

Null Controllability of Some Impulsive Evolution Equation in a Hilbert Space

R. BOUKHAMLA⁽¹⁾ & S. MAZOUZI⁽²⁾

(1) Centre Universitaire de Souk-Ahras, Souk-Ahras 41000, Algeria.

(2) Département de Mathématiques, Université d'Annaba, BP 12, ANNABA 23000, Algeria.

Abstract

We shall establish a necessary and sufficient condition under which we have the null controllability of some first order impulsive evolution equation in a Hilbert space.

MSC(2000) : 34A37, 93B05, 93C15.

Keywords: *Null-controllability, impulsive conditions, mild solutions, evolution equation.*

1 Introduction

The problem of exact controllability of linear systems represented by infinite conservative systems has been extensively studied by several authors A. Haraux [8], R.Triggiani [16], Z.H. Guan, T.H. Qian, and X.Yu [7], see also the references [1, 2, 6, 10,15]. In the sequel, we shall be concerned with the problem of null controllability of some first order evolution equation subject to impulsive conditions and so we shall derive a necessary and sufficient condition under which null controllability occurs. Actually, we shall establish an equivalence between the null-controllability and some "observability" inequality in somehow more general framework than that proposed by A Haraux [8]. Regarding the literature on the impulsive differential equations we refer the reader to the works of D.D. Bainov and P.S. Simeonov [3, 4] and

the references [5, 9,11, 12, 13]. We are going to study the following problem

$$y'(t) + Ay(t) = Bu(t), \quad t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \quad (1)$$

$$y(0) = y^0,$$

$$\Delta y(t_k) = I_k y(t_k) + D_k v_k, \quad k \in \sigma_1^m, \quad (1_k)$$

where the final time T is a positive number, y^0 is an initial condition in a Hilbert space H endowed with an inner product $\langle \cdot, \cdot \rangle_H$, $y(t) : [0, T] \rightarrow H$ is a vector function, σ_1^m is a subset of \mathbb{N} given by $\sigma_1^m = \{1, 2, \dots, m\}$, and finally, $\{t_k\}_{k \in \sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0, T)$, and $\Delta y(t_k)$ denotes the jump of $y(t)$ at $t = t_k$, *i.e.*,

$$\Delta y(t_k) = y(t_k^+) - y(t_k^-)$$

where $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits of $y(t)$ at $t = t_k$ respectively. On the other hand, the operators $A, B, I_k, D_k : H \rightarrow H$ are given linear bounded operators. Moreover, we set the following assumptions:

(H1) $A^* = -A$,

(H2) $I_k^* = -I_k$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$, the operator $\mathcal{I}_k = I_k + I$ is invertible,

(H3) $B^* = B \geq 0$ and there is $d_0 > 0$ such that

$$(Bu, u)_H \leq d_0 \|u\|_H^2, \quad \text{for all } u \in H,$$

(H4) $D_k^* = D_k \geq 0$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$ there is $d_k > 0$ such that

$$(D_k u, u)_H \leq d_k \|u\|_H^2, \quad \text{for all } u \in H.$$

In the sequel we shall designate by h the function

$$h(t) = \left(u(t), \{v_k\}_{k \in \sigma_1^m} \right),$$

where $u(t) \in L^2\left((0, T) \setminus \{t_k\}_{k \in \sigma_1^m}; H\right)$ and

$$\{v_k\}_{k \in \sigma_1^m} \in l^2(\sigma_1^m; H) \doteq \left\{ \{v_k\}_{k \in \sigma_1^m}, v_k \in H \right\}.$$

We point out that the space $\mathcal{K}_m = L^2\left((0, T) \setminus \{t_k\}_{k \in \sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)$ is a Hilbert space with respect to the inner product

$$(h, \tilde{h})_{\mathcal{K}_m} = \int_0^T (u(t), \tilde{u}(t))_H dt + \sum_{k=1}^m (v_k, \tilde{v}_k)_H,$$

defined for all $h = (u(t), \{v_k\}_{k=1}^m)$ and $\tilde{h} = (\tilde{u}(t), \{\tilde{v}_k\}_{k=1}^m) \in \mathcal{K}_m$. We shall denote by \mathcal{B} the control operator given by

$$\mathcal{B} = \left(B, \{D_k\}_{k \in \sigma_1^m}\right) \in \mathcal{L}\left(L^2\left((0, T) \setminus \{t_k\}_{k \in \sigma_1^m}; H\right) \times l^2(\sigma_1^m; H)\right),$$

so that

$$\mathcal{B}h(t) = \left(Bu(t), \{D_k v_k\}_{k \in \sigma_1^m}\right).$$

We have for every $h = (u(t), \{v_k\}_{k=1}^m) \in \mathcal{K}_m$

$$\begin{aligned} (\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\ &= \int_0^T (u(t), Bu(t))_H dt + \sum_{k=1}^m (v_k, D_k v_k)_H \\ &= (h, \mathcal{B}h)_{\mathcal{K}_m}, \end{aligned}$$

which shows that $\mathcal{B}^* = \mathcal{B}$, that is, \mathcal{B} is self-adjoint. On the other hand, we have

$$\begin{aligned} (\mathcal{B}h, h)_{\mathcal{K}_m} &= \int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^m (D_k v_k, v_k)_H \\ &\leq d_0 \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^m d_k \|v_k\|_H^2 \\ &\leq \delta \|h\|_{\mathcal{K}_m}^2, \end{aligned}$$

where $\delta = \max\{d_0, d_1, \dots, d_m\}$. Thus, the operator is \mathcal{B} bounded in \mathcal{K}_m .

Next, we consider the *homogeneous system* associated with (1) :

$$\varphi'(t) + A\varphi(t) = 0, \quad t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \quad (2)$$

$$\varphi(0) = \varphi^0,$$

$$\Delta\varphi(t_k) = I_k\varphi(t_k), \quad k \in \sigma_1^m. \quad (2_k)$$

We point out that on each interval $[t_k, t_{k+1})$, for $k = 0, \dots, m$, the solution φ is left continuous at each time t_k .

Consider the corresponding homogeneous backward problem :

$$-\tilde{\varphi}'(t) + \mathbf{A}\tilde{\varphi}(t) = 0, \quad t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \quad (3)$$

$$\tilde{\varphi}(T) = \varphi^0,$$

$$\Delta\tilde{\varphi}(t_{m-(k-1)}) = -\tilde{I}_{m-(k-1)}\tilde{\varphi}(t_{m-(k-1)}^+), \quad k \in \sigma_1^m, \quad (3_k)$$

where

$$\mathbf{A} = A^* = -A, \quad \tilde{I}_{m-(k-1)} = I_{m-(k-1)}^* = -I_{m-(k-1)}, \quad k \in \sigma_1^m.$$

We observe that the problem (3) on the interval $[t_m, T]$ is equivalent to the classical backward problem

$$\begin{aligned} -\tilde{\varphi}'(t) + \mathbf{A}\tilde{\varphi}(t) &= 0, \quad t \in [t_m, T], \\ \tilde{\varphi}(T) &= \varphi^0. \end{aligned}$$

We introduce the following space : $\mathcal{PC}([0, T]; H) = \{y, y : [0, T] \rightarrow H \text{ such that } y(t) \text{ is continuous at } t \neq t_k, \text{ and has discontinuities of first kind at } t = t_k, \text{ for every } k \in \sigma_1^m\}$.

Evidently, $\mathcal{PC}([0, T]; H)$ is a Banach space with respect to the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t \in (0, T)} \|y(t)\|.$$

On the other hand, we define the subspaces \mathcal{PLC} , (respectively, \mathcal{PRC}) = $\{y, y \in \mathcal{PC} \text{ such that } y(t) \text{ is left (respectively, right) continuous at } t = t_k, \text{ for every } k \in \sigma_1^m\}$.

Remark 1 1) The space \mathcal{PLC} , (respectively, \mathcal{PRC}) can be identified to a subspace of \mathcal{K}_m . That is, to each $y \in \mathcal{PLC}$, (respectively, $\tilde{y} \in \mathcal{PRC}$) is assigned the function \mathbf{h} (respectively, $\tilde{\mathbf{h}}$) defined by

$$\mathbf{h}(t) = \left(y(t), \{y(t_k)\}_{k \in \sigma_1^m} \right),$$

and

$$\tilde{\mathbf{h}}(t) = \left(\tilde{y}(t), \{\tilde{y}(t_k)\}_{k \in \sigma_1^m} \right).$$

The mapping $y \mapsto \mathbf{h}(t)$ (respectively, $\tilde{y} \mapsto \tilde{\mathbf{h}}$) is a linear injection.

2) Let $\tilde{y} \in \mathcal{PRC}$, the function y can be written as :

$$\tilde{y}(t) = \begin{cases} \tilde{y}_{[0]}(t) & \text{if } t \in [t_0, t_1) \\ \tilde{y}_{[k]}(t) & \text{if } t \in [t_k, t_{k+1}) \\ \tilde{y}_{[m]}(t) & \text{if } t \in [t_m, T]. \end{cases}$$

Next, let $\tau_k = t_k - t_{k-1}$, we define the operator $\mathcal{T} : D(\mathcal{T}) = \mathcal{PRC} \subset \mathcal{K}_m \rightarrow \mathcal{K}_m$ by

$$(\mathcal{T}\tilde{y})(t) = \begin{cases} \tilde{y}_{[0]}((T-t)\frac{\tau_1}{\tau_{m+1}} + t_0) & \text{if } t \in [t_m, T], \\ \tilde{y}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k) & \text{if } t \in [t_{m-k}, t_{m-(k-1)}), \quad k \in \sigma_1^{m-1}, \\ \tilde{y}_{[m]}((t_1 - t)\frac{\tau_{m+1}}{\tau_1} + t_m) & \text{if } t \in (0, t_1]. \end{cases} \quad (4)$$

We note that the range of \mathcal{T} is exactly $\mathcal{P}\mathcal{L}\mathcal{C}$. The function $(\mathcal{T}\tilde{y})(t)$ can be written as follows:

$$(\mathcal{T}\tilde{y})(t) = \begin{cases} y_{[0]}(t) & \text{if } t \in [t_0, t_1], \\ y_{[k]}(t) & \text{if } t \in (t_k, t_{k+1}], \quad k \in \sigma_1^{m-1}, \\ y_{[m]}(t) & \text{if } t \in (t_m, T]. \end{cases}$$

Let $X(t)$ be the resolvent solution of the operator system

$$\begin{aligned} X'(t) + AX(t) &= 0, \quad 0 = t_0 < t < t_{m+1} = T, \quad t \neq t_k, \quad k = 1, 2, \dots, m, \\ X(0) &= I, \\ X(t_k + 0) - X(t_k - 0) &= I_k X(t_k), \quad k = 1, 2, \dots, m, \end{aligned}$$

where $I : H \rightarrow H$ is the identity operator. We shall suppose that the operator $\mathcal{I}_k = I_k + I$ has a bounded inverse.

Definition 1 A function $y \in \mathcal{PC}([0, T]; H)$ is a mild solution to the impulsive problem (1) if the impulsive conditions are satisfied and

$$\begin{aligned} y(t) &= G(t, 0^+)y^0 + \int_0^t G(t, s)Bu(s) ds \\ &+ \sum_{0 < t_k \leq t} G(t, t_k)(D_k v_k), \quad \text{for every } t \in (0, T), \end{aligned}$$

where the evolution operator $G(t, s)$ is given by

$$G(t, s) = X(t)X^{-1}(s).$$

It is not hard to check that the operator $G(t, t_k)$ satisfies the operator system

$$\begin{aligned} G'(t, t_k) + AG(t, t_k) &= 0, \quad t \in [t_k, t_{k+1}), \quad k \in \sigma_0^m, \\ G(t_k, t_k) &= I, \\ G(t_{k+1}^+, t_k) - G(t_{k+1}^-, t_k) &= I_{k+1}G(t_{k+1}^-, t_k). \end{aligned}$$

It is well known that (1) has a unique solution y such that

$$y \in \mathcal{PLC}([0, T]; H) \cap C^1\left([0, T] \setminus \{t_k\}_{k \in \sigma_1^m}; H\right).$$

Now, we define the concept of mild solution for the backward impulsive system (3) associated with system (2).

Definition 2 We say that $\tilde{\varphi} \in \mathcal{PRC}([0, T]; H)$ is a mild solution for the backward impulsive system (3) if $\mathcal{T}\tilde{\varphi}$ is a mild solution for the homogeneous impulsive system (2).

Let us introduce the notion of the null controllability of the initial state as follows:

Definition 3 We say that the initial state $y^0 \in H$ is null controllable at time T , if there is a control function $h \in \mathcal{K}_m$ for which the solution y of system (1) satisfies $y(T) = 0$.

2 Main Results

First we begin by the following lemma.

Lemma 1 Assume that $\xi(t), \zeta(t) \in L^1([0, T]; H)$ and $\{\xi_k\}_{k=1}^m, \{\zeta_k\}_{k=1}^m \in l^1(\sigma_1^m, H)$. Then, for every vector functions

$$\gamma(t) \in \mathcal{PLC}([0, T]; H) \cap C^1\left([0, T] \setminus \{t_k\}_{k \in \sigma_1^m}; H\right)$$

and

$$\eta(t) \in \mathcal{PRC}([0, T]; H) \cap C^1\left([0, T] \setminus \{t_k\}_{k \in \sigma_1^m}; H\right)$$

satisfying the problem

$$\begin{aligned} \frac{d}{dt} \langle \gamma(t), \eta(t) \rangle &= \langle \xi(t), \zeta(t) \rangle, \quad t \neq t_k, \text{ for } k \in \sigma_1^m, \\ \Delta \langle \gamma(t_k), \eta(t_k) \rangle &= \langle \Delta \gamma(t_k), \eta(t_k) \rangle + \langle \gamma(t_k), \Delta \eta(t_k) \rangle = \langle \xi_k, \zeta_k \rangle, \quad k \in \sigma_1^m, \end{aligned}$$

we have the following identity

$$\begin{aligned} \langle \gamma(t), \eta(t) \rangle|_0^T &= \langle \gamma(T), \eta(T) \rangle - \langle \gamma(0), \eta(0) \rangle \\ &= \int_0^T \langle \xi(t), \zeta(t) \rangle dt + \sum_{k=1}^m \langle \xi_k, \zeta_k \rangle. \end{aligned} \quad (5)$$

Proof. It is straightforward. \square

We also need the following Lemmas.

Lemma 2 [14] *If $\mathcal{B} \in \mathcal{L}(\mathcal{K}_m)$ is self-adjoint and nonnegative, then*

$$\|\mathcal{B}h\| \leq \|\mathcal{B}\|^{1/2} (\mathcal{B}h, h)_{\mathcal{K}_m}^{1/2}, \quad h \in \mathcal{K}_m.$$

Lemma 3 *If $\tau_{k+1} = \tau_{m-(k-1)}$, $k \in \sigma_0^{m-1}$, then for the mild solution $\tilde{\varphi}$ of (3), the identity holds :*

$$\int_0^T |B\tilde{\varphi}|_H^2 dt + \sum_{k=1}^m |D_k \tilde{\varphi}(t_k^+)|_H^2 = \int_0^T |B\varphi|_H^2 dt + \sum_{k=1}^m |D_k \varphi(t_{m-(k-1)})|_H^2. \quad (6)$$

Proof. For each $k \in \sigma_0^m$, using the change of variable $t \rightarrow (t_{m-(k-1)} - t) \frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k$ we have

$$\begin{aligned} & \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)) dt \\ &= \int_{t_{m-k}}^{t_{m-(k-1)}} (B\tilde{\varphi}_{[k]}((t_{m-(k-1)} - t) \frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k), B\tilde{\varphi}_{[k]}((t_{m-(k-1)} - t) \frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k)) dt \\ &= \frac{-\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_{k+1}}^{t_k} (B\tilde{\varphi}_{[k]}(s), B\tilde{\varphi}_{[k]}(s)) ds \\ &= \int_{t_k}^{t_{k+1}} (B\tilde{\varphi}_{[k]}(s), B\tilde{\varphi}_{[k]}(s)) ds. \end{aligned}$$

Summing up with respect to k , we get

$$\sum_{k=0}^m \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)) dt = \sum_{k=0}^m \int_{t_k}^{t_{k+1}} (B\tilde{\varphi}_{[k]}(t), B\tilde{\varphi}_{[k]}(t)) dt.$$

Thus, we obtain

$$\int_0^T |B\tilde{\varphi}|_H^2 dt = \int_0^T |B\varphi|_H^2 dt.$$

On the other hand, by virtue of the definition of the function $\tilde{\varphi}$ we get

$$\varphi(t_{m-k}) = \tilde{\varphi}(t_{k+1}), \quad k \in \sigma_0^{m-1}.$$

Also, we have

$$\varphi(t_{m-(k-1)}) = \tilde{\varphi}(t_k), \quad k \in \sigma_1^m,$$

and

$$\tilde{\varphi}(t_{m-k}) = \varphi(t_{k+1}), \quad k \in \sigma_0^{m-1}.$$

This implies that

$$\begin{aligned} \sum_{k=1}^m |D_k \tilde{\varphi}(t_k)|_H^2 &= \sum_{k=0}^{m-1} \langle D_{m-k} \tilde{\varphi}(t_{m-k}), D_{m-k} \tilde{\varphi}(t_{m-k}) \rangle_H \\ &= \sum_{k=0}^{m-1} \langle D_{m-k} \varphi(t_{k+1}), D_{m-k} \varphi(t_{k+1}) \rangle_H \\ &= \sum_{l=1}^m \langle D_l \varphi(t_{m-(l-1)}), D_l \varphi(t_{m-(l-1)}) \rangle_H \\ &= \sum_{k=1}^m \langle D_k \varphi(t_{m-(k-1)}), D_k \varphi(t_{m-(k-1)}) \rangle_H \\ &= \sum_{k=1}^m |D_k \varphi(t_{m-(k-1)})|_H^2, \end{aligned}$$

which gives (6). □

Corollary 1 *If $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, and B, D_k are nonnegative in H , then the following holds:*

$$\begin{aligned} &\int_0^T \langle B\tilde{\varphi}(t), \tilde{\varphi}(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k \tilde{\varphi}(t_k), \tilde{\varphi}(t_k) \rangle \\ &= \int_0^T \langle B\varphi(t), \varphi(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)}) \rangle. \end{aligned}$$

Proof. This follows immediately from Lemma 3 if we substitute B by $B^{\frac{1}{2}}$, and D_k by $D_k^{\frac{1}{2}}$. \square

Now, we state and establish the following Theorem.

Theorem 1 *Let $y^0 \in H$ be a given initial state for the system (1), then y^0 is null controllable at time T if and only if there is a positive constant C such that*

$$|\langle y^0, \tilde{\varphi}^0 \rangle_H| \leq C \left\{ \int_0^T |B\varphi|_H^2 dt + \sum_{k=1}^m |D_k \varphi(t_{m-(k-1)})|_H^2 \right\}^{1/2}, \quad \forall \tilde{\varphi}^0 \in H, \quad (7)$$

where $\varphi \in \mathcal{P}\mathcal{L}\mathcal{C}([0, T]; H)$ is the unique mild solution to (2) with $\varphi(T) = \tilde{\varphi}^0$.

Proof. It suffices to prove this Theorem for the special case $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, because the norm $\|\cdot\| \doteq \left\{ \sum_{k=0}^m \frac{\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} |\cdot|_H^2 dt \right\}^{1/2}$ is equivalent to the usual norm of $L^2([0, T]; H)$.

We shall proceed in several steps.

Step 1: Let y and $\tilde{\varphi}$ be strong solutions to (1) and (3), respectively. Then, for $t \neq t_k$, $k \in \sigma_1^m$, we have

$$\begin{aligned} \frac{d}{dt} \langle y(t), \tilde{\varphi}(t) \rangle &= \langle y(t), \tilde{\varphi}'(t) \rangle + \langle y'(t), \tilde{\varphi}(t) \rangle & (8) \\ &= \langle y(t), -A\tilde{\varphi}(t) \rangle + \langle -Ay(t) + Bu(t), \tilde{\varphi}(t) \rangle \\ &= \langle y(t), -A\tilde{\varphi}(t) \rangle + \langle -Ay(t), \tilde{\varphi}(t) \rangle + \langle Bu(t), \tilde{\varphi}(t) \rangle \\ &= \langle Bu(t), \tilde{\varphi}(t) \rangle. \end{aligned}$$

Multiplying equation (3_k) in (3) from the left by $y(t_{m-(k-1)})$ the solution of (1), and multiplying equation (1_k) in (1) from the right by $\tilde{\varphi}(t_k)$ the solution of (3), and finally adding memberwise we get

$$\begin{aligned} \Delta \langle y(t), \tilde{\varphi}(t) \rangle|_{t=t_k} &= \langle y(t_k), \Delta \tilde{\varphi}(t_k) \rangle + \langle \Delta y(t_k), \tilde{\varphi}(t_k) \rangle & (9) \\ &= \langle y(t_k), I_k \tilde{\varphi}(t_k) \rangle + \langle I_k y(t_k) + D_k v_k, \tilde{\varphi}(t_k) \rangle \\ &= \langle y(t_k), I_k \tilde{\varphi}(t_k) \rangle + \langle I_k y(t_k), \tilde{\varphi}(t_k) \rangle + \langle D_k v_k, \tilde{\varphi}(t_k) \rangle \\ &= \langle D_k v_k, \tilde{\varphi}(t_k) \rangle. \end{aligned}$$

Setting $\gamma(t) = y(t)$, $\eta(t) = \tilde{\varphi}(t)$, $\xi(t) = Bu(t)$, $\zeta(t) = \tilde{\varphi}(t)$, $\xi_k = D_k v_k$, $\zeta_k = \tilde{\varphi}(t_k)$, then equations (5), (8) and (9) give

$$\langle y(T), \tilde{\varphi}(T) \rangle - \langle y(0), \tilde{\varphi}(0) \rangle = \int_0^T \langle Bu(t), \tilde{\varphi}(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k v_k, \tilde{\varphi}(t_k) \rangle. \quad (10)$$

Since \mathcal{B} is bounded, self-adjoint and $\mathcal{B} \geq 0$, then by density the latter identity is still valid for mild solutions y of (1). Identity (10) can be written as follows

$$\langle y(T), \tilde{\varphi}(T) \rangle - \langle y(0), \tilde{\varphi}(0) \rangle = \int_0^T \langle u(t), B\tilde{\varphi}(t) \rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k \tilde{\varphi}(t_k) \rangle. \quad (11)$$

Next, if there is a certain $h(t) \in \mathcal{K}_m$ such that the mild solution of (1) with $y(0) = y^0$ satisfies $y(T) = 0$, then

$$-\langle y(0), \tilde{\varphi}(0) \rangle = \int_0^T \langle u(t), B\tilde{\varphi}(t) \rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k \tilde{\varphi}(t_k) \rangle,$$

and so by Cauchy-Schwarz Inequality we obtain

$$\begin{aligned} |\langle y(0), \tilde{\varphi}(0) \rangle_H| &\leq \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\times \left\{ \int_0^T \|B\tilde{\varphi}(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|D_k \tilde{\varphi}(t_k)\|_H^2 \right\}^{1/2}. \end{aligned} \quad (12)$$

Using Lemma 3, and equation (12) we have

$$\begin{aligned} |\langle y(0), \tilde{\varphi}(0) \rangle_H| &\leq \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\times \left\{ \int_0^T \|B\varphi(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|D_k \varphi(t_{m-(k-1)})\|_H^2 \right\}^{1/2}. \end{aligned}$$

Setting

$$C = \|h(t)\|_{\mathcal{K}_m} = \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2}$$

we find that

$$|(\langle y(0), \tilde{\varphi}(0) \rangle_H)| \leq C \left\{ \int_0^T \|B\varphi(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|D_k\varphi(t_{m-(k-1)})\|_H^2 \right\}^{1/2}.$$

This shows the necessary condition of the Theorem.

Step 2: To prove the sufficiency we need the following result when $\mathcal{B} \geq \alpha > 0$.

Claim 1 Assume that there is $\alpha > 0$ such that

$$\left\{ \int_0^T \|Bu(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|D_k v_k\|_H^2 \right\} \geq \alpha \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}$$

then, for every $y^0 \in H$ there is $\varphi^0 \in H$ such that the mild solution of (1) with

$$h(t) = (\tilde{\varphi}(t), \tilde{\varphi}(t_1), \dots, \tilde{\varphi}(t_k), \dots, \tilde{\varphi}(t_m)) \in \mathcal{K}_m \text{ and } y(0) = y^0$$

satisfies $y(T) = 0$.

To prove this Claim, we consider for every $z \in H$ the solution φ of (2) satisfying $\varphi(T) = z$ and the unique mild solution y to the problem

$$\begin{aligned} y'(t) + Ay(t) &= B\tilde{\varphi}(t), t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \Delta y(t_k) &= I_k y(t_k) + D_k \tilde{\varphi}(t_k), \\ y(T) &= 0. \end{aligned}$$

Next, we introduce a bounded linear operator $\Lambda : H \rightarrow H$ defined by

$$\Lambda z = -y(0).$$

According to formula (11) and the Corollary 1 we have

$$\begin{aligned} |\langle \Lambda z, z \rangle| &= |-\langle y(0), \tilde{\varphi}(0) \rangle| = \left| \int_0^T \langle B\tilde{\varphi}(t), \tilde{\varphi}(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k \tilde{\varphi}(t_k), \tilde{\varphi}(t_k) \rangle \right| \\ &= \left| \int_0^T \langle B\varphi(t), \varphi(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)}) \rangle \right| \\ &\leq \varsigma \left\{ \int_0^T \|\varphi(t)\|^2 dt + \sum_{k=1}^{k=m} \|\varphi(t_k)\|^2 \right\}, \end{aligned}$$

where

$$\varsigma = \sup_{k \in \sigma_0^m} \{d_k\} < \infty.$$

We have

$$\int_0^T \|\varphi(t)\|^2 dt = \int_0^{t_1} \|\varphi(t)\|^2 dt + \int_{t_1}^{t_2} \|\varphi(t)\|^2 dt + \dots + \int_{t_m}^T \|\varphi(t)\|^2 dt.$$

Since there is no impulse in the interval $[t_k, t_{k+1})$ we have

$$\begin{aligned} \|\varphi(t)\| &= \|\varphi(t_k^+)\|, \text{ for every } t \in [t_k, t_{k+1}), k \in \sigma_0^m, \\ \|\varphi(t_{k+1}^-)\| &= \|\varphi(t_k^+)\|, \quad k \in \sigma_0^m. \end{aligned} \quad (13)$$

Therefore, there are $\tau_{k+1} = t_{k+1} - t_k > 0$, $k \in \sigma_0^m$ such that

$$\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \leq \rho_k \|\varphi(t_k^+)\|^2 = \tau_{k+1} \|I_k \varphi(t_k^-) + \varphi(t_k^-)\|^2, \quad k \in \sigma_1^m. \quad (14)$$

On the other hand, the continuity of I_k implies that

$$\|\varphi(t_k^+)\|^2 = \|(I_k + I)\varphi(t_k^-)\|^2 \leq (1 + L(I_k))^2 \|\varphi(t_k^-)\|^2, \quad k \in \sigma_1^m. \quad (15)$$

It follows from (14) and (15) that

$$\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \leq \tau_{k+1} (1 + L(I_k))^2 \|\varphi(t_k^-)\|^2, \quad k \in \sigma_1^m. \quad (16)$$

Since m is finite, and due to (13),(16), then there is a constant $0 < \mu < \infty$ such that $\langle \Lambda z, z \rangle \leq \mu \|z\|^2$, and thus, Λ is bounded.

Now, as \mathcal{B} is nonnegative in \mathcal{K}_m , we have

$$\|\mathcal{B}\xi(t)\| \geq \alpha \{(\xi(t), \xi(t))_{\mathcal{K}_m}\}^{1/2}$$

for all $\xi \in \mathcal{K}_m$; thus, by virtue of Lemma 2, we have

$$\begin{aligned} & \left\{ \int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^{k=m} (D_k v_k, v_k)_H \right\} \\ & \geq \alpha \|\mathcal{B}\| \left\{ \int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}. \end{aligned} \quad (17)$$

It follows from (11), (17) and Corollary 1 that

$$\begin{aligned}
 \langle \Lambda z, z \rangle &= -\langle y(0), \tilde{\varphi}(0) \rangle \\
 &= \int_0^T \langle B\varphi(t), \varphi(t) \rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)}) \rangle \\
 &\geq \alpha \|\mathcal{B}\| \left\{ \int_0^T \|\varphi(t)\|^2 dt + \sum_{k=1}^{k=m} \|\varphi(t_k)\|^2 \right\} \\
 &\geq \alpha \|\mathcal{B}\| \int_0^{t_1} \|\varphi(t)\|^2 dt = \|\mathcal{B}\| \alpha t_1 \|z\|^2 = \theta \|z\|^2,
 \end{aligned}$$

because there is no impulse before time t_1 . Therefore, Λ is coercive on H . To show that there is a bijection from H onto H , it suffices to prove that $\Lambda + I$ is a bijection from H onto H . Clearly, $\Lambda + I$ is injective since

$$\langle \Lambda z + z, z \rangle = \langle \Lambda z, z \rangle + \langle z, z \rangle \geq (\theta + 1) \|z\|^2.$$

On the other hand, let $y^0 \in H$, as the form $a(f, g) + \langle f, g \rangle = \langle \Lambda f, g \rangle + \langle f, g \rangle$ is symmetric and coercive, then, by virtue of Lax-Milgram Theorem, there is an element $f \in H$ such that

$$a(f, g) + \langle f, g \rangle = \langle y^0, g \rangle, \text{ for all } g \in H.$$

This implies that $\Lambda(H) = H$. Thus, for every $y^0 \in H$, there is a unique $z \in H$ such that $\Lambda(z) = -y^0$, which completes the proof of Claim 1.

Step 3: Assume that $B, D_k \geq 0$, then $\mathcal{B} \geq 0$,

$$\tilde{B}^2 = B, \tilde{D}_k^2 = D_k.$$

We define for $\varepsilon > 0$,

$$\beta^\varepsilon \doteq \tilde{B}^2 + \varepsilon I,$$

$$\delta_k^\varepsilon \doteq \tilde{D}_k^2 + \varepsilon I,$$

and

$$\mathcal{B}^\varepsilon \doteq (\beta^\varepsilon; \delta_1^\varepsilon, \dots, \delta_m^\varepsilon) = (\tilde{B}^2 + \varepsilon I; \tilde{D}_1^2 + \varepsilon I, \dots, \tilde{D}_m^2 + \varepsilon I).$$

According to Claim 1, there is $\tilde{\varphi}^{0,\varepsilon} \in H$ such that the mild solution y_ε of (1) with $y_\varepsilon(0) = y^0$ satisfies $y_\varepsilon(T) = 0$; where $\mathcal{B}(h)$ has been replaced by

$$\mathcal{B}^\varepsilon(\tilde{\varphi}(t), \tilde{\varphi}(t_1), \dots, \tilde{\varphi}(t_k), \dots, \tilde{\varphi}(t_m)) \in \mathcal{K}_m.$$

We obtain from (11) and Corollary 1

$$-\langle y(0), \tilde{\varphi}_\varepsilon(0) \rangle = \int_0^T \langle \beta_\varepsilon^\varepsilon \tilde{\varphi}(t), \tilde{\varphi}_\varepsilon(t) \rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \tilde{\varphi}_\varepsilon(t_k), \tilde{\varphi}_\varepsilon(t_k) \rangle, \quad (18)$$

and (7) gives

$$-\langle y(0), \tilde{\varphi}_\varepsilon(0) \rangle \leq C \left\{ \int_0^T \langle \tilde{B}^2 \varphi_\varepsilon(t), \varphi_\varepsilon(t) \rangle dt + \sum_{k=1}^{k=m} \langle \tilde{D}_k^2 \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)}) \rangle \right\}^{1/2}. \quad (19)$$

Whence,

$$-\langle y(0), \tilde{\varphi}_\varepsilon(0) \rangle \leq C \left\{ \int_0^T \langle \beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t) \rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)}) \rangle \right\}^{1/2}. \quad (20)$$

It follows at once from (18), (19) and (20) that

$$\begin{aligned} & \varepsilon \left\{ \int_0^T \|\varphi_\varepsilon(t)\|^2 dt + \sum_{k=1}^{k=m} \|\varphi_\varepsilon(t_k)\|^2 \right\} \\ & \quad + \int_0^T \langle \tilde{B} \varphi_\varepsilon(t), \tilde{B} \varphi_\varepsilon(t) \rangle dt + \sum_{k=1}^{k=m} \langle \tilde{D}_k \varphi_\varepsilon(t_{m-(k-1)}), \tilde{D}_k \varphi_\varepsilon(t_{m-(k-1)}) \rangle \\ & = \int_0^T \langle \beta^\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t) \rangle dt + \sum_{k=1}^{k=m} \langle \delta_k^\varepsilon \varphi_\varepsilon(t_{m-(k-1)}), \varphi_\varepsilon(t_{m-(k-1)}) \rangle \leq C^2. \end{aligned} \quad (21)$$

Step 4: According to the estimate (20) the family

$$\begin{aligned} b_\varepsilon &= \mathcal{B}^\varepsilon(\tilde{\varphi}_\varepsilon(t); \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{\varphi}_\varepsilon(t_m)) \\ &= (\tilde{B}_\varepsilon^2 \tilde{\varphi}(t); \tilde{D}_1^2 \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{D}_m \tilde{\varphi}_\varepsilon(t_m)) + \varepsilon(\tilde{\varphi}_\varepsilon(t); \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{\varphi}_\varepsilon(t_m)) \end{aligned}$$

is contained in a bounded subset \mathcal{K}_m .

Thus, both of the families

$$\sqrt{\varepsilon}(\tilde{\varphi}_\varepsilon(t); \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{\varphi}_\varepsilon(t_m)) \text{ and } (B\tilde{\varphi}_\varepsilon(t); D_1\tilde{\varphi}_\varepsilon(t_1), \dots, D_m\tilde{\varphi}_\varepsilon(t_m))$$

are bounded in \mathcal{K}_m . Therefore, we may extract a subfamily, say

$$(B\tilde{\varphi}_\varepsilon(t); D_1\tilde{\varphi}_\varepsilon(t_1), \dots, D_m\tilde{\varphi}_\varepsilon(t_m)) \rightharpoonup h, \text{ weakly in } \mathcal{K}_m.$$

Then clearly

$$(\tilde{B}^2 \tilde{\varphi}_\varepsilon(t); \tilde{D}_1^2 \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{D}_m^2 \tilde{\varphi}_\varepsilon(t_m)) + \varepsilon(\tilde{\varphi}_\varepsilon(t); \tilde{\varphi}_\varepsilon(t_1), \dots, \tilde{\varphi}_\varepsilon(t_m)) \rightharpoonup \mathcal{B}h, \text{ weakly in } \mathcal{K}_m.$$

Step 5: Taking the limit as $\varepsilon \rightarrow 0$, we see that the solution y of (1) with initial condition $y(0) = y^0$, h being as in **step 4** satisfies $y(T) = 0$. This completes the proof of Theorem 1. \square

As an immediate application of the foregoing Theorem we give the following example.

Example. One dimensional impulsive Schrödinger equation :

We consider the problem

$$\begin{aligned} \frac{\partial y(t, x)}{\partial t} + i \frac{\partial^2 y}{\partial x^2}(t, x) &= \chi_{\omega_0} u(t, x), \quad t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, x \in \Omega = (0, 2\pi), \\ y(t, 0) &= y(t, 2\pi) = 0, \\ y(0, x) &= y^0, \\ \Delta y(t_k, x) &= i\alpha_k y(t_k, x) + \chi_{\omega_k} v_k(x), \quad k \in \sigma_1^m, \end{aligned} \quad (22)$$

where

$$t_{k+1} - t_k > 2\pi, \quad \omega_k = (a_1^k, a_2^k) \subset \Omega, k \in \sigma_0^m, \quad \{\alpha_k\}_{k \in \sigma_1^m} \subset \mathbb{R}^+.$$

Let

$$H = L^2(\Omega, \mathbb{C}), Aw(x) = i \frac{\partial^2 w}{\partial x^2}(x), \quad D(A) = \left\{ w \in H, \frac{\partial^2 w}{\partial x^2} \in H, w(0) = w(\pi) = 0 \right\},$$

and $I_k w(x) = i\alpha_k w(x)$ and the control operator is given by $B = \chi_{\omega_0}$, $D_k = \chi_{\omega_k}$, then the system (22) becomes an abstract formulation of (1). As a consequence of Theorem 1, the initial state $y^0 \in L^2(\Omega, \mathbb{C}) = H$ of the solution of (22) is null-controllable at $t = T$, if and only if, there is $C > 0$ such that

$$\begin{aligned} & \left| \int_{\Omega} y^0(x) \tilde{\varphi}^0(x) dx \right| \\ & \leq C \left\{ \int_0^T \int_{\omega_0} |\varphi|^2(t, x) dx dt + \sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) \right\}^{\frac{1}{2}}, \quad \forall \tilde{\varphi}^0 \in L^2(\Omega, \mathbb{C}), \end{aligned} \quad (23)$$

where $\tilde{\varphi}^0(x) = \varphi(T, x)$ and φ is the mild solution of

$$\begin{aligned} \frac{\partial \varphi(t, x)}{\partial t} + i \frac{\partial^2 \varphi(t, x)}{\partial x^2} &= 0, \quad t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \quad x \in \Omega, \\ \varphi(t, 0) &= \varphi(t, 2\pi) = 0, \\ \varphi(0, x) &= \varphi^0(x), \quad x \in \Omega, \\ \Delta \varphi(t_k, x) &= i\alpha_k \varphi(t_k, x), \quad x \in \Omega, \quad k \in \sigma_1^m. \end{aligned}$$

Here φ is given by

$$\varphi(t) = \begin{cases} \varphi_{[0]}(t) & , \text{ if } t \in [t_0, t_1) \\ \varphi_{[k]}(t) & , \text{ if } t \in [t_k, t_{k+1}) \\ \varphi_{[m]}(t) & , \text{ if } t \in [t_m, T], \end{cases}$$

where $\varphi_{[k]}(t)$ is a solution of the classical Schrödinger equation

$$\begin{aligned} \frac{\partial \varphi_{[k]}(t, x)}{\partial t} + i \frac{\partial^2 \varphi_{[k]}(t, x)}{\partial x^2} &= \chi_{\omega_0} u(t, x), \quad t \in (t_0, t_1), \quad x \in \Omega = (0, 2\pi), \\ \varphi_{[k]}(t, 0) &= \varphi_{[k]}(t, 2\pi) = 0, \\ \varphi_{[0]}(t_0, x) &= \varphi^0(x), \quad x \in \Omega, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varphi_{[k]}(t, x)}{\partial t} + i \frac{\partial^2 \varphi_{[k]}(t, x)}{\partial x^2} &= \chi_{\omega_0} u(t, x), \quad t \in (t_k, t_{k+1}), \quad x \in \Omega = (0, 2\pi), \\ \varphi_{[k]}(t, 0) &= \varphi_{[k]}(t, 2\pi) = 0, \\ \varphi_{[k]}(t_k, x) &= (1 + i\alpha_k) \varphi_{[k-1]}(t_k, x), \quad x \in \Omega, \quad k \in \sigma_1^m. \end{aligned}$$

Then a standard application of a variant of Ingham's Inequality [8] shows that

$$\int_{t_k}^{t_{k+1}} \int_{w_0} |\varphi_{[k]}|(t, x) dt dx \geq c(\tau_k, w_0) \int_{\Omega} |\varphi_{[k]}|(t_k^+, x) dx,$$

for some positive constants $c(\tau_k, w_0) > 0$. Summing up we get

$$\begin{aligned} \sum_{k=0}^m \int_{t_k}^{t_{k+1}} \int_{w_0} |\varphi_{[k]}|(t, x) dt dx &= \int_0^T \int_{w_0} |\varphi|^2(t, x) dx dt \\ &\geq c_1 \sum_{k=1}^m \int_{\Omega} |\varphi_{[k]}|(t_k^+, x) dx, \end{aligned}$$

where $c_1 = \min_{k \in \sigma_0^m} c(\tau_k, w_0) > 0$.

On the other hand, there is a positive constant $c_2 > 0$ such that

$$\sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) \geq c_2 \sum_{k=1}^m \int_{\Omega} |\varphi_{[k]}|^2(t_k^+, x) dx.$$

It follows that

$$\begin{aligned} \int_0^T \int_{\omega_0} |\varphi|^2(t, x) dx dt &+ \sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) \\ &\geq (c_1 + c_2) \sum_{k=1}^m \int_{\Omega} |\varphi_{[k]}|^2(t_k^+, x) dx \\ &\geq (c_1 + c_2) \int_{\Omega} |\varphi_{[m]}|^2(t_m^+, x) dx \\ &= (c_1 + c_2) \int_{\Omega} |\varphi|^2(T, x) dx. \end{aligned}$$

Now, since $\tilde{\varphi}^0(x) = \tilde{\varphi}(0, x) = \varphi(T, x)$, then,

$$\int_0^T \int_{\omega_0} |\varphi|^2(t, x) dx dt + \sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) \geq m(c_1 + c_2) \int_{\Omega} |\tilde{\varphi}^0|^2(x) dx,$$

from which we get

$$\int_{\Omega} |\tilde{\varphi}^0|^2(x) dx \leq \frac{1}{m(c_1 + c_2)} \left(\int_0^T \int_{\omega_0} |\varphi|^2(t, x) dx dt + \sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) \right).$$

We conclude by Cauchy-Schwarz inequality that

$$\begin{aligned} \left| \int_{\Omega} y^0(x) \tilde{\varphi}^0(x) dx \right| &\leq \left\{ \int_{\Omega} |y^0|^2(x) dx \int_{\Omega} |\tilde{\varphi}^0|^2(x) dx \right\}^{1/2} \\ &\leq \left\{ \frac{\int_{\Omega} |y^0|^2(x) dx}{m(c_1 + c_2)} \right\}^{1/2} \left(\int_0^T \int_{\omega_0} |\varphi|^2(t, x) dx dt \right. \\ &\quad \left. + \sum_{k=1}^m \int_{\omega_k} |\varphi|^2(t_{m-(k-1)}, x) dx \right)^{1/2}, \end{aligned}$$

which establishes the necessary and sufficient condition of null controllability stated in Theorem 1.

We conclude our paper by a special case when our initial state is an eigensolution of the following linear operator $\Gamma : H \rightarrow H$ defined by

$$\Gamma(\psi) = \int_0^T X^{-1}(s)B^2X(s)\psi ds + \sum_{k=1}^{k=m} X^{-1}(t_k)D_k^2X(t_k)\psi.$$

We have the following result of null-controllability.

Proposition 1 *Let $\lambda > 0$ be an eigenvalue of Γ with eigenvector $\psi \in H$. Then, the solution y to the problem*

$$\begin{cases} y'(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ \Delta y(t_k) = I_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi, \end{cases} \quad (24)$$

satisfies

$$y(T) = 0.$$

Proof.

Write system (24) into the form

$$\begin{cases} y'(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0, T) \setminus \{t_k\}_{k \in \sigma_1^m}, \\ y(t_k^+) = \mathcal{I}_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi. \end{cases}$$

Therefore, this impulsive problem has a solution which can be represented explicitly as follows

$$y(t) = X(t)\psi + \int_0^t G(t, s) \left[-\frac{1}{\lambda}B^2(X(s)\psi) \right] ds + \sum_{0 < t_k \leq t} G(t, t_k) \left[-\frac{1}{\lambda}D_k^2X(t_k)\psi \right],$$

where the evolution operator $G(t, s)$ is given by

$$G(t, s) = X(t)X^{-1}(s).$$

On the other hand, the system (24) yields

$$\begin{aligned}
 y(T) &= X(T)\psi + \int_0^T G(T, s) \left\{ -\frac{1}{\lambda} B^2(X(s)\psi) \right\} ds \\
 &\quad + \sum_{0 < t_k \leq T} G(T, t_k) \left\{ -\frac{1}{\lambda} D_k^2 X(t_k)\psi \right\} \\
 &= X(T) \left[\psi + \int_0^T X^{-1}(T)G(T, s) \left\{ -\frac{1}{\lambda} B^2(X(s)\psi) \right\} ds \right. \\
 &\quad \left. - \frac{1}{\lambda} \sum_{0 < t_k \leq T} X^{-1}(T)G(T, t_k) \left\{ D_k^2 X(t_k)\psi \right\} \right] \\
 &= X(T) \left[\psi + \int_0^T X^{-1}(s) \left\{ -\frac{1}{\lambda} B^2(X(s)\psi) \right\} ds \right. \\
 &\quad \left. - \frac{1}{\lambda} \sum_{0 < t_k \leq T} X^{-1}(t_k) \left\{ D_k^2 X(t_k)\psi \right\} \right] \\
 &= X(T) \left[\psi - \frac{1}{\lambda} \Gamma(\psi) \right] = 0.
 \end{aligned}$$

This shows that the initial state ψ is null-controllable at time T with control

$$h(t) = \left(u(t), \{v_k\}_{k \in \sigma_1^m} \right) = \left(-\frac{1}{\lambda} X(t)\psi, \left\{ -\frac{1}{\lambda} X(t_k)\psi \right\}_{k \in \sigma_1^m} \right),$$

which completes the proof of the Proposition. □

References

- [1] N.U. Ahmed, *Optimal impulse control for impulsive systems in Banach spaces*. J. Math. Anal. Appl. Vol. 1 (No.1)(2000), 37-52.
- [2] M.U. Akhmetov, A. Zafer, *The controllability of boundary-value problems for quasilinear impulsive systems*, Nonlinear Analysis 34 (1998) 1055-1065.
- [3] D.D. Bainov and P.S. Simeonov, *Systems with impulse effect, theory and applications*, Ellis Hardwood series in Mathematics and its Applications, Ellis Hardwood, Chichester, 1989.

- [4] D.D. Bainov - P.S. Simeonov, *Impulsive differential equations: Asymptotic properties of the solutions*, World Scientific, Series on Advances in Math. for Applied Sciences, 28 (1995).
- [5] L. Berezansky and E. Braverman, *Boundedness and stability of impulsively perturbed systems in a Banach space*. Preprint functan/9312001.
- [6] R.K. George, A.K. Nandakumaran and A. Arapostathis, *A note on controllability of impulsive systems*. J. Math. Anal. Appl. 241, 276-283, 2000.
- [7] Z.H. Guan, T.H. Qian, and X. Yu, *Controllability and observability of linear time-varying impulsive systems*. IEEE Circuits Syst. I, vol. 49, pp. 1198-1208, 2002.
- [8] A. Haraux, *An alternative functional approach to exact controllability of conservative systems*, Portugaliae mathematica 61, 4 (2004), 399-437.
- [9] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*. World Scientific series in Modern Mathematics, Vol. 6, Singapore, 1989.
- [10] S. Leela, F.A. McRae, and S. Sivasundaram, *Controllability of impulsive differential equations*, J. Math. Anal. Appl. 177, 1993 , 24-30.
- [11] X. Liu, *Nonlinear boundary value problems for first order impulsive integrodifferential equations*, Appl. Anal. 36(1990), 119-130.
- [12] J.H. Liu, *Nonlinear impulsive evolution equations*, Dynamics Contin. Discr. Impulsive Syst., 6 (1999), 77-85.
- [13] A.M. Samoilenko and N.A. Perestyuk, *Impulsive differential equations*, World Scientific, Singapore, 1995.
- [14] R.E. Showalter, *Hilbert space methods for partial differential equations*, Electronic Journal of Differential Equations, Monograph 01, 1994.
- [15] M. Slemrod, *Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control*, Math. Control Signals Systems, Vol. 2, 265-285, 1989.

- [16] R. Triggiani, *Controllability and observability in Banach space with bounded operators*, SIAM J. Control Optimiz. 13(1) (1975), 462-490.

(Received October 17, 2006)