_1|x_1|_\infty + \cdots + |a_i|_1|x_1^{(i-1)}|_\infty + |a_{i+1}|_1|\varphi_q(x_2)|_\infty + \cdots$$

$$+|a_n|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty + |b|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty^\theta + C)$$

where $C = |r|_1 + |e|_1 + \dfrac{M}{1+\triangle}$.
From $|x_1|_\infty \leq \|x\|$,we obtain

$$|x_1|_\infty \leq \frac{1+\triangle}{1-(1+\triangle)|a_1|_1}[|a_2|_1|x_1'|_\infty+\cdots+|a_{i+1}|_1|\varphi_q(x_2)|_\infty+\cdots+|a_n|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty+|b|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty^\theta+C]$$

From $|x_1'| \leq \|x\|$,we obtain

$$|x_1'|_\infty \leq \frac{1+\triangle}{1-(1+\triangle)(|a_1|_1+|a_2|_1)}[|a_3|_1|x_1''|_\infty + \cdots$$

$$+|a_{i+1}|_1|\varphi_q(x_2)|_\infty + \cdots + |a_n|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty + |b|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty^\theta + C].$$

$$\cdots\cdots$$

$$|(\varphi_q(x_2))^{(n-i-1)}|_\infty \leq \frac{1+\triangle}{1-(1+\triangle)\sum_{k=1}^{n-1}|a_k|_1}[|a_n|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty + |b|_1|(\varphi_q(x_2))^{(n-i-1)}|_\infty^\theta + C],$$

then

$$|(\varphi_q(x_2))^{(n-i-1)}|_\infty \leq \frac{(1+\triangle)|b|_1}{1-(1+\triangle)\sum_{k=1}^{n-1}|a_k|_1}|(\varphi_q(x_2))^{(n-i-1)}|_\infty^\theta + \frac{2C}{1-2\sum_{k=1}^{n-1}|a_k|_1}.$$

Consider $\theta \in [0,1)$ together with $\sum_{k=1}^{n}|a_k|_1 < \dfrac{1}{1+\triangle}$, we claim that there exists constant $M_1 > 0$ such that

$$|(\varphi_q(x_2))^{(n-i-1)}|_\infty \leq M_1 \tag{3.17}$$

Then there exist constants $M_k > 0, k = 2,\cdots,i, M_j > 0, j = i+1,\cdots,n$ such that

$$|x_1^{(k)}|_\infty < M_k, |(\varphi_q(x_2))^{(n-j)}|_\infty < M_j,$$

thus there exists $N > 0$ such that $\|x\| < N$,therefor we show that Ω_1 is bounded.

Step 2. The set $\Omega_2 = \{x \in KerL : Nx \in ImL\}$ is bounded.

The fact $x \in \Omega_2$ implies that $x = (c_1, c_2)$ and

$$N(x) = (\varphi_q(c_2), f(t, c_1, 0, \cdots, 0, \varphi_q(c_2), \cdots, 0) + e(t))$$

From $QNx = 0$,we have

$$\int_\xi^1 \varphi_q(c_2)dt = 0, \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, 0, \varphi_q(c_2), \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i} = 0,$$

which implies $c_2 = 0$ and

$$\int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i} = 0.$$

Consider condition (C3), we obtain that $|c_1| \leq M^*$, then the set Ω_2 is bounded.

Step 3. If the first part of condition (C3) is satisfied,there exists $M^* > 0$ such that for any $c \in R$, if $c_1 > M^*$, then

$$c_2 \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, 0, \varphi_q(c_2), \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i} < 0.$$

Let $\Omega_3 = \{x \in KerL : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]$,here $J : ImQ \to KerL$ is the linear isomorphism given by $J(c_1, c_2) = (c_1, c_2)$,we obtain

$$\lambda c_1 = (1-\lambda)\varphi_q(c_2)$$
$$\lambda c_2 = (1-\lambda)\frac{(n-i)!}{1-\eta^{n-i}} \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, \varphi_q(c_2), 0, \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i}.$$

If $\lambda = 1$,it's easy to see $c_1 = c_2 = 0$.If $\lambda = 0, \varphi_q(c_2) = 0$ implies $c_2 = 0$, then

$$\int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i} = 0.$$

Considering condition $C3$,$|c_1| < M^*$.

For $\lambda \neq 0, \lambda \neq 1$, if $|c_1| \geq M^*$,we obtain that

$$\lambda c_2^2 = c_2(1-\lambda)\frac{(n-i)!}{1-\eta^{n-i}} \int_\eta^1 \int_0^{s_{n-i}} \cdots \int_0^{s_2} (f(s_1, c_1, 0, \cdots, \varphi_q(c_2), 0, \cdots, 0) + e(s_1))ds_1 \cdots ds_{n-i} < 0,$$

which contradicts toλ $c_2^2 \geq 0$.Thus $|c_1| < M^*$.From $\lambda c_1 = (1-\lambda)\varphi_q(c_2)$ and $\lambda \neq 0, \lambda \neq 1, |c_2| < (\frac{\lambda}{1-\lambda}M^*)^{p-1}$. Thus the set Ω_3 is bounded.

Step 4. If the second part of condition (C3) is satisfied, similar with above argument, the set $\Omega_4 = \{x \in KerL : \lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\}$ is bounded too.

Now we show all the conditions of Lemma 2.1 are satisfied.

Let Ω be a bounded open set of Y such that $\bigcup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. By the Ascoli-Arezela theorem, we can show that

$K_P(I-Q)N:\overline{\Omega}\to Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above arguments, we have

(i) $Lx\neq Nx$, for every $(x,\lambda)\in[(domL\backslash KerL)\bigcap\partial\Omega]\times(0,1)$;

(ii) $Nx\neq ImL$, for every $x\in KerL\bigcap\partial\Omega$;

(iii)If the first part of condition (C3) holds, we let

$$H(x,\lambda)=-\lambda x+(1-\lambda)JQNx.$$

According to the above argument, we know that $H(x,\lambda)\neq 0$, for $x\in KerL\bigcap\partial\Omega$, by the homotopy property of degree, we get

$$\begin{aligned}\deg(JQN|_{KerL},\Omega\cap KerL,0)=&\deg(H(x,0),\Omega\cap KerL,0)\\ =&\deg(H(x,1),\Omega\cap KerL,0)\\ =&\deg(-I,\Omega\cap KerL,0)\neq 0.\end{aligned}$$

If the second part of condition C3 holds, we let

$$H(x,\lambda)=\lambda x+(1-\lambda)JQNx,$$

Similar to argument above, we have $\deg(JQN|_{KerL},\Omega\cap KerL,0)$=$\deg(I,\Omega\cap KerL,0)\neq 0$.

Then by Lemma 2.1, $Lx=Nx$ has at least one solution in $domL\cap\overline{\Omega}$, so that problem (1.1), (1.2) has at least one solution in $C^{n-1}[0,1]$. The proof of Theorem 3.1 is completed.

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