Electronic Journal of Qualitative Theory of Differential Equations
2007, No. **20**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

On Rectifiable Oscillation of Euler Type Second Order Linear Differential Equations

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ABSTRACT

We study the oscillatory behavior of solutions of the second order linear differential equation of Euler type: (E) $y'' + \lambda x^{-\alpha} y = 0,\ x \in (0,1]$, where $\lambda > 0$ and $\alpha > 2$. **Theorem (a)** For $2 \leq \alpha < 4$, all solution curves of (E) have finite arc length; **(b)** For $\alpha \geq 4$, all solution curves of (E) have infinite arc length. This answers an open problem posed by M. Pasic [8]

§1. In a recent paper [8], M. Pasic introduced the concept of " rectifiable oscillation" in the study of oscillatory behavior of solutions of the second order linear differential equation of Euler type in the finite interval $(0,1]$:

$$y'' + \lambda x^{-\alpha} y = 0, \qquad x \in (0,1]. \tag{1}$$

In case of the Euler's equations, i.e. $\alpha = 2$, it is well known that equation (1) is oscillatory when $\lambda > \dfrac{1}{4}$ and nonoscillatory if $\lambda \leq \dfrac{1}{4}$. For $\alpha > 2$, it follows from the Sturm's Comparison Theorem that all solutions of (1) are oscillatory as $x \to 0$.

For a real function $y(x)$ defined on the closed interval $\overline{I} = [0,1]$, we denote its graph by $G(y) = \big\{\big(t, y(t)\big) : 0 \leq t \leq 1\big\}$ as a subset in $\mathbb{R}^2$. $G(y)$ is said to be a rectifiable curve in $\mathbb{R}^2$ if its arc length $L_G(y)$ is finite where $L_G(y)$ is defined by

$$L_G(y) = \sup \left\{ \sum_{i=1}^{m} \|\big(t_i, y(t_i)\big) - \big(t_{i-1}, y(t_{i-1})\big)\|_2 \right\},$$

where supremum is taken over all partitions:

$0 = t_0 < t_1 < \cdots < t_m = 1$ of the unit interval $[0,1]$ (See Apostol [1], p. 175). Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^2$.

An oscillatory function $y(x)$ on $I = (0,1]$ is said to be rectifiable (resp. unrectifiable) if its graph $G(y)$ is rectifiable (resp. unrectifiable). In other words, it is rectifiable if it has finite arc length on I and unrectifiable otherwise. Equation (1) is said to be rectifiable (resp. unrectifiable) oscillatory on I if all its nontrivial solutions are rectifiable (resp. unrectifiable).

In the familiar case of the Euler's equation, i.e. equation (1) when $\alpha = 2$ and $\lambda > \dfrac{1}{4}$, we can write down the explicit form of its general solutions

$$y(x) = c_1\sqrt{x}\cos(\rho\log x) + c_2\sqrt{x}\sin(\rho\log x), \tag{2}$$

where $\rho^2 = \lambda - \dfrac{1}{4}$.

Consider a typical section of the solution curve $y(x)$ of (1) between its two zeros a_{k+1} and a_k where $a_{k+1} < a_k$ and $a_k \to 0$ as $k \to \infty$. It is clear from the geometry that its arc length is bounded above by $2|y(s_k)| + a_k - a_{k+1}$ where $y'(s_k) = 0$ and $a_{k+1} < s_k < a_k$. In the special case (2) of the Euler equation, we can simply estimate $\sin(\rho\log x)$ as the case of $\cos(\rho\log x)$ is similar. For $x \in (0,1]$, $\log x$ is negative, so zeros of $\sin(\rho\log x)$ are given by $a_k = \exp(-k\pi/\rho)$ and maxima of $\sin(\rho\log x)$ occurs at $s_k = \exp(-k\pi/2\rho)$. The curve $\sin(\rho\log x)$ over interval (a_{k+1}, a_k) is bounded by $2|\sqrt{s_k}|$ and $a_k - a_{k+1}$, hence

$$L_G(y) \le 2\sum_{k=0}^{\infty}\exp(-k\pi/4\rho) + 1. < \infty,$$

so equation (1) when $\alpha = 2$, $\lambda > \frac{1}{4}$ is rectifiable oscillatory.

When $\alpha = 4$, equation (1) can also be solved in explicit form, namely, solutions are of the form

$$y(x) = c_1 x\cos(\sqrt{\lambda}/x) + c_2 x\sin(\sqrt{\lambda}/x). \tag{3}$$

The typical section of the solution curve $y(x)$ between two zeros a_{k+1} and a_k with its extrema at s_k, $a_{k+1} < s_k < a_k$, has arc length exceeding $2|y(s_k)|$. Let us again consider only the case $y(x) = x\cos(\sqrt{\lambda}/x)$ which has zeros at $a_k = 2\sqrt{\lambda}/(2k+1)\pi$ and at its extrema s_k, we have

$|y(s_k)| \geq |y(\sqrt{\lambda}/k\pi)| = \sqrt{\pi}/k\pi$. Clearly $\sum\limits_{k=0}^{\infty} |y(s_k)| > \dfrac{\sqrt{\lambda}}{\pi} \sum\limits_{k=0}^{\infty} \dfrac{1}{k} = \infty$, so the length of of the solution curve $x\cos(\sqrt{\lambda}/x)$ is infinite hence the oscillatory solution given in (3) is unrectifiable. This is interesting because the solution curves given in (3) exhibits a phenomenon whereby a one dimensional curve can have infinite length within a bounded region in $\mathbb{R}^2$, like the Koch's snowflake in fractal geometry, see Devancy [5] p. 182.

Pasic [8] introduced the following two boundary layer conditions:

(I) there exist $c > 0$ and $d \in (0, a]$ depending on $y(x)$ such that $|y'(x)| \leq cx^{-\alpha/4}$ for all $x \in (0, d)$.

(II) there exist $c > 0$ and $d \in (0, a]$ depending on $y(x)$ that $|y(x)| \leq cx^{\alpha/4}$ for all $x \in (0, d)$,

and proved

Theorem $\mathbb{P}$. *For solutions of (1) satisfying boundary layer condition (I) $\big($or (II)$\big)$, we have for $\lambda > 0$,*

(a) if $4 > \alpha > 2$; then equation (1) is rectifiable oscillatory:

(b) if $\alpha \geq 4$, then equation (1) is unrectifiable oscillatory provided that it also admits the existence of two linearly independent solutions satisfying boundary layer condition (I) $\big($or (II)$\big)$.

Pasic [8] posed as an open problem whether the additional assumptions of boundary layer conditions (I) and (II) can be removed for other values of α, $\alpha \neq 2$ and 4. The purpose of this paper is to show that both of these boundary layer conditions are superfluorous in the case of equation (1). We can therefore improve Theorem P to read

Theorem 1. *(a) Equation (1) is rectifiable oscillatory for $2 < \alpha < 4$, and (b) Equation (1) is unrectifiable oscillatory for $\alpha \geq 4$.*

§2. In this section, we shall give the proof of our main theorem. In view of Theorem $\mathbb{P}$, it suffices to prove that solutions of (1) satisfy either one of the boundary layer conditions (I) and (II). This can be accomplished by the use of Louiville transformation and an application of an asymptotic integration theorem in the elliptic case.

let $y(x) = x^{\alpha/4} z(x)$, then $z(x)$ satisfies, by equation (1), the following second order linear differential equation,

$$(4) \qquad (x^{\alpha/2} z')' + \left\{ \lambda x^{-\alpha/2} + \frac{\alpha}{4}\left(\frac{\alpha}{4} - 1\right) x^{-2+\alpha/2} \right\} z = 0, \;\; x \in (0, 1].$$

We make the Louiville transformation for $\alpha > 2$

$$(5) \qquad s = \left(\frac{\alpha}{2} - 1\right)^{-1} x^{1-\alpha/2} \quad \hat{z}(s) = z(x),$$

which transform the behavior of x near 0 to s near infinity. Using (5), equation (4) becomes

$$(6) \qquad \frac{d^2 \hat{z}}{ds^2} + \left[\lambda + \frac{\alpha}{4}\left(\frac{\alpha}{4} - 1\right)\left(\frac{\alpha}{2} - 1\right)^{-2} \frac{1}{s^2} \right] \hat{z} = 0, \quad s \geq 1.$$

Solutions of (6) have the form

$$(7) \qquad \hat{z}(s) = (c_1 + o(1)) \sin \sqrt{\lambda} s + (c_2 + o(1)) \cos \sqrt{\lambda} s$$

see Hartman [6; Chapter XI Theorem 8.1, p.370]. In particular, $\frac{d\hat{z}}{ds}(s)$ has the similar form (7) because the unperturbed equation

$$(8) \qquad \frac{d^2 \hat{u}}{ds^2} + \lambda \hat{u} = 0, \quad s \geq 1$$

has $\sin \sqrt{\lambda} s$ and $\cos \sqrt{\lambda} s$ as fundamental solutions. Since $\hat{z}(s)$ is bounded by (7) so by (5) and the transformation $y(x) = x^{\alpha/4} z(x)$, we deduce that $y(x) = 0\left(x^{\lambda/4}\right)$ satisfying boundary layer condition (II).

To see that boundary layer condition (I) is also satisfied, we note that

$$(9) \qquad y'(x) = x^{\alpha/4} z'(x) + \frac{\alpha}{4} x^{\alpha/4-1} z(x)$$

and

$$(10) \qquad z'(x) = -x^{-\alpha/2} \frac{dz}{ds}(s) = 0\left(x^{-\alpha/2}\right).$$

From (9) and (10), we obtain $y'(x) = 0\left(x^{-\alpha/4}\right)$ since for $\alpha \geq 2$, $\frac{\alpha}{4} - 1 \geq -\frac{1}{2} \geq -\frac{\alpha}{4}$ and $0\left(x^{\alpha/4-1}\right) = 0\left(x^{-\alpha/4}\right)$ as $x \to 0$. This completes the proof that all solutions of (1) when $\alpha > 2$ satisfy both the boundary layer conditions (I) and (II).

We can now apply Theorem $\mathbb{P}$ to complete the proof of our main Theorem. On the other hand, by using the transformed equation (4), we can also give a much simpler proof somewhat different from that gives by Pasic [8].

We first consider the unperturbed equation (8)

$$\left(x^{\alpha/2}u'\right)' + \lambda x^{-\alpha/2}u = 0, \quad \hat{u}(s) = u(x) \tag{11}$$

which has its general solution given by

$$u(x) = c_1 \sin\left(\sqrt{\lambda}/\sigma x^{\sigma}\right) + c_2 \cos\left(\sqrt{\lambda}/\sigma x^{\sigma}\right), \sigma = \left(\frac{\alpha}{2} - 1\right) > 0,$$

Let $\{x_n\}$ be the sequence of consecutive zeros of $u(x) = \sin\left(\sqrt{\lambda}/\sigma x^{\sigma}\right)$, i.e.

$$x_n = \left(\frac{\sqrt{\lambda}}{\sigma n\pi}\right)^{1/\sigma} \tag{12}$$

We denote $q(x) = \left\{\lambda x^{-\alpha/2} + \frac{\alpha\sigma^{-2}}{4}\left(\frac{\alpha}{4} - 1\right)x^{-2+\alpha/2}\right\}$ the coefficient function in the transformed equation (4) for $2 < \alpha < 4$, $q(x) \leq \lambda x^{-\alpha/2}$, so by Sturm Comparison Theorem between two consecutive zeros a_{k_0+1} and a_{k_0} of $y(x)$, $\big(\{a_k\}$ denotes the decreasing sequence of consecutive zeros of $y(x)\big)$ there exists at least one zero x_{n_0} such that $a_{k_0+1} < x_{n_0} < a_{k_0}$. Repeating this procedure to all pairs of consecutive zeros a_{a_0+j} and a_{k_0+j-1}, we obtain another zero $x_{n_0+n_j-1}$, $n_j \geq j$ such that $a_{k_0+j} < x_{n_0+n_j-1} < a_{k_0+j-1}$.

We note that for the segment of the solution curve $\Gamma_k = \big\{\big(x, y(x)\big) : a_{k+1} \leq x \leq a_k\big\}$ its arc length $L(\Gamma_k)$ satisfies

$$2\left|y\left(s_k\right)\right| \leq L\left(\Gamma_k\right) \leq 2\left|y\left(s_k\right)\right| + \left(a_k - a_{k+1}\right), \tag{13}$$

where $y(x)$ attains its extrema s_k between a_{k+1} and a_k, i.e. $y'(s_k) = 0$. Piecing together the segments Γ_k, we note that the arc length of the Graph of solution curve satisfies

$$L_G(y) = \sum_{k=0}^{\infty} L(\Gamma_k) + \text{ arc length } \{(x, y(x) : a_0 \leq x \leq 1\}. \tag{14}$$

the last term in (14) is a positive constant depending on the solution $y(x)$ which we denote by M_0. From (13) and (14), we obtain

$$2\sum_{k=0}^{\infty} |y(s_k)| \leq L_G(y) \leq 2\sum_{k=0}^{\infty} |y(s_k)| + M_0 + a_0, \tag{15}$$

since $\sum_{k=0}^{\infty}(a_k - a_{k+1}) = a_0$. Estimate (15) allows us to conclude that the solution $y(x)$ is rectifiable or unrectifiable depending on whether the series $\sum_{k=0}^{\infty}|y(s_k)|$ is convergent or divergent.

Writing $k = k_0 + j$, we obtain the estimate

$$a_k = a_{k_0+j} < x_{n_0+n_j-1} \leq x_{n_0+j-1} = k_\alpha \left(\frac{1}{n_0 + k - k_0 - 1}\right) \tag{16}$$

where $k_\alpha = \left(\frac{\sqrt{\lambda}}{\sigma\pi}\right)^{1/\sigma}$, $\sigma = \left(\frac{\alpha}{2} - 1\right)$. Since $|y(x)| \leq cx^{\alpha/4}$ for $x \in (0, b]$ and $a_{k+1} < s_k < a_k$, we find

$$\sum_{k=0}^{\infty}|y(s_k)| \leq \sum_{k=0}^{\infty} s_k^{\alpha/4} \leq c\sum_{k=0}^{\infty} a_k^{\alpha/4}. \tag{17}$$

Now consider $\alpha/4\sigma$ as a function of α which for $\alpha > 2$ is decreasing in α and equals to 1 at $\alpha = 4$. Combining (16)and (17), we obtain for $k_1 \geq n_0 - k_0 - 1$

$$\sum_{k=k_1}^{\infty}|y(s_k)| \leq ck_\alpha^{\alpha/4}\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{\alpha/4\sigma} < \infty,$$

since $\alpha/4\sigma > 1$ for $2 < \alpha < 4$. This proves (a).

Returning to the case when $\alpha \geq 4$, we have $q(x) \geq \lambda x^{-\alpha/2}$ so we can apply Sturm's Comparison theorem to equations (4) and (11) and conclude that between two zeros (12) of solutions of (11), $x_{k_0+1} < x_{k_0}$, there exists at least one zero a_{i_0}, i.e. $x_{k_0+1} < a_{i_0} < x_{k_0}$. Repeating this process to all pairs of consecutive zeros, $x_{k_0+k} < x_{k_0+k-1}$, we obtain $a_{i_0+i_k}$ satisfying $x_{k_0+k} < a_{i_0+i_k} < x_{k_0+k-1}$ where $i_k \geq k$. Hence by (12)

$$K_\alpha\left(\frac{1}{k_0 + k}\right)^{1/\sigma} = x_{k_0+k} < a_{i_0+i_k} < a_{i_0+k},\ k_\alpha = \left(\frac{\sqrt{\lambda}}{\sigma\pi}\right)^{1/\sigma}. \tag{18}$$

Let $w(x)$ be a linearly independent solution of $y(x)$ chosen such that the Wronskian $W(y,w)(x) = y(x)w'(x) - w(x)y'(x)$ is a constant, say $W(y,w) \equiv 1$. Evaluating $W(y,w)(x)$ at $x = s_k$ where $y'(s_k) = 0$, we obtain by boundary layer condition (II) when applied to $w(x)$ the following estimate :

$$|y(s_k)| \geq \frac{1}{|w'(s_k)|} \geq \frac{1}{c}s_k^{\alpha/4} \geq \frac{1}{c}a_{k+1}^{\alpha/4} \tag{19}$$

Combining (18) and (19), we have

$$\sum_{k=i_0+1}^{\infty}|y(s_k)| \geq K_\alpha^{\alpha/4}c^{-1}\sum_{k=i_0+1}^{\infty}\left(\frac{1}{k_0+k}\right)^{\alpha/4\sigma}=\infty, \tag{20}$$

since $\alpha/4\sigma \leq 1$ for $\alpha \geq 4$. The divergence of the series in (20) shows that solution $g(x)$ is unrectifiable. This completes the proof of (b).

§3. In this section, we show that our Theorem can be further extended to give a somewhat more general result for the harmonic oscillator equation:

$$y'' + f(x)y = 0, \qquad x \in (0,1] \tag{21}$$

where $f(x) > 0$ and $f(x) \sim \lambda x^{-\alpha}, \lambda > 0, \alpha > 2$, as $x \to 0$, i.e. $\lim_{x\to\infty} x^\alpha f(x) = \lambda$. If $f(x) \in C^2(0,1]$ and satisfies

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 f^{-\frac{1}{4}}(x)\left|\left(f^{-\frac{1}{4}}(x)\right)''\right|dx < \infty, \tag{22}$$

then conclusion of the Theorem remains valid.

Given the asymptotic behavior $f(x) \sim \lambda x^{-\alpha}, \lambda > 0, \alpha > 2$ and condition (22), we can employ the WKB type asymptotic integration formula for solutions of (21) due to Wintner [9] see Coppel [4; p.122], which states that $\lim_{\varepsilon\to 0}\int_\varepsilon^1 \sqrt{f(x)}dx = \infty$ plus condition (22) imply that all solutions of (21) satisfy the asymptotic formula:

$$y(x) = f^{-\frac{1}{4}}(x)\left\{c_1 \sin\int_x^1 \sqrt{f(\xi)}d\xi + c_2\cos\int_x^1\sqrt{f(\xi)}d\xi + o(1)\right\} \tag{23}$$

and

$$y'(x) = f^{\frac{1}{4}}(x)\left\{c_1 \sin\int_x^1 \sqrt{f(\xi)}d\xi + c_2\cos\int_x^1\sqrt{f(\xi)}d\xi + o(1)\right\}, \tag{24}$$

which together imply $y(x) = 0\left(f^{-\frac{1}{4}}(x)\right)$ and $y'(x) = 0\left(f^{\frac{1}{4}}(x)\right)$ as $x \to 0$. The asymptotic behaviour $f(x) \sim \lambda x^{-\alpha}$ shows that for $0 < \varepsilon < \lambda$, there exists $\delta > 0$ and $0 < b < 1$ such that

$$0 < \lambda - \varepsilon < f(x)x^\alpha < \lambda + \varepsilon \quad \text{for all} \quad x \in (0,b]. \tag{25}$$

Since $\alpha > 2$, (25) yields $\lim_{\varepsilon\to 0} \int_\varepsilon^1 \sqrt{f(x)}dx = \infty$ so that subject to (22) formula (23) and (24) are applicable. We note that zeros a_k of $y(x)$ occur at:

$$\int_{a_k}^1 \sqrt{f(\xi)}d\xi = k\pi$$

which by (25) gives the estimate

$$\text{(26)} \qquad \sqrt{\lambda - \varepsilon} \int_{a_k}^1 \frac{d\xi}{\xi^{\alpha/2+1=\sigma}} \le k\pi \quad \text{or} \quad \frac{\sqrt{\lambda-\varepsilon}}{\sigma}\{a_k^\sigma - 1\} \le k\pi$$

where $\sigma = \dfrac{\alpha}{2} - 1 > 0$. Denote $\lambda_1^{-1} = \sqrt{\lambda-\varepsilon}/\sigma$. It follows from (26) that

$$\text{(27)} \qquad a_k \ge \left(\frac{1}{\lambda_1 k\pi + 1}\right)^{\frac{1}{\sigma}}; \quad k = 1, 2, \cdots\cdots$$

Let $w(x)$ be another solution of (21) which is linearly independent from $y(x)$ and can be chosen so that the Wronskian $y(x)w'(x) - y'(x)w(x) \equiv 1$. As before, denote $\{s_k\}$ be sequence of consecutive extrema of $y(x)$, i.e. $y'(s_k) = 0$, $a_{k+1} < s_k < a_k$. Since $w'(x)$ also satisfies (24), so by (25)

$$\text{(28)} \qquad |y(s_k)| \ge \frac{1}{|w'(s_k)|} \ge \frac{1}{f^{1/4}(s_k)} > \frac{s_k^{\alpha/4}}{(\lambda+\varepsilon)^{1/4}} = \lambda_2 s_k^{\alpha/4},$$

where $\lambda_2 = (\lambda+\varepsilon)^{-1/4}$. Since $s_k > a_{k+1}$, we obtain form (27) and (28)

$$\text{(29)} \qquad \sum_{k=1}^\infty |y(s_k)| \ge \lambda_2 \sum_{k=1}^\infty a_{k+1}^{\alpha/4} \ge \lambda_2 \sum_{k=1}^\infty \left(\frac{1}{\lambda_1 k\pi + 1}\right)^{\alpha/4\sigma}.$$

Again we note that $\alpha/4\sigma = \alpha/(2\alpha - 4)$ is decreasing in α and equals to 1 when $\alpha = 4$. So if $\alpha \ge 4$ we have $\alpha/4\sigma \le 1$. Thus the infinite series appeared as the last term in (29) diverges, so does $\sum_{k=1}^\infty |y(s_k)|$. Returning to the estimate (15) which related the arc length of the solution curve $y(x)$ to $\sum_{k=1}^\infty |y(s_k)|$, the divergence of $\sum_{k=1}^\infty |y|$ proves that $y(x)$ has infinite arc length, i.e. $y(x)$ is unrectifiable.

The case $2 < \lambda < 4$ is similar. We leave the details to the interested reader and therefore conclude that solution curves of (21) have finite arclength. Hence we have proved

Theorem 2. *Suppose that $f(x) \in C^2(0,1]$ and $f(x) > 0$. If $f(x) \sim \lambda x^{-\alpha}, \lambda > 0, \alpha > 2$ as $x \to 0$ and $f(x)$ satisfies (22), thus*

(a) for $\alpha < 4$, all solution of (21) are rectifiable oscillatory; and

(b) for $\alpha \geq 4$, all solutions of (21) *are unrectifiable oscillatory.*

§4. In this last section, we give another example of unrectifiable oscillation and several remarks concerning results discussed in this paper.

Example. Consider a special case of equation (21)

$$y'' + x^{-4}\exp(2/x)y = 0, \tag{30}$$

where the coefficient $f(x)$ is highly singular at $x = 0$, but is not of Euler type, i.e. $f(x)$ is not asymptotic to a negative power of x. Furthermore, $f(x) = x^{-4}\exp\,(2/x)$ satisfies

(i) $\displaystyle\int_\varepsilon^1 \sqrt{f(x)}dx = \exp(1/\varepsilon) - e \to \infty$ as $\varepsilon \to 0$

(ii) $\displaystyle\lim_{\varepsilon\to 0}\int_\varepsilon^1 f^{-\frac14}(x)\left|\left(f^{-\frac14}(x)\right)''\right|dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{4x^3}\exp\left(-\frac{1}{x}\right)dx < \infty,$

so condition (22) of Wintner's asymptotic formula is satisfied. Hence all solutions of (30) satisfy the boundary layer conditions (I) and (II) introduced in Pasic [8].

Once again let $w(x)$ be the solution of (30) linearly independent of $y(x)$ such that the Wronkian $W(y,w)(x) = (yw' - wy')(x) \equiv 1$. Denote $\{x_n\}$ the sequence of consecutive zeros of $y(x), x_n \to 0$ as $n \to \infty$ and $\{s_n\}$ the corresponding consecutive extrema $\{s_n\}$, i.e. $y'(s_n) = 0$, such that $x_{n+1} < s_n < x_n$ (the uniqueness of s_n between the two zeros x_{n+1} and x_n is guaranteed by the concavity of $y(x)$, i.e. $y''(x) \leq 0$).

Using the asymptotic formula (23), (24) of any solution $y(x)$ and $w(x)$ of (30), we can determine its extrema s_n by the formula

$$e^{\frac{1}{s_n}} - e = \left(n + \frac12\right)\pi \quad \text{or} \quad s_n = \left[\log\left(e + \left(n+\frac12\right)\pi\right)\right]^{-1} \tag{31}$$

We can now estimate $|y(s_n)|$ by using (31) to obtain

$$\begin{aligned} |y(s_n)| = \frac{1}{|w'(s_n)|} &\geq s_n\exp\left(-\frac{1}{2s_n}\right) = \left[\log\left(e + n + \frac12\right)\pi\right]^{-1}\left[e + \left(n+\frac12\right)\pi\right]^{-\frac12} \\ &\geq \left[e + \left(n + \frac12\right)\pi\right]^{-1} \geq [(n+2)\pi]^{-1}. \end{aligned} \tag{32}$$

since $\log x \le \sqrt{x}$ for large value of x and $e + \frac{1}{2}\pi \le 2\pi$.

Summing up the terms $|y(s_n)|$ in (32), we find

$$\sum_{n=1}^{\infty} |y(s_n)| \ge \frac{1}{\pi} \sum_{n=3}^{\infty} \frac{1}{n} = \infty,$$

which proves that $y(x)$ is unrectifiable oscillatory.

Remark 1. Condition (22) required for Wintner's asymptotic formula has another form given by Atkinson [3];

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f^{-\frac{3}{2}}(x)|f''(x)|dx < \infty \tag{30}$$

which may be easier to apply. (See Coppel [4; p.122]).

Suppose that $f(x) \sim x^{-\alpha}, \alpha > 2$, as $x \to 0$ and $f(x)$ is sufficiently smooth such that its second derivative satisfies $f''(x) \sim x^{-\alpha-2}$. Then it is easy to see that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f^{-3/2}|f''|x = \lim_{\varepsilon \to 0} \left(\frac{1 - \varepsilon^{\alpha/2-1}}{\frac{\alpha}{2} - 1} \right) = \frac{2}{\alpha - 2} < \infty$$

which is of course finite because $\alpha > 2$.

Remark 2. The proof can be modified if we assume instead of $f(x) \sim \lambda x^{-\alpha}$, $\lambda > 0$, $\alpha > 2$, as $x \to 0$ the inequality.

$$m_1 x^{-\alpha} \le f(x) \le m_2 x^{-\alpha}, \ x \in (0, b]$$

where m_1, m_2, and b are positive constants and $0 < b < 1$.

Remark 3. It may be of interest to digress to a discussion on the oscillation criteria of equations (21) over the finite interval $(0, 1]$. If we are concerned with the semi-infinite interval $[1, \infty)$, then the standard Fite-Wintner oscillation criterion states that $\int_1^{\infty} x^{\mu} a(x)dx = \infty$ for any $\mu < 1$ implies oscillation of (21). However the condition that $\int_0^1 \sqrt{a(x)}dx = \infty$ imposed in Theorem 2 does not always ensure oscillation since the Euler equation $y'' + \frac{1}{4}x^{-2}y = 0$ has $y(x) = \sqrt{x}$ as a nonoscillatory solution. The corresponding Fite-Wintner oscillation criterion for the finite interval is $\int_0^1 x^{2-\mu} a(x)dx = \infty$ for any $\mu < 1$, which is satisfied for $a(x) = \lambda x^{-\alpha}$, $\lambda > 0$, $\alpha > 2$.

Remark 4. Oscillation of the harmonic oscillator (21) was traditionally discussed for the semi-infinite interval where the independent variable x relates to time and the dependent variable y relates to the travelling waves. However for studies in some areas such as nuclear physics, the independent variable x relates to radial distance from the centre of the nuclei and the dependent variable y relates to nuclear charge of the atom(s). How does one interpret the physical meaning of unrectifiable oscillation in physical problems is certainly a most interesting question.

Remark 5. The characteristic exhibited by solution $y(x)$ in (3) is interesting because it arises from a simple linear differential equations whilst its solution curve over a finite interval has infinite arclength. Such curves are also known as fractals which normally associated with chaos in nonlinear dynamical systems. See e.g., Le Mehaute [7]. Addison [2] for further discussions on fractal geometry and chaos.

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(Received April 16, 2007)