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Functional Differential Inclusions with Integral Boundary Conditions

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Abstract

In this paper, we investigate the existence of solutions for a class of second order functional differential inclusions with integral boundary conditions. By using suitable fixed point theorems, we study the case when the right hand side has convex as well as nonconvex values.

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1 Introduction

This paper is concerned with the existence of solutions of second order functional differential inclusions with integral boundary conditions. More precisely, in Section 3, we consider the second order functional differential inclusion,

$$x''(t) + \lambda x'(t) \in F(t, x_t), \text{ a.e. } \quad t \in [0, 1], \tag{1}$$

with initial function values,

$$x(t) = \phi(t), \ t \in [-r, 0], \tag{2}$$

and integral boundary condtions,

$$x(1) = \int_0^1 g(x(s))ds, \tag{3}$$

where $F : [0, 1] \times C([-r, 0], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a compact valued multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $\lambda < 0$, $\phi \in C([-r, 0], \mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous.

For any continuous function y defined on $[-r, 1]$, and any $t \in [0, 1]$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from $t - r$, up to the present time t.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [13], Karakostas and Tsamatos [16], Lomtatidze and Malaguti [21] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for instance, Brykalov [6], Denche and Marhoune [10], Jankowskii [15] and Krall [19]. Recently Ahmad, Khan and Sivasundaram [1, 17] have applied the generalized method of quasilinearization to a class of second order boundary value problem with integral boundary conditions. Some results on the existence of solutions for a class of boundary value problems for second order differential inclusions with integral conditions have been obtained by Belarbi and Benchohra [4]. In this paper we shall present three existence results for the problem (1)-(3), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler, while in the third result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposables values. These results extend to the multivalued case some previous results in the literature.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C([-r, 1], \mathbb{R})$ is the Banach space of all continuous functions from $[-r, 1]$ into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : -r \le t \le 1\}.$$

$L^1([0, 1], \mathbb{R})$ denotes the Banach space of measurable functions $x : [0, 1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt \quad \text{for all } \ x \in L^1([0, 1], \mathbb{R}).$$

$AC^1((0, 1), \mathbb{R})$ is the space of differentiable functions $x : (0, 1) \to \mathbb{R}$, whose first derivative, x', is absolutely continuous.

For a normed space $(X, |\cdot|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to P_{cl}(\mathrm{I\!R})$ is said to be measurable if for every $y \in \mathrm{I\!R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [2], Aubin and Frankowska [3], Deimling [9] and Hu and Papageorgiou [14].

Definition 2.1 *A multivalued map $F : [0,1] \times C([-r,0], \mathrm{I\!R}) \to \mathcal{P}(\mathrm{I\!R})$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto F(t,u)$ is measurable for each $u \in C([-r,0], \mathrm{I\!R})$;

(ii) $u \longmapsto F(t,u)$ is upper semicontinuous for almost all $t \in [0,1]$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1([0,1], \mathrm{I\!R}_+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leq \varphi_q(t) \ \text{ for all } \|u\| \leq q \text{ and for a.e. } t \in [0,1].$$

For each $x \in C([-r,1], \mathrm{I\!R})$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1([0,1], \mathrm{I\!R}) : v(t) \in F(t, x_t) \text{ a.e. } t \in [0,1]\}.$$

Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathrm{I\!R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $C([-r,0], \mathrm{I\!R})$. A subset A of $L^1([0,1], \mathrm{I\!R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,1]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathrm{I\!R}))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : [0,1] \times C([-r,0],\mathrm{I\!R}) \to \mathcal{P}(\mathrm{I\!R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([-r,1],\mathrm{I\!R}) \to \mathcal{P}(L^1([0,1],\mathrm{I\!R}))$$

by letting

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathrm{I\!R}) : w(t) \in F(t,x_t) \text{ for } a.e.\ t \in [0,1]\}.$$

The operator $\mathcal{F}$ is called the Nymetzki operator associated with F.

Definition 2.3 *Let $F : [0,1] \times \mathrm{I\!R} \to \mathcal{P}(\mathrm{I\!R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathrm{I\!R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [18]).

Definition 2.4 *A multivalued operator $N : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \le \gamma d(x,y), \quad \textit{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.1 *[20]. Let X be a Banach space. Let $F : [0,1] \times C([-r,0],X) \longrightarrow P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$, then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,1],X) &\longrightarrow P_{cp,c}(C([0,1],X)), \\ x &\longmapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Lemma 2.2 *[5]. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1], \mathrm{I\!R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g : Y \to L^1([0,1], \mathrm{I\!R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.3 *[8] Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Main Results

In this section, we are concerned with the existence of solutions for the problem (1)-(3) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

Definition 3.1 *A function $x \in C([-r, 1], \mathrm{I\!R}) \cap AC^1((0,1), \mathrm{I\!R})$ is said to be a solution of (1)–(3), if there exists a function $v \in L^1([0,1], \mathrm{I\!R})$ with $v(t) \in F(t, x_t)$, for a.e. $t \in [0,1]$, such that $x''(t) + \lambda x'(t) = v(t)$ a.e. on $[0,1]$, and the function x satisfies the conditions (2) and (3).*

We need the following auxiliary result. Its proof uses a standard argument.

Lemma 3.1 *For any $\sigma(t)$, $\rho(t) \in C([0,1], \mathrm{I\!R})$, the nonhomogeneous linear problem*

$$x''(t) + \lambda x'(t) = \sigma(t), \ \ a.e. \ t \in [0,1],$$

$$x(0) = a,$$

$$x(1) = \int_0^1 \rho(s)ds,$$

has a unique solution $x \in AC^1((0,1), \mathrm{I\!R})$ given by

$$x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)ds,$$

where

$$P(t) = \frac{1}{e^{-\lambda} - 1}\left[a(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 \rho(s)ds\right]$$

is the unique solution of the problem

$$x''(t) + \lambda x'(t) = 0, \ \ a.e. \ t \in [0,1],$$

$$x(0) = a, \ \ x(1) = \int_0^1 \rho(s)ds$$

and

$$G(t,s) = \frac{1}{(1-e^{-\lambda})} \begin{cases} \dfrac{(e^{-\lambda t}-1)(e^{-\lambda s}-e^{-\lambda})}{e^{-\lambda s}}, & 0 \le t \le s \le 1 \\ \dfrac{(e^{-\lambda s}-1)(e^{-\lambda t}-e^{-\lambda})}{e^{-\lambda s}}, & 0 \le s \le t \le 1 \end{cases}$$

is the Green's function associated to the corresponding homogeneous problem

$$x''(t) + \lambda x'(t) = 0, \quad a.e.\ t \in [0,1],$$

$$x(0) = 0, \quad x(1) = 0.$$

Let us introduce the following hypotheses which are assumed hereafter:

(H1) The function $F : [0,1] \times C([-r,0], \mathbb{R}) \to P_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;

(H2) There exists a constant $0 < c < 1$ such that

$$|g(x)| \le c|x|, \quad \text{for all } x \in \mathbb{R};$$

(H3) There exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and a function $p \in L^1([0,1], \mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|)$$

for each $(t,u) \in [0,1] \times C([-r,0], \mathbb{R})$;

(H4) There exists a number $M > 0$ such that

$$\frac{(1-c)M}{|\phi(0)| + G_0\psi(M)\displaystyle\int_0^1 p(s)ds} > 1,$$

where

$$G_0 = \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|.$$

Theorem 3.1 *Suppose that hypotheses (H1)–(H4) are satisfied. Then the BVP (1)–(3) has at least one solution.*

Proof. Transform the BVP (1)–(3) into a fixed point problem. Consider the operator

$$N(x) : \left\{ h \in C([-r,1], \mathbb{R}) : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ P(t) + \displaystyle\int_0^1 G(t,s)v(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

where

$$P(t) = \frac{1}{e^{-\lambda}-1}\left[\phi(0)(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t}-1)\int_0^1 g(x(s))ds\right] \text{ and } v \in S_{F,x}.$$

Remark 3.1 *Clearly, from Lemma 3.1, the fixed points of N are solutions to (1)–(3).*

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: *$N(x)$ is convex for each $x \in C([-r,1],\mathbb{R})$.*

Indeed, if $h_1,\ h_2$ belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [0,1]$ we have

$$h_i(t) = P(t) + \int_0^1 G(t,s)v_i(s)ds,\ (i = 1,2).$$

Let $0 \le d \le 1$. Then, for each $t \in [0,1]$, we have

$$(dh_1 + (1-d)h_2)(t) = P(t) + \int_0^1 G(t,s)[dv_1(s) + (1-d)v_2(s)]ds.$$

Since $S_{F,x}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2: *N maps bounded sets into bounded sets in $C([-r,1],\mathbb{R})$.*

Let $B_q = \{x \in C([-r,1],\mathbb{R}) : \|x\|_\infty \le q\}$ be a bounded set in $C([-r,1],\mathbb{R})$ and $x \in B_q$. Then for each $h \in N(x)$, there exists $v \in S_{F,x}$ such that

$$h(t) = P(t) + \int_0^1 G(t,s)v(s)ds.$$

From (H2) and (H3) we have

$$\begin{aligned}
|h(t)| &\le |P(t)| + \int_0^1 |G(t,s)||v(s)|ds \\
&\le |P(t)| + G_0\int_0^1 |v(s)|ds \\
&\le |\phi(0)| + cq + G_0\psi(q)\int_0^1 p(s)ds \\
&= |\phi(0)| + cq + G_0\psi(q)\|p\|_{L^1}.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([-r,1],\mathbb{R})$.*

Let $r_1, r_2 \in [0,1],\ r_1 < r_2$ and B_q be a bounded set of $C([-r,1],\mathbb{R})$ as in Step 2 and $x \in B_q$. For each $h \in N(x)$

$$\begin{aligned}
|h(r_2) - h(r_1)| &\le |P(r_2) - P(r_1)| + \int_0^1 |G(r_2,s) - G(r_1,s)||v(s)|ds \\
&\le |P(r_2) - P(r_1)| + \psi(q)\int_0^1 |G(r_2,s) - G(r_1,s)|p(s)ds.
\end{aligned}$$

The right hand side tends to zero as $r_2 - r_1 \to 0$. The equicontinuity for the cases $r_1,\ r_2 \in [-r, 0]$ is obvious. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem, we can conclude that $N : C([-r,1], \mathbb{R}) \longrightarrow \mathcal{P}(C([-r,1], \mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $x_n \to x_*,\ h_n \in N(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. $h_n \in N(x_n)$ means that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$h_n(t) = P_n(t) + \int_0^1 G(t,s)v_n(s)ds,$$

where

$$P_n(t) = \frac{1}{e^{-\lambda} - 1}\left[\phi(0)(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g(x_n(s))ds\right].$$

We must show that there exists $h_* \in S_{F,x_*}$ such that, for each $t \in [0,1]$,

$$h_*(t) = P_*(t) + \int_0^1 G(t,s)v_*(s)ds,$$

where

$$P_*(t) = \frac{1}{e^{-\lambda} - 1}\left[\phi(0)(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g(x_*(s))ds\right].$$

Clearly we have

$$\|(h_n - P_n) - (h_* - P_*)\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

Consider the continuous linear operator

$$\Gamma : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^1 G(t,s)v(s)ds.$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$\Big(h_n(t) - P_n(t)\Big) \in \Gamma(S_{F,x_n}).$$

Since $x_n \to x_*$, it follows from Lemma 2.1 that

$$h_*(t) = P_*(t) + \int_0^1 G(t,s)v_*(s)ds$$

for some $v_* \in S_{F,x_*}$.

Step 5: *A priori bounds on solutions.*

Let x be a possible solution of the problem (1)–(3). Then, there exists $v \in L^1([0,1],\mathrm{I\!R})$ with $v \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$x(t) = P(t) + \int_0^1 G(t,s)v(s)ds.$$

This implies by (H2) and (H3) that, for each $t \in [0,1]$, we have

$$\begin{aligned} |x(t)| &\leq |\phi(0)| + c\int_0^t |x(s)|ds + G_0\int_0^1 p(s)\psi(\|x_s\|)ds \\ &\leq |a| + c\|x\|_\infty + G_0\psi(\|x\|_\infty)\int_0^1 p(s)ds. \end{aligned}$$

Consequently

$$(1-c)\|x\|_\infty \leq \left(|\phi(0)| + G_0\psi(\|x\|_\infty)\int_0^1 p(s)ds\right).$$

Then by (H4), there exists M such that $\|x\|_\infty \neq M$.

Let

$$U = \{x \in C([-r,1],\mathrm{I\!R}) : \|x\|_\infty < M\}.$$

The operator $N : \overline{U} \to \mathcal{P}(C([0,1],\mathrm{I\!R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [11], we deduce that N has a fixed point x in $\overline{U}$ which is a solution of the problem (1)–(3). This completes the proof.

We present now a result for the problem (1)-(3) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for multivalued map given by Covitz and Nadler [8]. We need the following hypotheses:

(H5) $F : [0,1] \times C([-r,0],\mathrm{I\!R}) \longrightarrow P_{cp}(\mathrm{I\!R})$ has the property that $F(\cdot,u) : [0,1] \to P_{cp}(\mathrm{I\!R})$ is measurable for each $u \in C([-r,0],\mathrm{I\!R})$;

(H6) $H_d(F(t,u),F(t,\overline{u})) \leq l(t)\|u-\overline{u}\|$ for almost all $t \in [0,1]$ and $u,\ \overline{u} \in C([-r,0],\mathrm{I\!R})$ where $l \in L^1([0,1],\mathrm{I\!R})$ and $d(0,F(t,0)) \leq l(t)$ for almost all $t \in [0,1]$,

(H7) There exists a nonnegative constant d such that

$$|g(x) - g(\overline{x})| \leq d|x - \overline{x}|, \text{ for all } x,\ \overline{x} \in \mathrm{I\!R}.$$

Theorem 3.2 *Assume that (H5)-(H7) are satisfied. If*

$$c + G_0\|l\|_{L^1} < 1,$$

then the BVP (1)-(3) has at least one solution.

Remark 3.2 *For each $x \in C([-r,1],\mathbb{R})$, the set $S_{F,x}$ is nonempty since by (H5), F has a measurable selection (see [7], Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1: $N(x) \in P_{cl}(C([-r,1],\mathbb{R}))$ *for each* $x \in C([-r,1],\mathbb{R})$.

Indeed, let $(x_n)_{n\geq 0} \in N(x)$ such that $x_n \longrightarrow \tilde{x}$ in $C([-r,1],\mathbb{R})$. Then, $\tilde{x} \in C([-r,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$x_n(t) = P(t) + \int_0^1 G(t,s)v_n(s)ds.$$

Using the fact that F has compact values and from (H6), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,1],\mathbb{R})$ and hence $v \in S_{F,x}$. Then, for each $t \in [0,1]$,

$$x_n(t) \longrightarrow \tilde{x}(t) = P(t) + \int_0^1 G(t,s)v(s)ds.$$

So, $\tilde{x} \in N(x)$.

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(N(x), N(\overline{x})) \leq \gamma \|x - \overline{x}\|_\infty \text{ for each } x, \overline{x} \in C([-r,1],\mathbb{R}).$$

Let $x, \overline{x} \in C([-r,1],\mathbb{R})$ and $h_1 \in N(x)$. Then, there exists $v_1(t) \in F(t,x_t)$ such that for each $t \in [0,1]$

$$h_1(t) = P(t) + \int_0^1 G(t,s)v_1(s)ds.$$

From (H6) it follows that

$$H_d(F(t,x_t), F(t,\overline{x}_t)) \leq l(t)\|x_t - \overline{x}_t\|.$$

Hence, there exists $w \in F(t,\overline{x}_t)$ such that

$$|v_1(t) - w| \leq l(t)\|x_t - \overline{x}_t\|, \ t \in [0,1].$$

Consider $U : [0,1] \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)\|x_t - \overline{x}_t\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{x}_t)$ is measurable (see Proposition III.4 in [7]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t,\overline{x}_t)$, and for each $t \in [0,1]$,

$$|v_1(t) - v_2(t)| \leq l(t)\|x_t - \overline{x}_t\|.$$

Let us define for each $t \in [0,1]$

$$h_2(t) = \bar{P}(t) + \int_0^1 G(t,s)v_2(s)ds,$$

where

$$\bar{P}(t) = \frac{1}{e^{-\lambda}-1}\left[\phi(0)(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g(\bar{x}(s))ds\right].$$

We have

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq |P(t) - \bar{P}(t)| + \int_0^1 |G(t,s)||v_1(s) - v_2(s)|ds \\ &\leq c\|x - \bar{x}\|_\infty + \int_0^1 |G(t,s)|l(s)\|x_s - \overline{x}_s\|ds. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq (c + G_0\|l\|_{L^1})\,\|x - \overline{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of x and $\overline{x}$, it follows that

$$H_d(N(x), N(\overline{x})) \leq (c + G_0\|l\|_{L^1})\,\|x - \overline{x}\|_\infty.$$

So, N is a contraction and thus, by Lemma 2.3, N has a fixed point x which is solution to (1)–(3). The proof is complete.

In this part, by using the nonlinear alternative of Leray Schauder type combined with the selection theorem of Bresssan and Colombo for semi-continuous maps with decomposable values, we shall establish an existence result for the problem (1)–(3). We need the following hypothesis:

(H8) $F : [0,1] \times C([-r,0], \mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that:
a) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b)$u \mapsto F(t,u)$ is lower semi-continuous for each $t \in [0,1]$.

The following lemma is of great importance in the proof of our next result.

Lemma 3.2 *[12] Let $F : [0,1] \times C([-r,0], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assume $(H3)$ and $(H8)$ hold. Then F is of l.s.c. type.*

Theorem 3.3 *Assume that (H2), (H3), (H4) and (H8) hold. Then the BVP (1)–(3) has at least one solution.*

Proof. Note that (H3), (H8) and Lemma 3.2 imply that F is of l.s.c. type. Then from Lemma 2.2, there exists a continuous function $f : C([-r,1], \mathbb{R}) \to L^1([0,1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([-r,1], \mathbb{R})$.

Consider the problem

$$x''(t) + \lambda x'(t) = f(x_t), \text{ a.e. } t \in [0,1], \tag{4}$$

$$x(t) = \phi(t), \ t \in [-r, 0], \tag{5}$$

$$x(1) = \int_0^1 g(x(s))ds. \tag{6}$$

It is clear that if $x \in C([-r,1], \mathbb{R}) \cap AC^1((0,1), \mathbb{R})$ is a solution of (4)–(6), then x is a solution to the problem (1)–(3). Transform the problem (4)–(6) into a fixed point theorem. Consider the operator $\bar{N}$ defined by

$$(\bar{N}x)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ P(t) + \displaystyle\int_0^1 G(t,s) f(x(s))ds, & \text{if } t \in J. \end{cases}$$

We can easily show that $\bar{N}$ is continuous and completely continuous. The remainder of the proof is similar to that of Theorem 3.1.

References

[1] B. Ahmad, R. A. Khan and S. Sivasundaram, Generalized quasilinearization method for a first order differential equation with integral boundary condition, *J. Cont., Disc. Impulsive Sys. Ser. A Math. Anal.* **12** (2) (2005), 289–296.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[3] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[4] A. Belarbi and M. Benchohra, Existence results for nonlinear boundary-value problems with integral boundary conditions, *Electron. J. Differ. Eqns.* **6** (2005), 1-10.

[5] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

[6] S. A. Brykalov, A second order nonlinear problem with two-point and integral boundary conditions, *Georgian Math. J.* **1** (1994), 243-249.

[7] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[8] H. Covitz and S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.

[9] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[10] M. Denche, A. L. Marhoune, High order mixed-type differential equations with weighted integral boundary conditions, *Electron. J. Differ. Eqns.*, vol (2000), **60**, 1-10.

[11] J. Dugundji and A. Granas, *Fixed Point Theory*, Springer-Verlag, New York, 2005.

[12] M. Frigon and A. Granas, Théorèmes d'existence pour des inclusions différentielles sans convexité, *C. R. Acad. Sci. Paris Sér. I Math.* **310** (12) (1990), 819-822.

[13] J. M. Gallardo, Second order differential operators with integral boundary conditions and generation of semigroups, *Rocky Mountain J. Math.* **30** (2000), 1265-1292.

[14] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory* **I**, Kluwer, Dordrecht, 1997.

[15] T. Jankowskii, Differential equations with integral boundary conditions, *J. Comput. Appl. Math.* **147** (2002), 1-8.

[16] G. L. Karakostas and P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary value problems, *Electron. J. Diff. Eqns.*, vol. (2002), **30**, 1-17.

[17] R. A. Khan, The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions, *Elec. J. Qual. Theo. Diff. Eqns.* (2003), **10**, 1-15.

[18] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[19] A.M. Krall, The adjoint of a differential operator with integral boundary conditions, *Proc. Amer. Math. Soc.* **16** (1965), 738-742.

[20] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.

[21] A. Lomtatidze and L. Malaguti, On a nonlocal boundary value problems for second order nonlinear singular differential equations, *Georg. Math. J.* **7** (2000), 133-154.

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