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Hybrid Approximations via Second Order Combined Dynamic Derivatives on Time Scales

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Abstract. This article focuses on the approximation of conventional second order derivative via the combined (diamond-α) dynamic derivative on time scales with necessary smoothness conditions embedded. We will show the constraints under which the second order dynamic derivative provides a consistent approximation to the conventional second derivative; the cases where the dynamic derivative approximates the derivative only via a proper modification of the existing formula; and the situations in which the dynamic derivative can never approximate consistently even with the help of available structure correction methods. Constructive error analysis will be given via asymptotic expansions for practical hybrid modeling and computational applications.

Keywords. Time scale theory, dynamic derivatives, conventional derivatives, approximations, numerical errors, uniform and nonuniform grids, asymptotic expansions

AMS (MOS) Subject Classification: 34A45, 39A13, 74H15, 74S20

1. Introduction

Recent developments of the theory and methods of dynamic equations on time scales have not only bridged the discrepancies between traditional differential and difference equations, but also provided new tools for hybrid modeling and adaptive computations. Enthusiastic discussions of the literature can be found in many publications in the community, for instance, in [2, 3, 5-8, 15] and references therein. Latest topics of the studies include properties of the higher order and multidimensional dynamic derivatives, and numerical applications for solving challenging engineering and environmental problems [4, 9, 14-16].

Based on the well-known Δ (delta) and ∇ (nabla) dynamic derivatives [1, 2, 10], a combined dynamic derivative, that is, $\Diamond_\alpha$ (diamond-α) dynamic derivative, was introduced and applied for solving certain differential equations on adaptive grids [11-14]. Although, strictly speaking, the above-mentioned combined derivative is not a dynamic derivative for the absence of its anti-derivative [11], the formula is extremely useful in numerical applications and we may still call it a "dynamic derivative" [12, 14]. This paper will continue our explorations of connections between the diamond-α dynamic derivative and conventional derivatives. We will show that while the combined dynamic derivative is not a consistent approximation of the conventional derivative in general, it does offer reasonable results in certain numerical applications. A modification of the dynamic derivative formula will be introduced to achieve the consistency in some cases. Proper incorporations of the underlying time scale structures are keys to the success. Although modifications of the combined derivative may introduce substantial complexities to the numerical algorithms, the rewards received are often tremendous due to the improvements of approximation quality.

We assume that the reader has certain working experiences with the time scales theory as well as methods of numerical approximations. Our discussions will begin with Section 2, where a review of the time scales theory preliminaries will be given. Basic concepts of the approximations will also be introduced. Section 3 will be devoted to approximations involving the second order diamond-α derivative. We will demonstrate that the dynamic derivative possesses similar properties as that of the $\Delta,\ \nabla$ dynamic derivatives. Asymptotic expansions will be used to qualitatively predict the numerical error entertained. A modified dynamic derivative formula is proposed for potential applications on discrete time scales. Finally, conclusions and remarks on further investigations will be given in the Fourth Section.

2. Preliminaries

Let $\mathbb{T}$ be an one-dimensional time scale, and set $a = \sup \mathbb{T},\ b = \inf \mathbb{T}$. Thus, $\mathbb{T}$ can be viewed as a closed set of real numbers, or a hybrid grid, superimposed over the interval $[a, b]$ from approximation point of view. Further, let $\sigma,\ \rho$ be the forward and backward jump functions and $\mu,\ \eta$ be the forward and backward step functions for appropriate $t \in \mathbb{T}$. We write $f^\sigma(t) = f(\sigma(t)),\ f^\rho(t) = f(\rho(t))$ and denote $\lambda(t) = \mu(t)/\eta(t)$ if $\eta(t) \neq 0$.

We denote $\sigma^n(t) = \sigma(\sigma^{n-1}(t)),\ \rho^m(t) = \rho(\rho^{m-1}(t)),\ n, m = 1, 2, \ldots,$ under the agreement $\sigma^0(t) = \rho^0(t) = t$. Further, a point $t \in \mathbb{T}$ is called left-scattered, right-scattered if $\rho(t) < t,\ \sigma(t) > t$, respectively. A point $t \in \mathbb{T}$ is called left-dense, right-dense if $\rho(t) = t,\ \sigma(t) = t$, respectively. We define $\mathbb{T}^\kappa = \mathbb{T}$ if b is left-dense and $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$ if b is left-scattered. Similarly, we define $\mathbb{T}_\kappa = \mathbb{T}$ if a is right-dense and $\mathbb{T}_\kappa = \mathbb{T} \setminus \{a\}$ if a is right-scattered. We denote $\mathbb{T}^\kappa \cap \mathbb{T}_\kappa = \mathbb{T}^\kappa_\kappa$. By the same token, we may in general define extended time scales $\mathbb{T}^{\kappa^m},\ \mathbb{T}_{\kappa^n}$ and $\mathbb{T}^{\kappa^m}_{\kappa^n};\ m, n = 0, 1, 2, \ldots,$ under the notation $\mathbb{T}^{\kappa^0} = \mathbb{T}_{\kappa^0} = \mathbb{T}$. We say a time scale $\mathbb{T}$ is uniform if for all $t \in \mathbb{T}^\kappa_\kappa,\ \mu(t) = \eta(t)$. A uniform time scale is an interval if $\mu(t) = 0$, and is a uniform difference grid if $\mu(t) > 0$. For the convenience of discussions, we may decompose $\mathbb{T}$ into the following subsets [12, 14]:

$$\begin{aligned} A: &= \{t \in \mathbb{T} : t \text{ is left-dense and right-scattered}\}, \\ B: &= \{t \in \mathbb{T} : t \text{ is left-scattered and right-dense}\}, \\ C: &= \{t \in \mathbb{T} : t \text{ is left-scattered and right-scattered}\}, \\ D: &= \{t \in \mathbb{T} : t \text{ is left-dense and right-dense}\}. \end{aligned}$$

Without loss of generality, we may assume that $a \in A \cup D$ and $b \in B \cup D$.

Based on the Δ dynamic differentiation [2, 10] and ∇ dynamic differentiation [1, 2], we may define the combined diamond-α derivative [14] of $f,\ t \in \mathbb{T}^{\kappa}_{\kappa}$, as

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable. Let $\mathbb{T}$ be uniform. Then the Δ, ∇ and $\diamondsuit_\alpha$ derivatives of f reduce to the conventional derivative f' if $\mu(t) = 0$ and f' exists, or to appropriate finite difference formulae if $\mu(t) > 0$. Needless to mention that the dynamic differential operators Δ, ∇ and $\diamondsuit_\alpha$ do not commute in general.

We may define the second order combined dynamic derivative as $f^{\diamondsuit_\alpha \diamondsuit_\alpha} = \left(f^{\diamondsuit_\alpha}\right)^{\diamondsuit_\alpha}$. Higher order derivatives $f^{\diamondsuit_\alpha \diamondsuit_\alpha \cdots \diamondsuit_\alpha}$ can be defined in a similar way.

Finally, let functions f and g be defined on $\mathbb{T}$, and g be an approximation of f. If

$$|f(t) - g(t)| = O\left(\max\{\mu^r(t), \eta^r(t)\}\right), \quad t \in \mathbb{T}, \tag{2.1}$$

where $0 \le \mu,\ \eta < 1$, then we say that the approximation is accurate to the order r with respect to the step functions at t. From approximation point of view, an approximation g is consistent if and only if $r > 0$.

3. Main Results

For the simplicity in discussions, we may decompose $\mathbb{T}^{\kappa^2}_{\kappa^2}$ into the following subsets,

$$\begin{aligned} A_1 &= \{t:\ t \in A \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in B\}, \\ A_2 &= \{t:\ t \in A \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in C\}, \\ B_1 &= \{t:\ t \in B \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \rho(t) \in A\}, \\ B_2 &= \{t:\ t \in B \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \rho(t) \in C\}, \\ C_1 &= \{t:\ t \in C \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in B,\ \rho(t) \in A\}, \\ C_2 &= \{t:\ t \in C \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in B,\ \rho(t) \in C\}, \\ C_3 &= \{t:\ t \in C \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in C,\ \rho(t) \in A\}, \\ C_4 &= \{t:\ t \in C \cap \mathbb{T}^{\kappa^2}_{\kappa^2},\ \sigma(t) \in C,\ \rho(t) \in C\}, \\ D_1 &= \{t:\ t \in D \cap \mathbb{T}^{\kappa^2}_{\kappa^2}\}. \end{aligned}$$

Theorem 3.1. *Let f be twice continuously differentiable in $[a,\ b]$, μ be ∇ differentiable on $A_1 \cup A_2$, and η be Δ differentiable on $B_1 \cup B_2$, $0 \le \alpha \le 1$. Then*

$$
f^{\diamondsuit_\alpha \diamondsuit_\alpha}(t) =
\begin{cases}
\alpha^2 \dfrac{f'(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} + (1-\alpha)^2 f''(t) + \alpha(1-\alpha)\phi_1(\mu, f), \quad t \in A_1, \\
\alpha^2 \dfrac{\mu(t) f^{\sigma^2}(t) - (\mu(t) + \mu^\sigma(t))\, f^\sigma(t) + \mu^\sigma(t) f(t)}{\mu^2(t)\mu^\sigma(t)} + (1-\alpha)^2 f''(t) \\
\quad + \alpha(1-\alpha)\phi_2(\mu, f), \quad t \in A_2, \\
\alpha^2 f''(t) + (1-\alpha)^2 \dfrac{f^{\nabla}(t) - f'(\rho(t))}{\eta(t)} + \alpha(1-\alpha)\psi_2(\eta, f), \quad t \in B_1, \\
\alpha^2 f''(t) + (1-\alpha)^2 \dfrac{\eta^\rho(t) f(t) - (\eta^\rho(t) - \eta(t))\, f^\rho(t) + \eta(t) f^{\rho^2}(t)}{\eta^2(t)\eta^\rho(t)} \\
\quad + \alpha(1-\alpha)\psi_2(\eta, f), \quad t \in B_2, \\
\alpha^2 \dfrac{f'(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} + (1-\alpha)^2 \dfrac{f^{\nabla}(t) - f'(\rho(t))}{\eta(t)} + \alpha(1-\alpha)\phi_3(\mu, \eta, f), \quad t \in C_1, \\
\alpha^2 \dfrac{f'(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} + (1-\alpha)^2 \dfrac{\eta^\rho(t) f(t) - (\eta^\rho(t) + \eta(t))\, f^\rho(t) + \eta(t) f^{\rho^2}(t)}{\eta^2(t)\eta^\rho(t)} \\
\quad + \alpha(1-\alpha)\phi_3(\mu, \eta, f), \quad t \in C_2, \\
\alpha^2 \dfrac{\mu(t) f^{\sigma^2}(t) - (\mu(t) + \mu^\sigma(t))\, f^\sigma(t) + \mu^\sigma(t) f(t)}{\mu^2(t)\mu^\sigma(t)} + (1-\alpha)^2 \dfrac{f^{\nabla}(t) - f'(\rho(t))}{\eta(t)} \\
\quad + \alpha(1-\alpha)\phi_3(\mu, \eta, f), \quad t \in C_3, \\
\alpha^2 \dfrac{\mu(t) f^{\sigma^2}(t) - (\mu(t) + \mu^\sigma(t))\, f^\sigma(t) + \mu^\sigma(t) f(t)}{\mu^2(t)\mu^\sigma(t)} + \alpha(1-\alpha)\phi_3(\mu, \eta, f) \\
\quad + (1-\alpha)^2 \dfrac{\eta^\rho(t) f(t) - (\eta^\rho(t) + \eta(t))\, f^\rho(t) + \eta(t) f^{\rho^2}(t)}{\eta^2(t)\eta^\rho(t)}, \quad t \in C_4, \\
f''(t), \quad t \in D_1,
\end{cases}
$$

where

$$
\begin{aligned}
\phi_2(\mu, f) &= \frac{\left(1 - \mu^{\nabla}(t)\right) f^{\Delta}(t) + f'(\sigma(t)) - 2f'(t)}{\mu(t)}, \\
\psi_2(\eta, f) &= \frac{2f'(t) - f'(\rho(t)) - \left(1 + \eta^{\Delta}(t)\right) f^{\nabla}(t)}{\eta(t)}, \\
\phi_3(\mu, \eta, f) &= \frac{(\mu(t) + \eta(t))\,[\eta(t) f^\sigma(t) - (\mu(t) + \eta(t)) f(t) + \mu(t) f^\rho(t)]}{\mu^2(t)\eta^2(t)}.
\end{aligned}
$$

Proof. Note that, for $t \in \mathbb{T}^{\kappa^2}_{\kappa^2}$ we have [12, 14]

$$
f^{\diamondsuit_\alpha \diamondsuit_\alpha}(t) = \alpha^2 f^{\Delta\Delta}(t) + \alpha(1-\alpha)\left(f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t)\right) + (1-\alpha)^2 f^{\nabla\nabla}(t). \quad (3.1)
$$

Therefore the proof of the theorem can be accomplished by proper combinations of the results stated in [12, 13] for each of the distinguished sub-time scale cases, respectively. Since proofs of the nine cases are similar, we only need to present one of them as an illustration. Without loss of generality, we may consider the case for $t \in C_2$ as an example. According to Theorems 3.1-2 in [13] we have

$$\begin{aligned} f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t) &= \frac{\eta(t) f^{\sigma}(t) - (\eta(t) + \mu(t)) f(t) + \mu(t) f^{\rho}(t)}{\mu(t)\eta^2(t)} \\ &\quad + \frac{\eta(t) f^{\sigma}(t) - (\eta(t) + \mu(t)) f(t) + \mu(t) f^{\rho}(t)}{\mu^2(t)\eta(t)} \\ &= \frac{\mu(t) + \eta(t)}{\mu^2(t)\eta^2(t)} \left[\eta(t) f^{\sigma}(t) - (\mu(t) + \eta(t)) f(t) + \mu(t) f^{\rho}(t)\right] \\ &= \phi_3(\mu, \eta, f). \end{aligned} \tag{3.2}$$

On the other hand, we may observe that

$$f^{\Delta\Delta}(t) = \frac{f'(\sigma(t)) - f^{\Delta}(t)}{\mu(t)}, \tag{3.3}$$

$$f^{\nabla\nabla}(t) = \frac{\eta^{\rho}(t) f(t) - (\eta^{\rho}(t) + \eta(t)) f^{\rho}(t) + \eta(t) f^{\rho^2}(t)}{\eta^2(t)\eta^{\rho}(t)}. \tag{3.4}$$

Substituting (3.2)-(3.4) into (3.1), we obtained the identity we wish immediately after a straightforward simplification. The proof is thus completed. □

Theorem 3.2. *Let f be four times continuously differentiable in $[a,\ b]$, μ be ∇ differentiable on $A_1 \cup A_2$, and η be Δ differentiable on $B_1 \cup B_2$, $0 \le \alpha \le 1$. Then the $\diamondsuit_\alpha \diamondsuit_\alpha$ dynamic derivative of f*

(i) *reduces to the cases of $\Delta\Delta$ or $\nabla\nabla$ derivative when $\alpha = 1$ or $\alpha = 0$, respectively;*

(ii) *is not a consistent approximation to f'' at $t \in \mathbb{T}^{\kappa^2}_{\kappa^2}$, $0 < \alpha < 1$, except in the case that $t \in D_1$, or that $t \in C_1 \cup C_2 \cup C_3 \cup C_4$ and there exists a function $\alpha = \alpha(t)$ such that*

$$\begin{array}{ll} (2 - r(t))\alpha^2(t) - (2 - r(t))\alpha(t) = 1, & t \in C_1, \\ (1 + p(t) - r(t))\alpha^2(t) - (2p(t) - r(t))\alpha(t) + p(t) = 2, & t \in C_2, \\ (1 + q(t) - r(t))\alpha^2(t) - (2 - r(t))\alpha(t) = 1, & t \in C_3, \\ (p(t) + q(t) - r(t))\alpha^2(t) - (2p(t) - r(t))\alpha(t) + p(t) = 2, & t \in C_4, \end{array} \tag{3.5}$$

where

$$p(t) = 1 + \frac{1}{\lambda^{\rho}(t)},\ q(t) = 1 + \lambda^{\sigma}(t),\ r(t) = 2 + \lambda(t) + \frac{1}{\lambda(t)}.$$

Further,

$$
f^{\diamondsuit_\alpha \diamondsuit_\alpha}(t) =
\begin{cases}
\frac{1}{2}\left[\alpha^2 + 2(1-\alpha)^2 + \alpha(1-\alpha)\left(3 - \mu^\nabla(t)\right)\right] f''(t) \\
\quad - \alpha(1-\alpha)\frac{\mu^\nabla(t)}{\mu(t)} f'(t) + \mu(t)\Phi_1(\alpha,\mu,f), \quad t \in A_1, \\
\frac{1}{2}\left[\alpha^2\left(1+\lambda^\sigma(t)\right) + 2(1-\alpha)^2 + \alpha(1-\alpha)^2\left(3 - \mu^\nabla(t)\right)\right] f''(t) \\
\quad - \alpha(1-\alpha)\frac{\mu^\nabla(t)}{\mu(t)} f'(t) + \mu(t)\Phi_2(\alpha,\mu,f), \quad t \in A_2, \\
\frac{1}{2}\left[2\alpha^2 + (1-\alpha)^2 + \alpha(1-\alpha)\left(3 + \eta^\Delta(t)\right)\right] f''(t) \\
\quad - \alpha(1-\alpha)\frac{\eta^\Delta(t)}{\eta(t)} f'(t) + \eta(t)\Psi_1(\alpha,\eta,f), \quad t \in B_1, \\
\frac{1}{2}\left[2\alpha^2 + (1-\alpha)^2\left(1 + \frac{1}{\lambda^\rho(t)}\right) + \alpha(1-\alpha)\left(3 + \eta^\Delta(t)\right)\right] f''(t) \\
\quad - \alpha(1-\alpha)\frac{\eta^\Delta(t)}{\eta(t)} f'(t) + \eta(t)\Psi_2(\alpha,\eta,f), \quad t \in B_2, \\
\frac{1}{2}\left[\alpha^2 + (1-\alpha)^2 + \alpha(1-\alpha)\left(2 + \lambda(t) + \frac{1}{\lambda}\right)\right] f''(t) \\
\quad + \mu(t)\Phi_3(\alpha,\mu,f) + \eta(t)\Psi_3(\alpha,\eta,f), \quad t \in C_1, \\
\frac{1}{2}\left[\alpha^2 + (1-\alpha)^2\left(1 + \frac{1}{\lambda^\rho(t)}\right) + \alpha(1-\alpha)\left(2 + \lambda(t) + \frac{1}{\lambda(t)}\right)\right] f''(t) \\
\quad + \mu(t)\Phi_3(\alpha,\mu,f) + \eta(t)\Psi_4(\alpha,\eta,f), \quad t \in C_2, \\
\frac{1}{2}\left[\alpha^2\left(1+\lambda^\sigma(t)\right) + (1-\alpha)^2 + \alpha(1-\alpha)\left(2 + \lambda(t) + \frac{1}{\lambda}\right)\right] f''(t) \\
\quad + \mu(t)\Phi_4(\alpha,\mu,f) + \eta(t)\Psi_3(\alpha,\eta,f), \quad t \in C_3, \\
\frac{1}{2}\left[\alpha^2\left(1+\lambda^\sigma(t)\right) + (1-\alpha)^2\left(1 + \frac{1}{\lambda^\rho(t)}\right) + \alpha(1-\alpha)\left(2 + \lambda(t) + \frac{1}{\lambda}\right)\right] f''(t) \\
\quad + \mu(t)\Phi_4(\alpha,\mu,f) + \eta(t)\Psi_4(\alpha,\eta,f), \quad t \in C_4, \\
f''(t), \quad t \in D_1,
\end{cases}
$$

where

$$
\begin{aligned}
\Phi_1(\alpha,\mu,f) &= \frac{\alpha}{6}\left\{\alpha\left(3f'''(\xi_4) - f'''(\zeta_4)\right)\right. \\
&\quad \left. +(1-\alpha)\left[\left(1-\mu^\nabla(t)\right) f'''(\xi_1) + 3f'''(\zeta_1)\right]\right\}, \\
\Phi_2(\alpha,\mu,f) &= \frac{\alpha}{6}\left\{3\alpha\left(1+\lambda^\sigma(t)\right)\phi_1(\mu,f)\right. \\
&\quad \left. +(1-\alpha)\left[\left(1-\mu^\nabla(t)\right) f'''(\xi_1) + 3f'''(\zeta_1)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3(\alpha,\mu,f) &= \frac{\alpha}{6}\left\{\alpha\left(3f'''(\xi_4)-f'''(\zeta_4)\right)\right.\\
&\left.+(1-\alpha)\left[\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)+\frac{\mu(t)}{2}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right]\right\},\\
\Phi_4(\alpha,\mu,f) &= \frac{\alpha}{6}\left\{3\alpha\left(1+\lambda^\sigma(t)\right)\phi_1(\mu,f)\right.\\
&\left.+(1-\alpha)\left[\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)+\frac{\mu(t)}{2}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right]\right\},\\
\Psi_1(\alpha,\eta,f) &= \frac{1-\alpha}{6}\left\{(1-\alpha)\left(f'''(\xi_5)-3f'''(\zeta_5)\right)\right.\\
&\left.-\alpha\left[3f'''(\xi_2)-\left(1+\eta^\Delta(t)\right)f'''(\zeta_2)\right]\right\},\\
\Psi_2(\alpha,\eta,f) &= \frac{1-\alpha}{6}\left\{3(1-\alpha)\left(1+\frac{1}{\lambda^\rho(t)}\right)\psi_1(\eta,f)\right.\\
&\left.-\alpha\left[3f'''(\xi_2)-\left(1+\eta^\Delta(t)\right)f'''(\zeta_2)\right]\right\},\\
\Psi_3(\alpha,\eta,f) &= \frac{1-\alpha}{6}\left\{(1-\alpha)\left(f'''(\xi_5)-3f'''(\zeta_5)\right)\right.\\
&\left.+\alpha\left[\left(\lambda^2(t)-1\right)f'''(t)+\frac{\eta(t)}{2}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right]\right\},\\
\Psi_4(\alpha,\eta,f) &= \frac{1-\alpha}{6}\left\{3(1-\alpha)\left(1+\frac{1}{\lambda^\rho(t)}\right)\psi_1(\eta,f)\right.\\
&\left.+\alpha\left[\left(\lambda^2(t)-1\right)f'''(t)+\frac{\eta(t)}{2}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right]\right\}.
\end{aligned}
$$

The identities obtained characterize asymptotically the approximation error involved.

Proof. Let us show the identities for the $\diamondsuit_\alpha\diamondsuit_\alpha$ derivative first. To achieve so, recall [13] and Theorem 3.1. For $t \in A_1$, we have

$$
\begin{aligned}
f^{\diamondsuit_\alpha\diamondsuit_\alpha}(t) &= \alpha^2\left[\frac{1}{2}f''(t)+\frac{\mu(t)}{6}\left(3f'''(\xi_4)-f'''(\zeta_4)\right)\right]+(1-\alpha)^2 f''(t)\\
&+\alpha(1-\alpha)\left[\frac{1}{2}\left(1-\mu^\nabla(t)\right)f''(t)-\frac{\mu^\nabla(t)}{\mu(t)}f'(t)\right.\\
&\left.+\frac{\mu(t)}{6}\left(1-\mu^\nabla(t)\right)f'''(\xi_1)+f''(t)+\frac{\mu(t)}{2}f'''(\zeta_1)\right]\\
&= \left[\frac{\alpha^2}{2}+(1-\alpha)^2+\frac{\alpha(1-\alpha)}{2}\left(1-\mu^\nabla(t)+2\right)\right]f''(t)\\
&-\alpha(1-\alpha)\frac{\mu^\nabla(t)}{\mu(t)}f'(t)+\mu(t)\Phi_1(\alpha,\mu,f)
\end{aligned}
$$

which quickly yields the result we want. Here we must notice that, due to the existence of the nontrivial term involving $\mu^{\nabla}(t)f'(t)/\mu(t)$, no matter how we select the parameter $\alpha,\ 0<\alpha<1$, the dynamic derivative under the investigation can never be a consistent approximation to f'' as $t\in A_1$. Further, let $t\in A_2$. We may deduce that

$$\begin{aligned}
f^{\diamond_\alpha\diamond_\alpha}(t) &= \alpha^2\left[\frac{1}{2}\left(1+\lambda^\sigma(t)\right)f''(t)+\frac{1}{2}\left(1+\lambda^\sigma(t)\right)\phi_1(\mu,f)\mu(t)\right]+(1-\alpha)^2f''(t)\\
&\quad+\alpha(1-\alpha)\left[\frac{1}{2}\left(1-\mu^{\nabla}(t)\right)f''(t)-\frac{\mu^{\nabla}(t)}{\mu(t)}f'(t)\right.\\
&\quad\left.+\frac{\mu(t)}{6}\left(1-\mu^{\nabla}(t)\right)f'''(\xi_1)+f''(t)+\frac{\mu(t)}{2}f'''(\zeta_1)\right]\\
&= \left[\frac{\alpha^2}{2}\left(1+\lambda^\sigma(t)\right)+(1-\alpha)^2+\frac{\alpha(1-\alpha)}{2}\left(1-\mu^{\nabla}(t)+2\right)\right]f''(t)\\
&\quad-\alpha(1-\alpha)\frac{\mu^{\nabla}(t)}{\mu(t)}f'(t)+\mu(t)\Phi_2(\alpha,\mu,f).
\end{aligned}$$

Now, a straightforward simplification of the above result leads to the identity we want. Again, we are able to conclude that the formula acquired is not consistent for $0<\alpha<1$ in general due to the term involving the nontrivial function $\mu^{\nabla}(t)f'(t)/\mu(t)$. As for the situation when $t\in B_1$, it is found that

$$\begin{aligned}
f^{\diamond_\alpha\diamond_\alpha}(t) &= \alpha^2f''(t)+(1-\alpha)^2\left[\frac{1}{2}f''(t)+\frac{\eta(t)}{6}\left(f'''(\xi_5)-3f'''(\zeta_5)\right)\right]\\
&\quad+\alpha(1-\alpha)\left[f''(t)-\frac{\eta(t)}{2}f'''(\xi_2)+\frac{1}{2}\left(1+\eta^{\Delta}(t)\right)f''(t)\right.\\
&\quad\left.-\frac{\eta^{\Delta}(t)}{\eta(t)}f'(t)+\frac{\eta(t)}{6}\left(1+\eta^{\Delta}(t)\right)f'''(\zeta_2)\right]\\
&= \left[\alpha^2+\frac{(1-\alpha)^2}{2}+\alpha(1-\alpha)+\frac{\alpha(1-\alpha)}{2}\left(1+\eta^{\Delta}(t)\right)\right]f''(t)\\
&\quad+(1-\alpha)^2\frac{\eta(t)}{6}\left(f'''(\xi_5)-3f'''(\zeta_5)\right)-\alpha(1-\alpha)\frac{\eta^{\Delta}(t)}{\eta(t)}f'(t)\\
&\quad+\alpha(1-\alpha)\left[-\frac{\eta(t)}{2}f'''(\xi_2)+\frac{\eta(t)}{6}\left(1+\eta^{\Delta}(t)\right)f'''(\zeta_2)\right]\\
&= \frac{1}{2}\left[2\alpha^2+(1+\alpha)^2+\alpha(1-\alpha)\left(3+\eta^{\Delta}(t)\right)\right]f''(t)\\
&\quad-\alpha(1-\alpha)\frac{\eta^{\Delta}(t)}{\eta(t)}f'(t)+\eta(t)\Psi_1(\alpha,\eta,f).
\end{aligned}$$

Thus our expected result holds. The nontrivial term involving $\eta^{\Delta}(t)f'(t)/\eta(t)$ again prevents the dynamic derivative formula from being a consistent approximation to

f'' for $0 < \alpha < 1$. Next, we consider the case that $t \in B_2$. It is not difficult to observe that

$$
\begin{aligned}
f^{\diamondsuit_\alpha \diamondsuit_\alpha}(t) &= \alpha^2 f''(t) + \frac{(1-\alpha)^2}{2}\left(1 + \frac{1}{\lambda^\rho(t)}\right)\left(f''(t) + \eta(t)\psi_1(\eta, f)\right) \\
&\quad + \alpha(1-\alpha)\left[f''(t) - \frac{\eta(t)}{2} f'''(\xi_2) + \frac{1}{2}\left(1 + \eta^\Delta(t)\right) f''(t)\right. \\
&\quad \left. - \frac{\eta^\Delta(t)}{\eta(t)} f'(t) + \frac{\eta(t)}{6}\left(1 + \eta^\Delta(t)\right) f'''(\zeta_2)\right] \\
&= \frac{1}{2}\left[2\alpha^2 + (1-\alpha)^2\left(1 + \frac{1}{\lambda^\rho(t)}\right) + \alpha(1-\alpha)(2 + 1 + \eta^\Delta(t))\right] f''(t) \\
&\quad - \alpha(1-\alpha)\frac{\eta^\Delta(t)}{\eta(t)} f'(t) + (1-\alpha)^2 \frac{\eta(t)}{2}\left(1 + \frac{1}{\lambda^\rho(t)}\right)\psi_1(\eta, f) \\
&\quad - \alpha(1-\alpha)\left[\frac{\eta(t)}{2} f'''(\xi_2) - \frac{\eta(t)}{6}\left(1 + \eta^\Delta(t)\right) f'''(\zeta_2)\right] \\
&= \frac{1}{2}\left[2\alpha^2 + (1-\alpha)^2\left(1 + \frac{1}{\lambda^\rho(t)}\right) + \alpha(1-\alpha)(2 + 1 + \eta^\Delta(t))\right] f''(t) \\
&\quad - \alpha(1-\alpha)\frac{\eta^\Delta(t)}{\eta(t)} f'(t) + \eta(t)\Psi_2(\alpha, \eta, f)
\end{aligned}
$$

which clearly indicates the identity we need. The dynamic derivative does not approximate $f''(t)$ due to the nontrivial term involving the function $\eta^\Delta(t) f'(t)/\eta(t)$, no matter what value of α, $0 < \alpha < 1$, is selected. Now, for the situation where $t \in C_1$, we observe that

$$
\begin{aligned}
f^{\diamondsuit_\alpha \diamondsuit_\alpha}(t) &= \alpha^2\left[\frac{1}{2} f''(t) + \frac{\mu(t)}{6}\left(3 f'''(\xi_4) - f'''(\zeta_4)\right)\right] \\
&\quad + (1-\alpha)^2\left[\frac{1}{2} f''(t) + \frac{\eta(t)}{6}\left(f'''(\xi_5) - 3 f'''(\zeta_5)\right)\right] \\
&\quad + \alpha(1-\alpha)\left[\frac{1}{2}(1 + \lambda(t)) f''(t) - \frac{\eta(t)}{6}\left(1 - \lambda^2(t)\right) f'''(t) + \frac{\eta^2(t)}{12}\left(1 + \lambda^3(t)\right) f^{(4)}(\xi_3)\right. \\
&\quad \left. + \frac{1}{2}\left(1 + \frac{1}{\lambda(t)}\right) f''(t) + \frac{\mu(t)}{6}\left(1 - \frac{1}{\lambda^2(t)}\right) f'''(t) + \frac{\mu^2(t)}{12}\left(1 + \frac{1}{\lambda^3(t)}\right) f^{(4)}(\zeta_3)\right] \\
&= \left[\frac{\alpha^2}{2} + \frac{(1-\alpha)^2}{2} + \frac{\alpha(1-\alpha)}{2}\left(1 + \lambda(t) + 1 + \frac{1}{\lambda(t)}\right)\right] f''(t) \\
&\quad + \frac{\mu(t)}{6}\left\{\alpha^2\left(3 f'''(\xi_4) - f'''(\zeta_4)\right) + \alpha(1-\alpha)\left[\left(1 - \frac{1}{\lambda^2(t)}\right) f'''(t)\right.\right. \\
&\quad \left.\left. + \frac{\mu(t)}{2}\left(1 + \frac{1}{\lambda^3(t)}\right) f^{(4)}(\zeta_3)\right]\right\} + \frac{\eta(t)}{6}\left\{(1-\alpha)^2\left(f'''(\xi_5) - 3 f'''(\zeta_5)\right)\right.
\end{aligned}
$$

$$+\alpha(1-\alpha)\left[-\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta(t)}{2}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right]\Bigg\}.$$

The above result leads directly to our favorable identity in the situation. By the similar token, for $t \in C_2$, we find via an asymptotic expansion that

$$\begin{aligned}
f^{\diamondsuit_\alpha\diamondsuit_\alpha}(t) &= \alpha^2\left[\frac{1}{2}f''(t)+\frac{\mu(t)}{6}\left(3f'''(\xi_4)-f'''(\zeta_4)\right)\right]\\
&+(1-\alpha)^2\left[\frac{1}{2}\left(1+\frac{1}{\lambda^\rho(t)}\right)\left(f''(t)+\eta(t)\psi_1(\eta,f)\right)\right]\\
&+\alpha(1-\alpha)\left[\frac{1}{2}(1+\lambda(t))f''(t)-\frac{\eta(t)}{6}\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta^2(t)}{12}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right.\\
&\left.+\frac{1}{2}\left(1+\frac{1}{\lambda(t)}\right)f''(t)+\frac{\mu(t)}{6}\left(1-\frac{1}{\lambda(t)}\right)f'''(t)+\frac{\mu^2(t)}{12}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right]\\
&= \left[\frac{\alpha^2}{2}+\frac{(1-\alpha)^2}{2}\left(1+\frac{1}{\lambda^\rho(t)}\right)+\frac{\alpha(1-\alpha)}{2}\left(1+\lambda(t)+1+\frac{1}{\lambda(t)}\right)\right]f''(t)\\
&+\frac{\mu(t)}{6}\left\{\alpha^2\left(3f'''(\xi_4)-f'''(\zeta_4)\right)+\alpha(1-\alpha)\left[\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)\right.\right.\\
&\left.\left.+\frac{\mu(t)}{2}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right]\right\}+\frac{\eta(t)}{6}\left\{3(1-\alpha)^2\left(1+\frac{1}{\lambda^\rho(t)}\right)\psi_1(\eta,f)\right.\\
&\left.+\alpha(1-\alpha)\left[-\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta(t)}{2}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right]\right\}
\end{aligned}$$

and the equations obtained above yield immediately the identity we want. Further, for $t \in C_3$, we must have

$$\begin{aligned}
f^{\diamondsuit_\alpha\diamondsuit_\alpha}(t) &= \alpha^2\left[\frac{1}{2}\left(1+\lambda^\sigma(t)\right)\left(f''(t)+\mu(t)\phi_1(\mu,f)\right)\right]\\
&+(1-\alpha)^2\left[\frac{1}{2}f''(t)+\frac{\eta(t)}{6}\left(f'''(\xi_5)-3f'''(\zeta_5)\right)\right]\\
&+\alpha(1-\alpha)\left[\frac{1}{2}(1+\lambda(t))f''(t)-\frac{\eta(t)}{6}\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta^2(t)}{12}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right.\\
&\left.+\frac{1}{2}\left(1+\frac{1}{\lambda(t)}\right)f''(t)+\frac{\mu(t)}{6}\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)+\frac{\mu^2(t)}{12}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right]\\
&= \left[\frac{\alpha^2}{2}\left(1+\lambda^\sigma(t)\right)+\frac{(1-\alpha)^2}{2}+\frac{\alpha(1-\alpha)}{2}\left(1+\lambda(t)+1+\frac{1}{\lambda(t)}\right)\right]f''(t)\\
&+\frac{\mu(t)}{6}\left\{3\alpha^2\phi_1(\mu,f)+\alpha(1-\alpha)\left[\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)\right.\right.
\end{aligned}$$

$$+\frac{\mu(t)}{2}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\Big]\Big\}+\frac{\eta(t)}{6}\left\{(1-\alpha)^2\left(f'''(\xi_5)-3f'''(\zeta_5)\right)\right.$$
$$\left.+\alpha(1-\alpha)\left[-\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta(t)}{2}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right]\right\}.$$

Now, for $t \in C_4$, by means of Theorem 3.1 we arrive at

$$\begin{aligned}
f^{\diamondsuit_\alpha\diamondsuit_\alpha}(t) &= \alpha^2\left[\frac{1}{2}\left(1+\lambda^\sigma(t)\right)\left(f''(t)+\mu(t)\phi_1(\mu,f)\right)\right] \\
&+(1-\alpha)^2\left[\frac{1}{2}\left(1+\frac{1}{\lambda^\rho(t)}\right)\left(f''(t)+\eta(t)\psi_1(\mu,f)\right)\right] \\
&+\alpha(1-\alpha)\left[\frac{1}{2}(1+\lambda(t))f''(t)-\frac{\eta(t)}{6}\left(1-\lambda^2(t)\right)f'''(t)+\frac{\eta^2(t)}{12}\left(1+\lambda^3(t)\right)f^{(4)}(\xi_3)\right. \\
&\left.+\frac{1}{2}\left(1+\frac{1}{\lambda(t)}\right)f''(t)+\frac{\mu(t)}{6}\left(1-\frac{1}{\lambda^2(t)}\right)f'''(t)+\frac{\mu^2(t)}{12}\left(1+\frac{1}{\lambda^3(t)}\right)f^{(4)}(\zeta_3)\right] \\
&= \left[\frac{\alpha^2}{2}\left(1+\lambda^\sigma(t)\right)+\frac{(1-\alpha)^2}{2}\left(1+\frac{1}{\lambda^\rho(t)}\right)+\frac{\alpha(1-\alpha)}{2}\left(1+\lambda(t)+1+\frac{1}{\lambda(t)}\right)\right]f''(t) \\
&+\mu\Phi_4(\alpha,\mu,f)+\eta(t)\Psi_4(\alpha,\eta,f).
\end{aligned}$$

Needless to say that the last identity for the case when $t \in D_1$ can be obtained by using the above as well as the definition of the diamond-α dynamic derivative introduced in [14] and Section 2. Let $t \in C_1 \cup C_2 \cup C_3 \cup C_4$ and recall (2.1). For the sake of consistency in approximations, according to the identities obtained, we must require

$$\begin{aligned}
&\alpha^2+(1-\alpha)^2+\alpha(1-\alpha)\left(2+\lambda(t)+\frac{1}{\lambda(t)}\right)=2, && t\in C_1,\\
&\alpha^2+(1-\alpha)^2\left(1+\frac{1}{\lambda^\rho(t)}\right)+\alpha(1-\alpha)\left(2+\lambda(t)+\frac{1}{\lambda(t)}\right)=2, && t\in C_2,\\
&\left(1+\lambda^\sigma(t)\right)\alpha^2+(1-\alpha)^2+\alpha(1-\alpha)\left(2+\lambda(t)+\frac{1}{\lambda(t)}\right)=2, && t\in C_3,\\
&\left(1+\lambda^\sigma(t)\right)\alpha^2+(1-\alpha)^2\left(1+\frac{1}{\lambda^\rho(t)}\right)+\alpha(1-\alpha)\left(2+\lambda(t)+\frac{1}{\lambda(t)}\right)=2, && t\in C_4.
\end{aligned}$$

Rewrite the above four quadratic equations in α into the standard form, and we obtain subsequently (3.5). The proof is therefore completed. □

Remark. It is not difficult to see that solutions of (3.5) may not exist on an arbitrary discrete time scale, although the solutions always exist for particularly chosen $\mathbb{T}$. An especially interesting case may be when $\mathbb{T}$ is uniformly discrete, for which $p(t) \equiv q(t) \equiv 2$ and $r(t) \equiv 4$. In the situation, the second order diamond-α derivative yields a first order five-point approximation to f'', provided that $\mu(t) = \eta(t)$ is sufficiently small.

Let $\mathbb{T}$ be discrete. In the event that the solution of (3.5) does not exist, we may define the following *modified dynamic derivative formula* for computational applications.

$$\omega_{\diamondsuit_\alpha\diamondsuit_\alpha}(f) = s_1(t) f^{\diamondsuit_\alpha\diamondsuit_\alpha}(t), \quad t \in \mathbb{T}^{\kappa^2}_{\kappa^2},$$

where

$$s_1(t) = \begin{cases} 1/\left((2-r(t))\alpha^2(t) - (2-r(t))\alpha(t)\right), & t \in C_1, \\ 2/\left((1+p(t)-r(t))\alpha^2(t) - (2p(t)-r(t))\alpha(t) + p(t)\right), & t \in C_2, \\ 1/\left((1+q(t)-r(t))\alpha^2(t) - (2-r(t))\alpha(t)\right), & t \in C_3, \\ 2/\left((p(t)+q(t)-r(t))\alpha^2(t) - (2p(t)-r(t))\alpha(t) + p(t)\right), & t \in C_4. \end{cases}$$

Clearly, the modified formula offers a first order approximation to f''.

4. Conclusions

Let $\mathbb{T}$ be a set of real numbers superimposed on $[a,\ b]$, $\mathbb{T}$ is nonempty, and $f(t)$ is sufficiently smooth on $[a,\ b]$. Then the second order combined dynamic derivative, that is, the $\diamondsuit_\alpha$ dynamic derivative, is not a consistent approximation to the conventional derivative f'' in general. However, the dynamic derivative can be consistent approximations in certain special cases, in particular when discrete time scales are employed. In many cases, the second order dynamic derivative discussed can be reformulated to produce a consistent approximation to f'' too. The key for achieving so includes a proper incorporation of the time scale structures into the underlying dynamic derivative formula. However, the order of a modified dynamic derivative formula is usually low. This may prevent further generalizations of the combined derivative formulae, in particularly in approximating third or higher order conventional derivatives on hybrid grids.

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References

[1] D. R. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, Nabla dynamic equations, Chapter 3, *Advances in Dynamic Equations on Time Scales,* edited by M. Bohner and A. Peterson, Birkhäuser, Boston and Berlin (2003).

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston and Berlin (2001).

[3] M. Bohner and A. Peterson, First and second order linear dynamic equations on time scales, *J. Difference Eqns Appl.,* **7**, 767-792 (2001).

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston and Berlin (2003).

[5] J. M. Davis, J. Henderson, P. Rajendra and W. Yin, Solvability of a nonlinear second order conjugate eigenvalue problem on a time scale, *Abstr. Appl. Anal.,* **5**, 91-99 (2000).

[6] P. W. Eloe, S. Hilger and Q. Sheng, A qualitative analysis on nonconstant graininess of the adaptive grid via time scales, *Rocky Mountain J. Math.,* **36**, 115-133 (2006).

[7] P. W. Eloe, Q. Sheng and J. Henderson, Notes on crossed symmetry solution of the two-point boundary value problems on time scales, *J. Difference Eqns, Appl.,* **9**, 29-48 (2003).

[8] L. Erbe and A. Peterson, Boundedness and oscillation for nonlinear dynamic equations on a time scale, *Proc. Amer. Math. Soc.,* **132**, 735-744 (2004).

[9] I. Gravagne, J. M. Davis and R. J. Marks, How deterministic must a real-time controller be? *Proc. IEEE/RSJ Interna. Conf. Intelligent Robots and Sys,* Alberta, 2005, 3856-3861.

[10] S. Hilger, Analysis on measure chain – a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18-56.

[11] J. W. Rogers, Jr. and Q. Sheng, Notes on the diamond-α dynamic derivative on time scales, *J. Math. Anal. Appl.,* in press.

[12] Q. Sheng, A view of dynamic derivatives on time scales from approximations, *J. Diff. Eqn. Appl.,* **11**, 63-82 (2005).

[13] Q. Sheng, A second view of dynamic derivatives on time scales from hybrid approximations, preprint (2006).

[14] Q. Sheng, M. Fadag, J. Henderson and J. Davis, An exploration of combined dynamic derivatives on time scales and their applications, *Nonlinear Anal.: Real World Appl.,* in press.

[15] V. Spedding, Taming nature's numbers, *New Scientist,* **179**, 2404, 28-31 (2003).

[16] D. M. Thomas and B. Urena, A model describing the evolution of West Nile-like encephalitis in New York City, *Math. Comput. Modelling,* **34** (2001), 771-781.

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