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Computation of radial solutions of semilinear equations

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Abstract

We express radial solutions of semilinear elliptic equations on R^n as convergent power series in r, and then use Pade approximants to compute both ground state solutions, and solutions to Dirichlet problem. Using a similar approach we have discovered existence of singular solutions for a class of subcritical problems. We prove convergence of the power series by modifying the classical method of majorants.

Key words: Semilinear elliptic equations, Pade approximants, convergence of solutions.

AMS subject classification: 65L10, 35J60.

1 Introduction

Solving initial value problems by power series provides an attractive alternative to classical methods of numerical analysis, like Runge-Kutta method. The idea of Runge-Kutta method is to replace a five term Taylor approximation with function evaluations. This method was developed way before the advent of computers. Similarly, the main numerical methods of mathematical physics, the finite differences and the finite elements, have been initially developed for solving linear problems by hand. Today people solve nonlinear problems on computers. Since symbolic capabilities have become widely available recently, one can easily perform repeated differentiations needed for the Taylor approximations. A major attraction of power series method is that one can easily estimate the truncation error, making possible *exact* computations, see e.g. O. Aberth [1]. In this work we develop series approximations of radial solutions of semilinear elliptic equations on R^n.

There is a lot of interest in the radial solutions of semilinear problems of the type

$$\Delta u + f(u) = 0,$$

both on balls in R^n, and on the entire space. With $r = |x|$, and together with initial conditions, the equation becomes

$$\begin{aligned} &u'' + \tfrac{n-1}{r}u' + f(u) = 0 \\ &u(0) = u_0, \;\; u'(0) = 0, \end{aligned} \tag{1.1}$$

and we wish to compute the solution as a power series in r, for any initial value of u_0. We begin by showing that the power series solution contains only even powers of r, i.e. $u(r) = \Sigma_{i=0}^{\infty} a_i \, r^{2i}$, with $a_0 = u_0$. We then derive a recursive formula for a_i, which is easy to implement in *Mathematica.* One can quickly compute a lot of terms of the series, giving highly accurate approximation for small r, practically an exact solution. The problem is that the power series solution usually has a finite radius of convergence, which can be small (if $f(u_0)$ is large). We need another approximation of solutions, which is defined for all r, and which agrees with the power series approximation for small r. A classical way of doing so is to use Pade approximations. These are rational functions, whose power series expansion agrees, as far as possible, with that of the function being approximated, see e. g. G. A. Baker [2]. We found this approach to be remarkably successful, providing us with practically closed form solutions for both Dirichlet problem, and ground state case. It is important for us that highly accurate Pade approximants are provided by *Mathematica*, with just two easy commands.

A natural question is why Pade approximants are close to the solutions of (1.1). It is known that real analytic functions, which vanish on some interval, are identically zero. It is known that solution $u(r)$ of (1.1) is analytic, in case $f(u)$ is analytic. For r small, Pade approximants are close to power series approximants, which are close to $u(r)$. Hence the left hand side of (1.1), which is analytic, is small for small r, and it has to remain small for a much wider range of r (at least until $u(r)$ becomes zero, in case of Dirichlet problem). Thus the question of justification of a numerical method is naturally connected to a theoretical issue of analyticity of solutions. We prove in Section 4 that solution $u(r)$ of (1.1) is analytic for non-integer n too, by modifying the classical methods of majorants. The crucial step involves majorization of solutions of (1.1) by solutions of certain first order equation.

Using a similar approach we have discovered existence of singular positive solution for the problem

$$u'' + \frac{n-1}{r}u' + \lambda u + u^p = 0, \quad u'(0) = 0, \quad u(1) = 0$$

not only in the supercritical range $p > \frac{n+2}{n-2}$, which is well known (see F. Merle and L. A. Peletier [6]), but also for subcritical p's, satisfying $p > \frac{n}{n-2}$, which is rather surprising. We obtain the singular solution in the form $u(r) = r^{-2/(p-1)} \Sigma_{i=1}^{\infty} a_i r^{2i}$, and compute the unique $\lambda = \bar{\lambda}$ at which this solution occurs. Our computations indicate convergence of the above series for small $r > 0$, but we did not carry out the proof.

2 Series solution

Let $u = u(r)$ be a radial solution of the initial value problem

$$\begin{aligned} u'' + \tfrac{n-1}{r}u' + f(u) &= 0 \\ u(0) = u_0, \quad u'(0) &= 0, \end{aligned} \tag{2.1}$$

where $f(u) \in C^\infty[0, \infty)$, and $u_0 > 0$. It will be convienient for us to consider solutions of (2.1) for both positive and negative r, even though one usually has $r \geq 0$ for the polar coordinate r. We wish to compute the solution as a series near $r = 0$, i.e. $u(r) = \Sigma_{i=0}^{\infty} b_i\, r^i$. The following lemma shows that $b_i = 0$ for all odd i.

Lemma 2.1 *Any solution of the problem (2.1) is an even function.*

Proof: Even though the initial value problem has a singularity at $r = 0$, it was shown in L. A. Peletier and J. Serrin [7] that it has the properties of regular initial value problems, in particular there is uniqueness of solutions to (2.1) for small $|r|$. We may also consider solutions of (2.1) in case $r < 0$. Observe that the change of variables $r \to -r$ leaves (2.1) invariant. If solution was not even, we would have two different solutions of (2.1) for $r > 0$, contradicting the just mentioned uniqueness result. $\diamondsuit$

In view of the lemma, we may express the solution of (2.1) as $u(r) = \Sigma_{i=0}^{\infty} a_i\, r^{2i}$. Numerically, we shall of course be computing the partial sums

$u(r)=\Sigma_{i=0}^{k} a_i\, r^{2i}$, which we expect to provide us with a solution, up to the terms of order $O(r^{2k+2})$. Write

$$u(r)=\bar{u}(r)+a_k r^{2k}, \tag{2.2}$$

where we regard $\bar{u}(r)$ as already computed, and the question is how to get a_k. (We begin with $\bar{u}(r)=u_0$.) If we plug our ansatz (2.2) into (2.1), we have

(2.3)
$$2k\,(n+2k-2)\,a_k r^{2k-2}=-f\left(\bar{u}(r)+a_k r^{2k}\right)-\bar{u}''-\frac{n-1}{r}\bar{u}'+O(r^{2k+2}),$$

and on the surface it may appear that we need to solve a nonlinear equation for a_k. It turns out that we can get an explicit formula for a_k. Express

$$a_k=-\frac{1}{\gamma_k}\lim_{r\to 0}\frac{f\left(\bar{u}(r)+a_k r^{2k}\right)+\bar{u}''+\frac{n-1}{r}\bar{u}'}{r^{2k-2}}, \tag{2.4}$$

where $\gamma_k=2k\,(n+2k-2)$. Using two term Taylor expansion of the first function, we have

$$\lim_{r\to 0}\frac{f(\bar{u}(r)+a_k r^{2k})+\bar{u}''+\frac{n-1}{r}\bar{u}'}{r^{2k-2}}=\lim_{r\to 0}\frac{f(\bar{u}(r))+\bar{u}''+\frac{n-1}{r}\bar{u}'}{r^{2k-2}}.$$

Observe that when we repeatedly apply the L'Hôpital's rule, all terms involving $\bar{u}$ will vanish, and we have

$$\lim_{r\to 0}\frac{f(\bar{u}(r))+\bar{u}''+\frac{n-1}{r}\bar{u}'}{r^{2k-2}}=\frac{1}{(2k-2)!}\frac{d^{2k-2}}{dr^{2k-2}}f(\bar{u}(r))\big|_{r=0}.$$

We arrive at an explicit formula for the coefficients:

$$a_k=-\frac{1}{2k\,(n+2k-2)\,(2k-2)!}\frac{d^{2k-2}}{dr^{2k-2}}f(\bar{u}(r))\big|_{r=0}, \tag{2.5}$$

where $\bar{u}(r)=\Sigma_{i=1}^{k-1} a_i\, r^{2i}$. The repeated differentiations in (2.5) can be easily performed using symbolic software like *Mathematica*. (Actually, *Mathematica* is able to compute $\lim_{r\to 0}\frac{f(\bar{u}(r))+\bar{u}''+\frac{n-1}{r}\bar{u}'}{r^{2k-2}}$ too, in case u_0 is rational, presumably by using the above formula.)

Example We have solved the problem

$$u''+\frac{3}{r}u'+u^3=0,\quad u(0)=1,\ u'(0)=0,$$

and obtained a series solution, corresponding to a known explicit ground-state solution $u(r) = \frac{8}{8+r^2}$.

We got lucky in this example. After computing a series solution, we have identified the sum of the series, $u(r) = \frac{8}{8+r^2}$. This function gives solution of the problem for all $r > 0$, while the corresponding series has radius of convergence $\sqrt{8}$. In general, the sum of a series solution cannot be identified, and moreover the radius of convergence may be small (if $f(u(0))$ is large). The problem is that power series approximation has a limited range. We need another approximation of $u(r) = \frac{8}{8+r^2}$, which is defined for all r. A classical way is to use Pade approximations.

3 Pade approximations

Example We have solved the problem

$$u'' + \frac{2}{r}u' + u^5 = 0, \quad u(0) = 1,\ u'(0) = 0.$$

It is connected to the previous example in the sense that they both give radial solutions of the critical PDE, with applications to geometry,

$$\Delta u + u^{\frac{n+2}{n-2}} = 0,$$

where n is the dimension of the space. (We use series to solve serious PDE's.) Similarly to the previous example, we wish to compute the ground-state solution. We computed a series solution, which begins with

$$u(r) = 1 - \frac{1}{6}r^2 + \frac{1}{24}r^4 - \frac{5}{432}r^6 + \frac{35}{10368}r^8 - \frac{7}{6912}r^{10} + \dots . \tag{3.1}$$

It seems unlikely that one can identify the sum of this series through elementary functions (even though there is a closed formula for this sum, which we mention below). After computing 55 terms of this series (powers of up to r^{110}), we have used a built in *Mathematica* command to approximate this polynomial of power 110 by its Pade approximation, centered at $r = 0$, and using polynomials of power 50 for both the numerator and denominator of Pade approximation. We called this rational approximation $p(r)$. To see how well it approximates the solution, we plugged it into our equation, and considered the "defect"

$$q(r) \equiv p''(r) + \frac{2}{r}p'(r) + p^5(r).$$

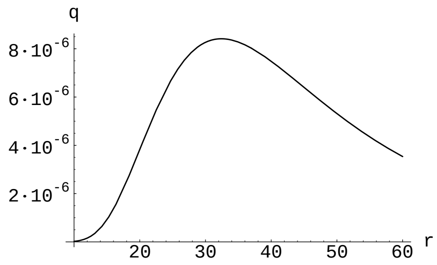

Figure 1: The defect of Pade approximation

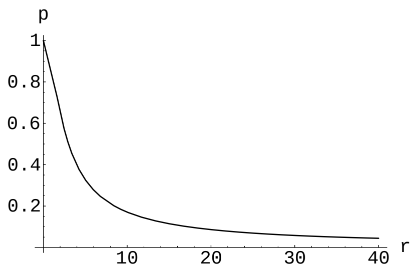

Figure 2: Pade approximation of the solution

The graph of $q(r)$ is given in Figure 1. One can see that $q(r)$ is small. It gets even smaller for $r > 60$, and it is practically zero when $r \in (0, 10)$. We see that $p(r)$ is an excellent approximation to the solution. The graph of $p(r)$ is given in Figure 2. This problem has an exact solution, $u(r) = \dfrac{\sqrt{3}}{\sqrt{3 + r^2}}$. We used it to confirm both the validity of the series (3.1), and the accuracy of the Pade approximants (this accuracy can be improved, increasing powers of both numerator and denominator, with the power of denominator chosen larger than numerator to allow decay to zero for large r).

Remark There is a closed form solution for the critical initial value problem

$$u'' + \frac{n-1}{r}u' + u^{\frac{n+2}{n-2}} = 0, \quad u(0) = \alpha, \;\; u'(0) = 0.$$

It is $u(r) = \alpha \left(\dfrac{n(n-2)}{n(n-2) + \alpha^{4/(n-2)}\, r^2} \right)^{\frac{n-2}{2}}$, see e.g. [8] or [5]. Our previous examples involved the cases of $n = 3, \, 4$.

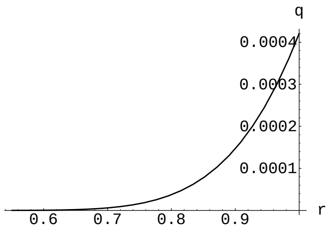

Figure 3: Defect of the approximation of the solution

The same approach works equally well for Dirichlet problems.

Example On a unit ball in R^3 we consider a sub-critical problem

$$\Delta v + v^4 = 0, \quad \text{for } |x| < 1, \;\; v = 0 \;\text{ when } |x| = 1. \tag{3.2}$$

By the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [4] the problem (3.2) has a unique positive solution, which is moreover radially symmetric, i.e. $u = u(r)$, where $r = |x|$. It follows that (3.2) may be replaced by the ODE

$$v'' + \frac{2}{r}v' + v^4 = 0, \quad v'(0) = 0, \;\; v(1) = 0. \tag{3.3}$$

We begin by solving the initial-value problem

$$u'' + \frac{2}{r}u' + u^4 = 0, \quad u(0) = 1, \;\; u'(0) = 0.$$

Similarly to the previous example, we have computed 55 even terms of the power series approximation at $r = 0$, and then approximated this polynomial of power 110 by its Pade approximation, centered at $r = 0$, and using polynomials of power 50 for both the numerator and denominator of Pade approximation. We used this approximation of $u(r)$ to find its first root $R \approx 14.9716$. Then by standard scaling, $v(r) = R^{2/3}u(R\,r)$ gives an approximation to the solution of (3.3). To see how good is this approximation, we again computed $q(r) = v'' + \frac{2}{r}v' + v^4$. In Figure 3 we give the graph of $q(r)$, which shows that the accuracy is high. ($q(r)$ is even smaller when $r \in (0, 0.6)$.) In Figure 4 we graph the approximation of $v(r)$.

4 Analyticity of solutions

In all examples we tried, the series solutions for the problem (2.1) appeared to be convergent, although with finite (sometimes small) radius of conver-

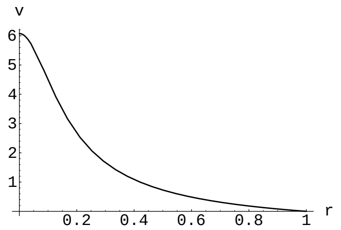

Figure 4: Solution of the Dirichlet problem (3.2)

gence (evaluating a_{k+1}/a_k for large k allows one to approximate the radius of convergence). In this section we prove convergence of the series solution for the problem (2.1), with any real $n \geq 1$.

We begin by showing an alternative derivation for the coefficients of the series solution. We rewrite the equation (2.1) as

$$ru'' + (n-1)u' + rf(u) = 0, \quad u(0) = u_0, \; u'(0) = 0. \tag{4.1}$$

Differentiating,

$$ru''' + nu'' + f(u) + rf'(u)u' = 0$$

Setting $r = 0$, we express $u''(0) = -\frac{1}{n}f(u_0)$. Differentiating (4.1) three times, we can express $u''''(0)$, and in general, differentiating $2k-1$ times, we express

$$(n + 2k - 2)u^{(2k)}(0) = -(2k-1)\frac{d^{2k-2}}{dr^{2k-2}}f(u(r))|_{r=0}, \tag{4.2}$$

which means that the coefficients of the Maclauren series for $u(r)$, $a_{2k} = \frac{u^{(2k)}(0)}{(2k)!}$, are the same as given by (2.5).

In the numerical examples above, solution was given by alternating power series. The following result describes a class of $f(u)$, for which this is true. (We do not claim yet that the series is convergent.)

Theorem 4.1 *Assume that $f^k(u) > 0$ for all $k \geq 0$ and $u > 0$. Then the coefficients of the series solution are alternating.*

Proof: Suffices to show that the series solution has positive coefficients in $\xi = -r^2$. Making this change of variables in (2.1), we have

$$4\xi u_{\xi\xi} + 2nu_\xi = f(u), \quad u(0) = u_0 > 0, \; u_\xi(0) = \frac{1}{2n}f(u_0) > 0. \tag{4.3}$$

Differentiating the equation (4.3), and setting $\xi = 0$, the way we just did, we see that $\frac{D^k u}{d\xi^k}(0) > 0$ for all $k \geq 1$, and hence the Maclauren series for $u(\xi)$ has all of its coefficients positive. $\diamondsuit$

We recall the classical notion of majorizing series. Let $u(r) = \Sigma_{k=0}^{\infty} u_k r^k$ and $v(r) = \Sigma_{k=0}^{\infty} v_k r^k$ be two power series. We say that v *majorizes* (is a *majorant* of) u if $|u_k| \leq v_k$ for all k. The following lemma is standard, but we include its simple proof.

Lemma 4.1 *Assume that the series $u(r) = \Sigma_{k=0}^{\infty} u_k r^k$ converges for $|r| \leq s$. Then it has a majorant of the form $\frac{Cs}{s-r}$.*

Proof: Since the series $\Sigma_{k=0}^{\infty} u_k s^k$ converges, we can find a constant C, so that $|u_k| s^k \leq C$. This means that the series $u(r) = \Sigma_{k=0}^{\infty} u_k r^k$ has a majorant $\Sigma_{k=0}^{\infty} \frac{C}{s^k} r^k = \frac{Cs}{s-r}$. $\diamondsuit$

Theorem 4.2 *Assume that $f(u)$ is a real analytic function. Then the solution of the problem (2.1), with any real $n \geq 1$, is analytic, i.e. the series $u(r) = \Sigma_{k=0}^{\infty} \frac{u^{(2k)}(0)}{(2k)!} r^{2k}$, where $u^{(2k)}(0)$ are computed using formula (4.2), converges for small r.*

Proof: Suffices to show that the series in $\xi = -r^2$ converges. Recall that the coefficients of the series in ξ can be computed by repeated differentiations of (4.3). Letting $u - u_0 = v$, we may assume that $u_0 = 0$ in (4.3). Assume that $f(u) = \Sigma_{k=0}^{\infty} f_k u^k$, and this series converges for $|u| \leq s$. The problem (4.3) takes the form

$$4\xi u_{\xi\xi} + 2nu_{\xi} = \Sigma_{k=0}^{\infty} f_k u^k, \quad u(0) = 0, \;\; u_{\xi}(0) = \frac{1}{2n} f(u_0) > 0. \tag{4.4}$$

By Lemma 4.1, $f(u)$ has a majorant $\frac{Cs}{s-u}$, for C large enough, i.e.

$$f_k \leq \frac{C}{s^k}. \tag{4.5}$$

We claim that the solution of (4.4) $u(\xi)$ is majorized by $v(\xi)$, which is the solution of

$$4\xi v_{\xi\xi} + 2nv_{\xi} = \frac{Cs}{s-v} = \Sigma_{k=0}^{\infty} \frac{C}{s^k} v^k, \;\; v(0) = 0, \;\; v_{\xi}(0) = \frac{1}{2n} f(u_0) > 0. \tag{4.6}$$

The proof is by induction. Assume that $|u^{(i)}(0)| \leq v^{(i)}(0)$ for all $i \leq m-1$, and we claim that

$$|u^{(m)}(0)| \leq v^{(m)}(0). \tag{4.7}$$

(Observe that $v^{(i)}(0) > 0$ for all $i \geq 1$, as follows by the differentiating of (4.6), since the function $\frac{Cs}{s-v}$ has positive derivatives.) To calculate $u^{(m)}(0)$, we differentiate (4.4) $m-1$ times and set $\xi = 0$, obtaining

$$(4m-4+2n)u^{(m)}(0) = \Sigma_{k=0}^{\infty} f_k \, \frac{d^{m-1}}{dr^{m-1}} u^k(r)\,|_{r=0}.$$

Similarly,

$$(4.8) \qquad (4m-4+2n)v^{(m)}(0) = \Sigma_{k=0}^{\infty} \frac{C}{s^k} \, \frac{d^{m-1}}{dr^{m-1}} v^k(r)\,|_{r=0}.$$

The term $\frac{d^{m-1}}{dr^{m-1}} u^k(r)\,|_{r=0}$ is a (possibly long) sum of products of powers of u and its various derivatives at $r = 0$, while $\frac{d^{m-1}}{dr^{m-1}} v^k(r)\,|_{r=0}$ is *exactly the same* type of a sum, involving v and its derivatives at $r = 0$. Using our inductive hypothesis and (4.5), we conclude the claim (4.7).

We claim that solution of (4.6) is in turn majorized by the solution of the first order problem

$$(4.9) \qquad 2nw_\xi = \frac{Cs}{s-w} = \Sigma_{k=0}^{\infty} \frac{C}{s^k} w^k, \quad w(0) = 0,$$

provided that we increase C (if necessary, and the increase is made in both (4.8) and (4.9)) so that the value of $w_\xi(0)$ computed from (4.9), $w_\xi(0) = \frac{C}{2n}$, is greater than $v_\xi(0) = \frac{1}{2n} f(u_0)$. The proof is by induction, as before. We assume that $v^{(i)}(0) \leq w^{(i)}(0)$ for $i \leq m-1$. Compute

$$2nw^{(m)}(0) = \Sigma_{k=0}^{\infty} \frac{C}{s^k} \, \frac{d^{m-1}}{dr^{m-1}} w^k(r)\,|_{r=0}.$$

By the inductive hypothesis the right hand side here is greater than that in (4.8) (both involve only positive terms), and hence $v^{(m)}(0) \leq w^{(m)}(0)$ for all $m \geq 1$.

Finally, we explicitly integrate the equation (4.9), obtaining

$$w(\xi) = s - \sqrt{s^2 - \frac{Cs}{n}\,\xi},$$

which can be represented by a convergent power series for small r (since the binomial series converges for small r). $\diamondsuit$

Remark For integer $n \geq 1$ this result has been known for a long time. Indeed, we have seen that a formal power series solution exists for the PDE version of (2.1), and then the proof of Cauchy-Kovalevskaya theorem implies convergence of the series.

5 Application to singular solutions

We consider positive solution of the Dirichlet problem on a unit ball in R^n

$$(5.1) \qquad \Delta u + \lambda u + u^p = 0, \text{ for } |x| < 1, \; u = 0 \text{ on } |x| = 1.$$

Here λ is a positive parameter, and $p > 1$. It is well known that when $p > \frac{n+2}{n-2}$, the critical exponent, there is a unique $\bar{\lambda}$ at which a singular solution exists, see F. Merle and L. A. Peletier [6]. Moreover, the significance of the critical solution is by now well understood: the curve of positive solutions of (5.1) tends to infinity, as $\lambda \to \bar{\lambda}$, and it approaches the singular solution for $r \neq 0$. Using power series expansion, we show that there is a unique $\bar{\lambda}$ at which a singular solution exists for $p > \frac{n}{n-2}$, i.e. for some sub-critical equations too.

It is easy to see that a singular solution must blow up like $r^{-\frac{2}{p-1}}$ near $r = 0$. We denote $\theta = -\frac{2}{p-1}$, and $A = -\theta(n + \theta - 2)$. Observe that $A > 0$ if and only if $p > \frac{n}{n-2}$. We shall obtain a solution of (5.1) in the form

$$(5.2) \qquad u(r) = r^{-\frac{2}{p-1}} v(r),$$

where $v(r) \in C^2[0, 1)$ satisfies

$$(5.3) \qquad v(0) = A^{\frac{1}{p-1}}.$$

We assume that $n \geq 3$, and $p > \frac{n}{n-2}$. In case $p < \frac{n+4}{n}$ we assume additionally that (this assumption is not needed if $p \geq \frac{n+4}{n}$)

$$(5.4) \qquad 2i\,(2i + n + 2\theta - 2) + (p - 1)A > 0 \text{ for all integers } i \geq 1.$$

Observe that (5.4) gives a finite set of conditions, since the condition $p > \frac{n}{n-2}$ implies that $\theta > -n + 2$, and then (5.4) holds if $i > \frac{n-2}{2}$. If we assume a stronger condition $p \geq \frac{n+4}{n}$ (which is larger than $\frac{n-2}{2}$ for $n \geq 4$), then $\theta > -n/2$, and the condition (5.4) holds. Since $\frac{n+4}{n} < \frac{n+2}{n-2}$, we see that the condition (5.4) is satisfied for supercritical p.

Radial solutions of (5.1) satisfy

$$(5.5) \qquad u'' + \frac{n-1}{r} u' + \lambda u + u^p = 0, \text{ for } r < 1, \; u(1) = 0.$$

Using the ansatz (5.2) in (5.1), we see that $v(r)$ satisfies

$$(5.6) \qquad r^2 v'' + (n - 1 + 2\theta) r v' + \left[\theta(n + \theta - 2) + \lambda r^2\right] v + v^p = 0.$$

We are looking for a solution of (5.6) satisfying

$$v'(0) = v(1) = 0. \tag{5.7}$$

We now scale λ out of (5.6), by letting $r = \frac{1}{\sqrt{\lambda}}\xi$. Then $v(\xi)$ satisfies

$$\xi^2 v'' + (n-1+2\theta)\xi v' + \left[-A+\xi^2\right] v + v^p = 0, \tag{5.8}$$

and $\dfrac{dv}{d\xi}(0) = 0$. Setting here $\xi = 0$, we see that $v(0)$ satisfies the condition (5.3). Hence this condition is necessary for existence of singular solution. We shall produce a power series solution of (5.8).

The equation (5.8) can be considered for $\xi < 0$ too. Since this equation is invariant under the transformation $\xi \to -\xi$, it follows that any classical solution is an even function, and so its power series expansion contains only even powers. We then look for solution of (5.8) in the form

$$v(\xi) = \Sigma_{i=0}^{\infty} a_i \xi^{2i}. \tag{5.9}$$

Plugging into (5.8), we have

$$a_0 + \Sigma_{i=1}^{\infty} c_i \xi^{2i} + \left(a_0 + \Sigma_{i=1}^{\infty} a_i \xi^{2i}\right)^p = 0, \tag{5.10}$$

where

$$c_i = \left[2i(2i+n+2\theta-2) - A\right] a_i + a_{i-1}. \tag{5.11}$$

The constant terms in (5.10) will vanish if we choose $a_0 = A^{\frac{1}{p-1}}$. We show that setting the power of ξ^{2i} to zero produces a linear equation for a_i of the form

$$\left[2i(2i+n+2\theta-2) + (p-1)A\right] a_i + g\left(a_0, a_1, \ldots, a_{i-1}\right) = 0, \tag{5.12}$$

where g is some function of previously computed coefficients. (Actual computations are easy to program, using *Mathematica*.) Indeed, denoting $t = \Sigma_{i=1}^{\infty} a_i \xi^{2i}$, we rewrite the last term in (5.10) as

$$\left(a_0 + \Sigma_{i=1}^{\infty} a_i \xi^{2i}\right)^p = a_0^p \, \Sigma_{j=0}^{\infty} \binom{p}{j} \frac{t^j}{a_0^j} \,.$$

The term ξ^{2i} with a coefficient involving a_i occurs when $j = 1$ in this sum, justifying (5.12). Using (5.12), we compute all a_i's.

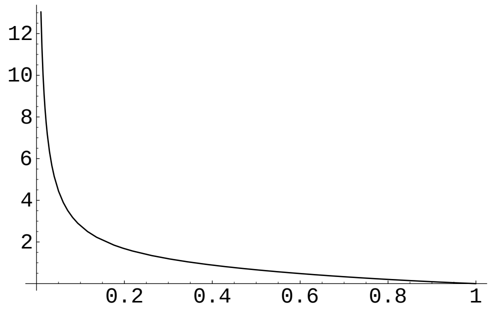

Figure 5: Singular solution of the problem (5.13)

Assume we could prove convergence of the series (5.9). (Our computations strongly suggest convergence.) This would give us solution for small ξ. Standard existence result gives us solution for all ξ. It is easy to see that $v(\xi)$ will eventually vanish. ($v''(0) = 2a_1 < 0$, so $v(\xi)$ is decreasing for small ξ. By maximum principle solution cannot turn around, and hence it tends to zero. Then v^p term is negligible, and $v(\xi)$ can be expressed through Bessel functions.) Let γ denote the first root of $v(\xi)$. Then $\bar{\lambda} = \gamma^2$, and $u(r) = r^\theta v(\gamma r)$ is the singular solution of (5.8) at $\lambda = \bar{\lambda}$.

Example If $n = 3$, then $\frac{n+2}{n-2} = 5$ and $\frac{n}{n-2} = 3$, $\frac{n+4}{n} = 7/3$. We took a subcritical $p = 4$, and found that the singular solution of

$$u'' + \frac{n-1}{r}u' + \lambda u + u^4 = 0, \quad \text{for } r < 1, \quad u(1) = 0 \tag{5.13}$$

occurs at $\bar{\lambda} \simeq 5.26295$. Here are the first few terms of the singular solution

$$u(r) \simeq r^{-2/3}(0.60571 - 0.79695r^2 + 0.18236r^4 + 0.02615r^6 - 0.02201r^8 \\ +0.00471r^{10} + 0.00054r^{12}).$$

Its graph is shown in Figure 5. When we plugged this solution into (5.13) at $\lambda = \bar{\lambda}$, we obtained the left hand side practically zero for all $r \in (0, 1)$. If the reader wishes to compute this solution using a standard ODE solver, he can start with $u(1) = 0$, $u'(1) \simeq -0.836857$, and solve the equation (5.5) (with $\lambda = 5.26295$) for decreasing r over $(0, 1)$. (Or one can start with $u(-1) = 0$, $u'(-1) \simeq 0.836857$, and solve the equation (5.5) forward for $-1 < r < 0$, since $u(r)$ is an even function.)

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