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Positive Periodic Solutions In Neutral Nonlinear Differential Equations

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Abstract

We use Krasnoselskii's fixed point theorem to show that the nonlinear neutral differential equation with delay

$$\frac{d}{dt}[x(t) - ax(t-\tau)] = r(t)x(t) - f(t, x(t-\tau))$$

has a positive periodic solution. An example will be provided as an application to our theorems.

AMS Subject Classifications: 34K20, 45J05, 45D05

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1 Introduction

Motivated by the papers [19], [21], [22], [23], and the references therein, we consider the nonlinear neutral differential equation with constant delay

$$\frac{d}{dt}[x(t) - ax(t-\tau)] = r(t)x(t) - f(t, x(t-\tau)), \tau \in \mathbb{R} \tag{1.1}$$

which arises in a food-limited population models (see [3], [4]- [7], [9], [10], [11]), [17] and blood cell models, (see [2], [26], [28]). For system (1.1), there may be a stable equilibrium point of the population. In the case the equilibrium point becomes unstable, there may exist a nontrivial periodic solution. Then the oscillation of solutions occurs. The existence of such stable periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of qualitative theory of differential equations. For historical background, basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models we refer the interested reader to [8], [13], [14], [15], [16], [18], [20], [24], [25] and [27].

One of the most used models, a prototype of (1.1), is the system of Volterra integrodifferential equations (see [27])

$$\dot{N}(t) = -\gamma(t)N(t) + \alpha(t)\int_0^\infty B(s)e^{-\beta(t)N(t-s)}ds$$

where $N(t)$ is the number of red blood cell at time t, and $\alpha, \beta, \gamma \in C(\mathbb{R}, \mathbb{R})$ are $T-$periodic, and $B \in L^1(\mathbb{R}^+)$ and piecewise continuous. This is a generalized model of the red cell system introduced by Wazewska-Czyzewska and Lasota [26]

$$\dot{n}(t) = -\gamma n(t) + \alpha e^{-\beta n(t-r)}$$

where α, β, γ, r are constants with $r > 0$. In [21] the authors established criteria for the existence of positive periodic solutions for the periodic neutral logistic equation, with distributed delays,

$$x'(t) = x(t)\Big[a(t) - \sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) - \sum_{j=1}^{m} b_j(t)\int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big],$$

where the coefficients a, a_i, b_j are continuous and periodic functions, with the same period. The values $T_i, \hat{T}_j$ are positive, and the functions μ_i, ν_j are nondecreasing with $\int_{-T_i}^{0}\mathrm{d}\mu_i = 1$ and $\int_{-\hat{T}_j}^{0}\mathrm{d}\nu_j = 1$. The above equation is of Logistic form and hence the method that were used to obtain the existence of positive periodic solutions will not work for our model (1.1). The same is true for the paper of [17]. In [22] the author used Krasnoselskii's fixed point theorem to show that the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q\big(t, x(t), x(t-g(t)\big)$$

has a periodic solution. Also, by transforming the problem to an integral equation the author was able, using the contraction mapping principle, to show that the periodic solution is unique. Finally, in [21] the authors considered the neutral differential equation with periodic coefficients

$$\frac{d}{dt}(x(t) - cx(t-\tau(t))) = -a(t)x(t) + g(t, x(t-\tau(t))), \tag{1.2}$$

and attempted to show that (1.2) has a positive periodic solution by appealing to cone theory. We point out that the results in [21] are not correct since the two sets Ω_1 and Ω_2 that was constructed by the authors are not open in the Banach space. For the same reason, an addendum has been added to the paper listed in reference [21].

The main aim of this research is to give a correct proof for the existence of a positive periodic solution by using a different fixed point theorem from the one used in [21].

2 Preliminaries

Krasnosel'skiĭ fixed point theorem has been extensively used in differential and functional differential equations, by Burton in proving the existence of periodic solutions. Also, Burton was the first to use the theorem to obtain stability results regarding solutions of integral equations and functional differential equations. For a collection of different type of results concerning stability, the existence of periodic solutions and boundedness of solutions, using fixed point theory, we refer the reader to the new published book [1] and the references therein. The author is unaware of any results regarding the use of Krasnosel'skiĭ to prove the existence of a positive periodic solution.

Theorem 2.1 (Krasnosel'skiĭ) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathbb{B}, \|\cdot\|\big)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) A is compact and continuous,

(ii) B is a contraction mapping.

(iii) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

For $T > 0$ define $P_T = \{\phi \in C(\mathbb{R}, \mathbb{R}), \phi(t+T) = \phi(t)\}, t \in \mathbb{R}$ where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t \in [0,T]} |x(t)| = \max_{t \in \mathbb{R}} |x(t)|.$$

We assume that all functions are continuous with respect to their arguments and for all $t \in \mathbb{R}$,

$$r(t+T) = r(t), \quad f(t+T, \cdot) = f(t, \cdot). \tag{2.1}$$

In addition to (2.1), we ask that $r(t)$ satisfies the average condition

$$\int_0^T r(s)\, ds > 0. \tag{2.2}$$

We begin with the following lemma.

Lemma 2.2. Suppose (2.1) and (2.2) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of equation (1.1) if and only if

$$x(t) = ax(t-\tau) + \int_t^{t+T} G(t,u)[f(u, x(u-\tau)) - ar(u)x(u-\tau)]\, du \tag{2.3}$$

where

$$G(t,u) = \frac{e^{\int_u^t r(s)ds}}{1 - e^{-\int_0^T r(s)ds}}. \tag{2.4}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). To be able to invert (1.1), we put it in the form

$$\frac{d}{dt}[x(t) - ax(t-\tau)] = r(t)\big(x(t) - ax(t-\tau)\big) - \big(f(t, x(t-\tau)) - ar(t)x(t-\tau)\big).$$

Next we multiply both sides of the resulting equation with $e^{-\int_0^t r(s)ds}$ and then integrate from t to $t+T$ to obtain

$$\begin{aligned} &\big(x(t+T) - ax(t+T-\tau)\big)e^{-\int_0^{t+T} r(s)ds} - \big(x(t) - ax(t-\tau)\big)e^{-\int_0^t r(s)ds} \\ = &- \int_t^{t+T} \big(f(u, x(u-\tau)) - ar(u)x(u-\tau)\big)e^{-\int_0^u r(s)ds} du. \end{aligned}$$

Using the fact that $x(t+T) = x(t)$, the above expression can be put in the form

$$\begin{aligned} x(t) &= ax(t-\tau) \\ &+ \int_t^{t+T} \frac{e^{\int_u^t r(s)ds}}{1 - e^{-\int_0^T r(s)ds}} \big(f(u, x(u-\tau)) - ar(u)x(u-\tau)\big)du. \end{aligned} \tag{2.5}$$

This completes the proof.

To simplify notation, we let

$$M = \frac{e^{\int_0^{2T} |r(s)|ds}}{1 - e^{-\int_0^T r(s)ds}}, \tag{2.6}$$

and

$$m = \frac{e^{-\int_0^{2T} |r(s)|ds}}{1 - e^{-\int_0^T r(s)ds}}. \tag{2.7}$$

It is easy to see that for all $(t,s) \in [0,2T] \times [0,2T]$,

$$m \le G(t,s) \le M$$

and for all $t, s \in \mathbb{R}$ we have ,

$$G(t+T, s+T) = G(t,s).$$

3 Main Results

In this section we obtain the existence of a positive periodic solution by considering the two case; $0 \le a < 1$, $-1 < a \le 0$. For some non-negative constant L and a positive constant K we define the set

$$\mathbb{M} = \{\phi \in P_T : L \le ||\phi|| \le K\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that

$$0 \le a < 1, \tag{3.1}$$

and for all $u \in \mathbb{R}, \rho \in \mathbb{M}$

$$\frac{(1-a)L}{mT} \le f(u,\rho) - ar(u)\rho \le \frac{(1-a)K}{MT}, \tag{3.2}$$

where M and m are defined by (2.6) and (2.7), respectively. To apply Theorem 2.1, we will need to construct two mappings; one is contraction and the other is compact. Thus, we set the map $\mathbf{A} : \mathbb{M} \to P_T$

$$(\mathbf{A}\varphi)(t) = \int_t^{t+T} G(t,u)[f(u,\varphi(u-\tau)) - ar(u)\varphi(u-\tau)]\, du, t \in \mathbb{R}. \tag{3.3}$$

In a similar way we set the map $\mathbf{B} : \mathbb{M} \to P_T$

$$(\mathbf{B}\varphi)(t) = a\varphi(t-\tau), t \in \mathbb{R}. \tag{3.4}$$

It is clear from condition (3.1) that $\mathbf{B}$ defines a contraction mapping under the supremum norm.

Lemma 3.1. If (2.1), (2.2), (3.1) and (3.2) hold, then the operator $\mathbf{A}$ is completely continuous on $\mathbb{M}$.

Proof. For $t \in [0,T]$ which implies that $u \in [t, t+T] \subseteq [0, 2T]$ and for $\varphi \in \mathbb{M}$ we have by (3.3) that

$$\begin{aligned} |(\mathbf{A}\varphi)(t)| &\le |\int_t^{t+T} G(t,u)[f(u,\varphi(u-\tau)) - ar(u)\varphi(u-\tau)]\, du| \\ &\le TM\frac{(1-a)K}{MT} = (1-a)K. \end{aligned}$$

From the estimation of $|\mathbf{A}\varphi(t)|$ it follows that

$$||\mathbf{A}\varphi|| \le (1-a)K.$$

This shows that $\mathbf{A}(\mathbb{M})$ is uniformly bounded. Left to show that $\mathbf{A}(\mathbb{M})$ is equicontinuous. Let $\varphi \in \mathbb{M}$. Then a differentiation of (3.3) with respect to t yields

$$(\mathbf{A}\varphi)'(t) = G(t, t+T)[f(t, \varphi(t-\tau)) - ar(t)\varphi(t-\tau)] + r(t)(\mathbf{A}\varphi)(t).$$

Hence, by taking the supremum norm in the above expression we have

$$||(\mathbf{A}\varphi)'|| \leq \frac{(1-a)K}{T} + ||r||(1-a)K.$$

Thus the estimation on $|(\mathbf{A}\varphi)'(t)|$ implies that $\mathbf{A}(\mathbb{M})$ is equicontinuous. Then using Ascoli-Arzela theorem we obtain that A is a compact map. Due to the continuity of all terms in (3.3), we have that $\mathbf{A}$ is continuous. This completes the proof of Lemma 3.1.

Theorem 3.2. If (2.1), (2.2), (3.1) and (3.2) hold, then Equation (1.1) has a positive periodic solution z satisfying $L \leq \|z\| \leq K$.

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then, by (3.3) and (3.4) we have that

$$\begin{aligned}
(\mathbf{B}\varphi)(t) + (\mathbf{A}\psi)(t) &= a\varphi(t-\tau) + \int_t^{t+T} G(t,u)[f(u,\psi(u-\tau)) - ar(u)\psi(u-\tau)]\,du \\
&\leq aK + M\int_t^{t+T} [f(u,\psi(u-\tau)) - ar(u)\psi(u-\tau)]\,du \\
&\leq aK + MT\frac{(1-a)K}{MT} = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathbf{B}\varphi)(t) + (\mathbf{A}\psi)(t) &= a\varphi(t-\tau) + \int_t^{t+T} G(t,u)[f(u,\psi(u-\tau)) - ar(u)\psi(u-\tau)]\,du \\
&\geq aL + m\int_t^{t+T} [f(u,\psi(u-\tau)) - ar(u)\psi(u-\tau)]\,du \\
&\geq aL + mT\frac{(1-a)L}{mT} = L.
\end{aligned}$$

This shows that $\mathbf{B}\varphi + \mathbf{A}\psi \in \mathbb{M}$. All the hypothesis of Theorem 2.1 are satisfied and therefore equation (1.1) has a periodic solution, say z residing in $\mathbb{M}$. This completes the proof.

For the next theorem we substitute conditions (3.1) and (3.2) with

$$-1 < a \leq 0 \tag{3.5}$$

and for all $u \in \mathbb{R}, \rho \in \mathbb{M}$

$$\frac{L - aK}{mT} \leq f(u, \rho) - ar(u)\rho \leq \frac{K - aL}{MT}, \tag{3.6}$$

where M and m are defined by (2.6) and (2.7), respectively.

Theorem 3.3. If (2.1), (2.2), (3.5) and (3.6) hold, then Equation (1.1) has a positive periodic solution z satisfying $L \leq \|z\| \leq K$.

The proof follows along the lines of Theorem 3.2, and hence we omit.

4 Example

Let $a = -\frac{1}{50}$. Then the neutral differential equation

$$\frac{d}{dt}[x(t) - ax(t - \pi)] = \frac{1}{2}sin^2(t)x(t) - \frac{cos^2(t)}{x^2(t - \pi) + 100} - \frac{1}{25} \tag{4.1}$$

has a positive π-periodic solution x satisfying $0 \leq \|x\| \leq 2$. To see this, we have

$$f(u, \rho) = \frac{cos^2(u)}{\rho^2 + 100} + \frac{1}{25},\ r(u) = \frac{1}{2}sin^2(u) \text{ and } T = \pi.$$

A simple calculation yields

$$8.923 < M < 8.925, \text{ and } 0.382 < m < 0.383.$$

Let $K = 2$, and $L = 0$ and define the set $\mathbb{M} = \{\phi \in P_\pi : 0 \leq \|\phi\| \leq 2\}$. Then for $\rho \in [0, 2]$ we have

$$\begin{aligned} f(u, \rho) - ar(u)\rho &= \frac{cos^2(u)}{\rho^2 + 100} + \frac{1}{100}sin^2(u)\rho + \frac{1}{25} \\ &\leq \frac{1}{100} + \frac{1}{50} + \frac{1}{25} = 0.07 < \frac{K - aL}{MT}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(u, \rho) - ar(u)\rho &= \frac{cos^2(u)}{\rho^2 + 100} + \frac{1}{100}sin^2(u)\rho + \frac{1}{25} \\ &> \frac{1}{25} > \frac{L - aK}{mT}. \end{aligned}$$

By Theorem 3.3, Equation (4.1) has a positive π-periodic solution x such that $0 \leq \|x\| \leq 2$.

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