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UNIFORM CONTINUITY OF THE SOLUTION MAP FOR NONLINEAR WAVE EQUATION IN REISSNER-NORDSTRÖM METRIC

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Abstract

In this paper we study the properties of the solutions to the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0,1], x \in \mathcal{R}^3, \tag{1}$$

$$u(1,x) = u_0 \in \dot{H}^1(\mathcal{R}^3), \quad u_t(1,x) = u_1 \in L^2(\mathcal{R}^3), \tag{2}$$

where g_s is the Reissner-Nordström metric (see [2]); $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen.

When $g(r) \equiv 0$ we prove that the Cauchy problem (1), (2) has a nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{H}^1(\mathcal{R}^+))$, where $r = |x|$, and the solution map is not uniformly continuous.

When $g(r) \neq 0$ we prove that the Cauchy problem (1), (2) has a nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{H}^1(\mathcal{R}^+))$, where $r = |x|$, and the solution map is not uniformly continuous.

Subject classification: Primary 35L10, Secondary 35L50.

1. Introduction

In this paper we study the properties of the solutions to the Cachy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0,1], x \in \mathcal{R}^3, \tag{1}$$

$$u(1,x) = u_0 \in \dot{H}^1(\mathcal{R}^3), \quad u_t(1,x) = u_1 \in L^2(\mathcal{R}^3), \tag{2}$$

where g_s is the Reissner-Nordström metric (see [2])

$$g_s = \frac{r^2 - Kr + Q^2}{r^2} dt^2 - \frac{r^2}{r^2 - Kr + Q^2} dr^2 - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta^2,$$

K and Q are positive constants, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen.

The Cauchy problem (1), (2) we may rewrite in the form

(1)
$$\frac{r^2}{r^2 - Kr + Q^2} u_{tt} - \frac{1}{r^2}\partial_r((r^2 - Kr + Q^2)u_r) - \frac{1}{r^2 \sin\phi}\partial_\phi(\sin\phi u_\phi) - \frac{1}{r^2 \sin^2\phi} u_{\theta\theta} = f(u) + g(r),$$

(2) $u(1, r, \phi, \theta) = u_0 \in \dot{H}^1(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi]), u_t(1, r, \phi, \theta) = u_1 \in L^2(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi])$.

When g_s is the Minkowski metric; $u_0, u_1 \in \mathcal{C}_0^\infty(\mathcal{R}^3)$ in [5](see and [1], section 6.3) is proved that there exists $T > 0$ and a unique local solution $u \in \mathcal{C}^2([0, T) \times \mathcal{R}^3)$ for the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad f \in \mathcal{C}^2(\mathcal{R}), \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

for which

$$\sup_{t<T, x\in\mathcal{R}^3} |u(t, x)| = \infty.$$

When g_s is the Minkowski metric, $1 \leq p < 5$ and initial data are in $\mathcal{C}_0^\infty(\mathcal{R}^3)$, in [5](see and [1], section 6.3) is proved that the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u|u|^{p-1}, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Minkowski metric and initial data are in $\mathcal{C}_0^\infty(\mathcal{R}^3)$, in [4](see and [1], section 6.3) is proved that there exists a number $\epsilon_0 > 0$ such that for any data $(u_0, u_1) \in \mathcal{C}_0^\infty(\mathcal{R}^3)$ with $E(u(0)) < \epsilon_0$, the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u^5, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Minkowski metric in [6] is proved that the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad t \in [0, 1], x \in \mathcal{R}^3,$$

$$u(1, x) = u_0, \quad u_t(1, x) = u_1,$$

has global solution. Here $f \in \mathcal{C}^2(\mathcal{R})$, $f(0) = f'(0) = f''(0) = 0$,

$$|f''(u) - f''(v)| \leq B|u - v|^{q_1}$$

for $|u| \leq 1$, $|v| \leq 1$, $B > 0$, $\sqrt{2} - 1 < q_1 \leq 1$, $u_0 \in \mathcal{C}^5_\circ(\mathcal{R}^3)$, $u_1 \in \mathcal{C}^4_\circ(\mathcal{R}^3)$, $u_0(x) = u_1(x) = 0$ for $|x - x_0| > \rho$, x_0 and ρ are suitable chosen.

When g_s is the Reissner - Nordström metric, $n = 3$, $p > 1$, $q \geq 1$, $\gamma \in (0,1)$ are fixed constants, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen, in [7] is proved that the initial value problem (1), (2), has nontrivial solution $u \in \mathcal{C}((0,1]\dot{B}^\gamma_{p,q}(\mathcal{R}^+))$ in the form

$$u(t,r) = \begin{cases} v(t)\omega(r) & for \quad r \leq r_1, \quad t \in [0,1], \\ 0 \quad for & r \geq r_1, \quad t \in [0,1], \end{cases}$$

where $r = |x|$, for which $\lim_{t \longrightarrow 0} ||u||_{\dot{B}^\gamma_{p,q}(\mathcal{R}^+)} = \infty$.

In this paper we will prove that the Cauchy problem (1), (2) has nontrivial solution $u = u(t,r) \in \mathcal{C}((0,1]\dot{H}^1(\mathcal{R}^+))$ and the solution map is not uniformly continuous. When we say that the solution map $(u_\circ, u_1, g) \longrightarrow u(t,r)$ is uniformly continuous we understand: *for every positive constant ϵ there exist positive constants δ and R such that for any two solutions u, v of the Cauchy problem* (1), (2), *with right hands $g = g_1$, $g = g_2$ of (1), so that*

$$(2') \qquad E(1, u - v) \leq \delta, \quad ||g_1||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_2||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_1 - g_2||_{L^2(\mathcal{R}^+)} \leq \delta,$$

the following inequality holds

$$(2'') \qquad E(t, u - v) \leq \epsilon \quad for \quad \forall t \in [0,1],$$

where

$$E(t,u) := ||\partial_t u(t,\cdot)||^2_{L^2(\mathcal{R}^+)} + \Big|\Big|\frac{\partial}{\partial r} u(t,\cdot)||^2_{L^2(\mathcal{R}^+)}.$$

Our main results are

Theorem 1.1. *Let K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, \quad \frac{1}{1-K+Q^2} \geq 1, \\ 1 - K + Q^2 > 0 \quad is \quad enough \quad small \quad such \quad that \quad \frac{K - \sqrt{K^2 - 4Q^2}}{2} - 2\sqrt{1 - K + Q^2} > 0. \end{cases}$$

Let also $g \equiv 0$, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, a and b are positive constants. Then the homogeneous Cauchy problem (1), (2) has nontrivial solution $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{H}^1(\mathcal{R}^+))$. Also there exists $t_\circ \in [0,1)$ for which exists constant $\epsilon > 0$ such that for every positive constant δ exist solutions u, v of (1), (2), *so that*

$$E(1, u - v) \leq \delta,$$

and

$$E(t_\circ, u - v) \geq \epsilon.$$

Theorem 1.2. *Let K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, \quad \frac{1}{1-K+Q^2} \geq 1, \\ 1 - K + Q^2 > 0 \quad \textit{is enough small such that} \quad \frac{K-\sqrt{K^2-4Q^2}}{2} - 2\sqrt{1-K+Q^2} > 0. \end{cases}$$

Let also $g \neq 0$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \geq 0$ for $r \geq 0$, $g(r) = 0$ for $r \geq r_1$, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, a and b are positive constants. Then the nonhomogeneous Cauchy problem (1), (2) has nontrivial solution $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{H}^1(\mathcal{R}^+))$. Also there exists $t_\circ \in [0,1)$ for which exists constant $\epsilon > 0$ such that for every pair positive constants δ and R exist solutions u, v of (1), (2), with right hands $g = g_1$, $g = g_2$ of (1), so that

$$E(1, u - v) \leq \delta, \quad ||g_1||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_2||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_1 - g_2||_{L^2(\mathcal{R}^+)} \leq \delta,$$

and

$$E(t_\circ, u - v) \geq \epsilon.$$

The paper is organized as follows. In section 2 we prove theorem 1.1. In section 3 we prove theorem 1.2.

2. Proof of theorem 1.1.

For fixed $q \geq 1$ and $\gamma \in (0,1)$ we put

$$C = \Big(\frac{q\gamma 2^{q\gamma}}{2^{q\gamma} - 1}\Big)^{\frac{1}{q}}$$

For fixed $p > 1$, $q \geq 1$, $\gamma \in (0,1)$ and $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \geq 0$ for $r \geq 0$ we suppose that the constants $A>0$, $Q>0$, $a>0$, $b>0$, $B>0$, $K>0$, $1<\beta<\alpha$ satisfy the conditions

$$
\begin{aligned}
& i1) \tfrac{1}{1-K+Q^2}\Big(\tfrac{1}{1-K+Q^2}\tfrac{2a}{A^2}+\tfrac{b}{AB}\Big)\le 1, \quad A>1, \quad \tfrac{A^2}{a}>1;\\
& i2)\begin{cases}
\tfrac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\tfrac{a}{2A^4}-\tfrac{br_1^2}{AB}\ge 0,\\
\tfrac{a}{2A^6\alpha^2(1-\alpha K+\alpha^2Q^2)}-\tfrac{2br_1^2}{A^2B^2}\ge 0,\\
\tfrac{a}{2A^4(1-\alpha K+\alpha^2Q^2)}\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)-\tfrac{4ar_1}{A^3B(1-K+Q^2)^2}\ge 0,\\
\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)^2\tfrac{1}{(1-\alpha K+\alpha^2Q^2)^2}\tfrac{a}{4A^6}-r_1^2\tfrac{2b}{(1-K+Q^2)A^2B^2}-r_1^2\tfrac{1}{1-K+Q^2}\max_{r\in[0,r_1]}g(r)\ge\tfrac{1}{A^2},\\
\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)^2\tfrac{1}{(1-\alpha K+\alpha^2Q^2)^2}\tfrac{a}{2A^4}-\tfrac{2ar_1^4}{AB(1-K+Q^2)}>0,
\end{cases}\\
& i3) C\Big(\tfrac{1}{(1-K+Q^2)^2}\tfrac{2a}{A^2}+\tfrac{2b}{AB(1-K+Q^2)}+\tfrac{1}{A^2}\Big)\tfrac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}+C\tfrac{2^{1-\gamma}}{A^2(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}}<1,\\
& i4)\begin{cases}
\Big(\tfrac{1}{\alpha^2}\tfrac{1}{1-\alpha K+\alpha^2Q^2}\tfrac{a}{4A^6}-\tfrac{1}{\beta^2}\tfrac{a}{4A^4}\Big)>0,\\
\Big(\tfrac{1}{\alpha^2}\tfrac{1}{1-\alpha K+\alpha^2Q^2}\tfrac{a}{2A^6}-\tfrac{1}{\beta^2}\tfrac{a}{2A^4}\Big)>0,
\end{cases}\\
& i5)\begin{cases}
K^2>4Q^2, \quad A\ge\tfrac{1}{1-K+Q^2}>1,\\
1>\tfrac{2Q^2}{K}>\tfrac{K-\sqrt{K^2-4Q^2}}{2},\\
1-K+Q^2>0 \quad is \quad enough \quad small \quad such \quad that\\
1>\tfrac{K-\sqrt{K^2-4Q^2}}{2}-2\sqrt{1-K+Q^2}>0,\\
\tfrac{2}{K-\sqrt{K^2-4Q^2}-2(1-K+Q^2)}<\beta<\alpha\le 3,\\
\tfrac{a}{4A^6\alpha^2(1-\alpha K+\alpha^2Q^2)}-\tfrac{2br_1^2}{A^2B^2}-r_1^2\max_{r\in[0,r_1]}g(r)\ge 0,
\end{cases}\\
& i6) \tfrac{1}{1-K+Q^2}\Big(\tfrac{2}{AB}\Big(\tfrac{1}{1-K+Q^2}\tfrac{2a}{A^2}+\tfrac{b}{AB}\Big)+r_1^2\max_{s\in[0,r_1]}g(s)\Big)\le\tfrac{2}{AB};\\
& i7) \quad \max_{s\in[0,r_1]}g(s)\le\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)^2\tfrac{1}{A^4},
\end{aligned}
$$

where

$$r_1=\frac{K-\sqrt{K^2-4Q^2}}{2}-\sqrt{1-K+Q^2}.$$

Example. Let $0<\epsilon<<\frac{1}{3}$ is enough small,

$$
\begin{aligned}
& A=\tfrac{1}{\epsilon^4}, \quad B=\tfrac{1}{\epsilon}, \quad p=\tfrac{3}{2}, \quad q=\tfrac{3}{2}, \quad \gamma=\tfrac{1}{3}, \quad \alpha=3, \quad \tfrac{1}{\beta}=\tfrac{K-\sqrt{K^2-4Q^2}}{2}-2\sqrt{1-K+Q^2},\\
& g(r)=\begin{cases}\epsilon^{11}(r-r_1)^2 \quad for \quad r\le r_1,\\ 0 \quad for \quad r\ge r_1,\end{cases}\\
& K=\tfrac{4}{3}+\tfrac{1}{6}\epsilon^{20}-\tfrac{3}{2}\epsilon^2,\\
& Q^2=\tfrac{1}{3}+\tfrac{1}{6}\epsilon^{20}-\tfrac{1}{2}\epsilon^2,\\
& a=\epsilon^4, \quad b=\epsilon^3.
\end{aligned}
$$

Then

$$
\begin{aligned}
& 1-\alpha K+\alpha^2Q^2=1-3K+9Q^2=\epsilon^{20},\\
& 1-K+Q^2=\epsilon^2\bullet.
\end{aligned}
$$

When $g(r)\equiv 0$ we put

$$u_0:=v(1)\omega(r)= \tag{1'}$$

$$= \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2-Ks+Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s))\right) ds d\tau \quad for \quad r \le r_1, \\ 0 \quad for \quad r \ge r_1, \end{cases}$$

and $u_1 \equiv 0$. Here $v(t)$ is fixed function which satisfies the conditions

$$\begin{array}{ll} (H1) & v(t) \in \mathcal{C}^3[0,\infty), \quad v(t) > 0 \quad for \quad \forall t \in [0,1]; \\ (H2) & v''(t) > 0 \quad for \quad \forall t \in [0,1], \quad v'(1) = v'''(1) = 0, \quad v(1) \ne 0; \\ (H3) & \begin{cases} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{a}{2A^4}, \quad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2a}{A^2}; \\ v''(t) - \frac{a}{2A^4} v(t) \ge 0 \quad for \quad t \in [0,1]. \end{cases} \end{array}$$

Bellow we will prove that the equation ($1'$) has unique nontrivial solution $\omega(r)$ for which $\omega(r) \in \mathcal{C}^2[0,r_1]$, $\omega(r) \in \dot{H}^1[0,r_1]$, $|\omega(r)| \le \frac{2}{AB}$ for $r \in [0,r_1]$, $\omega(r) \ge \frac{1}{A^2}$ for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\omega(r_1) = \omega'(r_1) = \omega''(r_1) = 0$.

Example.

There exists function $v(t)$ for which (H1)-(H3) are hold. Really, let us consider the function

$$v(t) = \frac{(t-1)^2 + \frac{4A^4}{a} - 1}{\frac{A^3}{a}}, \tag{3}$$

where the constants A and a satisfy the conditions $A > 1$, $\frac{A^2}{a} > 1$. Then

1) $v(t) \in \mathcal{C}^3[0,\infty)$ and $v(t) > 0$ for all $t \in [0,1]$, i.e. (H1) is hold.

2)

$$\begin{array}{l} v'(t) = \frac{2(t-1)}{\frac{A^3}{a}}, \quad v'(1) = 0, \\ v''(t) = \frac{2}{\frac{A^3}{a}} \ge 0 \quad \forall \quad t \in [0,1], \\ v'''(t) = 0, \quad v'''(1) = 0, \end{array}$$

consequently (H2) is hold. On the other hand we have

$$\frac{v''(t)}{v(t)} = \frac{2}{(t-1)^2 + \frac{4A^4}{a} - 1}.$$

From here

$$\begin{array}{l} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{a}{2A^4}, \\ \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2a}{A^2}, \\ v''(t) - \frac{a}{2A^4} v(t) = \frac{1}{\frac{2A^7}{a^2}} (2-t)t, \end{array}$$

i.e. (H3) is hold.

2.1. Local existence of nontrivial solutions of homogeneous Cauchy problem (1), (2)

Let $v(t)$ is fixed function which satisfies the hypothesis $(H1) - (H3)$.

In this section we will prove that the homogeneous Cauchy problem (1), (2) has nontrivial solution in the form

$$u(t,r)=\begin{cases} v(t)\omega(r) \quad for \quad r\le r_1, \quad t\in[0,1],\\ 0 \quad for \quad r\ge r_1, \quad t\in[0,1]. \end{cases}$$

Let us consider the integral equation

$(\star)$

$$u(t,r)=\begin{cases} \int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u(t,s))\Big)dsd\tau, 0\le r\le r_1, \quad t\in[0,1],\\ 0 \quad for \quad r\ge r_1, \quad t\in[0,1], \end{cases}$$

where $u(t,r)=v(t)\omega(r)$.

Theorem 2.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1)-(H3). Let also $p>1$, $q\in[1,\infty)$ and $\gamma\in(0,1)$ are fixed and the positive constants A, a, b, B, Q,K, $\alpha>\beta>1$ satisfy the conditions i1)-i6) and $f\in\mathcal{C}^1(\mathcal{R}^1)$, $f(0)=0$, $a|u|\le f'(u)\le b|u|$. Then the equation $(\star)$ has unique nontrivial solution $u(t,r)=v(t)\omega(r)$ for which $w\in\mathcal{C}^2[0,r_1]$, $u(t,r)=u_r(t,r)=u_{rr}(t,r)=0$ for $r\ge r_1$, $u(t,r)\in\mathcal{C}((0,1]\dot{H}^1[0,r_1])$, for $r\in\Big[\frac{1}{\alpha},\frac{1}{\beta}\Big]$ and $t\in[0,1]$ $u(t,r)\ge\frac{1}{A^2}$, for $r\in\Big[0,r_1\Big]$ and $t\in[0,1]$ $|u(t,r)|\le\frac{2}{AB}$.*

Proof. In [7, p. 303-309,theorem 3.1] is proved that the equation $(\star)$ has solution $u(t,r)$ in the form $u(t,r)=v(t)\omega(r)$ for which

$$\begin{array}{l} u(t,r)\in\mathcal{C}([0,1]\times[0,r_1]);\\ u(t,r)=u_r(t,r)=u_{rr}(t,r)=0 \quad for \quad r\ge r_1 \quad and \quad t\in[0,1],\\ u(t,r)\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}[0,r_1]);\\ for \quad r\in\Big[\frac{1}{\alpha},\frac{1}{\beta}\Big] \quad and \quad t\in[0,1] \quad u(t,r)\ge\frac{1}{A^2};\\ u(t,r)\ge 0 \quad for \quad t\in[0,1] \quad and \quad r\in\Big[\frac{1}{\alpha},r_1\Big];\\ for \quad r\in\Big[0,r_1\Big] \quad and \quad t\in[0,1] \quad |u(t,r)|\le\frac{2}{AB}. \end{array}$$

In [7] is used the following definition of the $\dot{B}^{\gamma}_{p,q}(M)$-norm ($\gamma\in(0,1)$, $p>1$, $q\ge1$) (see [3, p.94, def. 2], [1])

$$||u||_{\dot{B}^{\gamma}_{p,q}(M)}=\left(\int_0^2 h^{-1-q\gamma}||\Delta_h u||^q_{L^p(M)}dh\right)^{\frac{1}{q}},$$

where

$$\Delta_h u=u(x+h)-u(x).$$

Let $t\in[0,1]$ is fixed. Then

$$\begin{array}{l} \Big|\Big|\frac{\partial}{\partial r}u\Big|\Big|^2_{L^2([0,\infty))}=\\ =\int_0^{r_1}\Big(\frac{1}{r^2-Kr+Q^2}\int_r^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u(t,s))\Big)dsd\tau\Big)^2dr\le \end{array}$$

here we use that from i5) we have that $r_1 < 1$, $\frac{s^4}{s^2-Ks+Q^2} \leq \frac{1}{1-K+Q^2}$ for $s \in [0, r_1]$, $\frac{1}{r^2-Kr+Q^2} \leq \frac{1}{1-K+Q^2}$ for $r \in [0, r_1]$(see [7, Remark, p.300])

$$\leq \int_0^{r_1} \Big(\frac{1}{1-K+Q^2} \int_r^{r_1} \Big(\frac{1}{1-K+Q^2} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} |u(t,s)| + s^2 |f(u(t,s))| \Big) ds d\tau \Big)^2 dr \leq$$

here we use that $f(0) = 0$, $|f(u)| \leq \frac{b}{2}|u|^2$,

$$\leq \int_0^{r_1} \Big(\frac{1}{1-K+Q^2} \int_r^{r_1} \Big(\frac{1}{1-K+Q^2} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} |u(t,s)| + s^2 \frac{b}{2} |u(t,s)|^2 \Big) ds d\tau \Big)^2 dr \leq$$

here we use that $|u(t,r)| \leq \frac{2}{AB}$ for $r \in [0, r_1]$, $t \in [0,1]$,

$$\leq \int_0^{r_1} \Big(\frac{1}{1-K+Q^2} \int_r^{r_1} \Big(\frac{1}{1-K+Q^2} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \frac{2}{AB} + r_1^2 \frac{b}{2} \frac{4}{A^2B^2} \Big) ds d\tau \Big)^2 dr \leq$$

here we use that $\max_{t\in[0,1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}$,

$$\begin{array}{l} \leq \int_0^{r_1} \Big(\frac{1}{1-K+Q^2} \int_r^{r_1} \Big(\frac{1}{1-K+Q^2} \frac{2a}{A^2} \frac{2}{AB} + r_1^2 \frac{b}{2} \frac{4}{A^2B^2} \Big) ds d\tau \Big)^2 dr \leq \\ \leq r_1^3 \Big(\frac{1}{1-K+Q^2} \Big(\frac{1}{1-K+Q^2} \frac{4a}{A^3B} + \frac{2r_1^2 b}{A^2B^2} \Big) \Big)^2 < \infty. \end{array}$$

From here

$$\Big|\Big| \frac{\partial}{\partial r} u \Big|\Big|_{L^2([0,\infty))} < \infty$$

for every fixed $t \in (0,1]$. Therefore $u(t,r) \in \mathcal{C}((0,1]\dot{H}^1([0,\infty)))$. •

Let $\tilde{u}$ is the solution from the theorem 2. 1, i.e $\tilde{u}$ is the solution to the equation $(\star)$. From proposition 2.1([7]) $\tilde{u}$ satisfies the equation (1). Then $\tilde{u}$ is solution to the Cauchy problem (1), (2) with initial data

$$u_0 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2-Ks+Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) \Big) ds d\tau & for \quad r \leq r_1, \\ 0 \quad for \quad r \geq r_1, \end{cases}$$

$$u_1 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2-Ks+Q^2} v'''(1)\omega(s) - s^2 f'(u) v'(1)\omega(s) \Big) ds d\tau = 0 & for \quad r \leq r_1, \\ 0 \quad for \quad r \geq r_1, \end{cases}$$

$u_0 \in \dot{H}^1(\mathcal{R}^+)$, $u_1 \in L^2(\mathcal{R}^+)$, $\tilde{u} \in \mathcal{C}((0,1]\dot{H}^1[0, r_1])$.

2.2. Uniformly continuity of the solution map for the homogeneous Cauchy problem (1), (2)

Let $v(t)$ is same function as in Theorem 2.1.

Theorem 2.2. *Let $p > 1$, $q \geq 1$ and $\gamma \in (0,1)$ are fixed and the positive constants a, b, A, B, Q, K, $1 < \beta < \alpha$ satisfy the conditions i1)-i6). Let $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$,*

$a|u| \le f'(u) \le b|u|$. Then there exists $t_\circ \in [0,1)$ for which there exists constant $\epsilon > 0$ such that for every positive constant δ exist solutions u_1 and u_2 so that

$$E(1, u_1 - u_2) \le \delta$$

and

$$E(t_\circ, u_1 - u_2) \ge \epsilon.$$

Proof. Let us suppose that the solution map $(u_\circ, u_1, g) \longrightarrow u(t,r)$ is uniformly continuous.

Let

$$0 < \epsilon < \Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)^3 \Big(\frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2}\Big(\frac{a}{2A^6\alpha^2(1 - \alpha L + \alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2}\Big)\Big)^2. \tag{4}$$

Let also

$$u_1 = \tilde{u}, \quad u_2 = 0.$$

Then there exists positive constant δ such that

$$E(1, u_1 - u_2) \le \delta,$$

and

$$E(t, u_1 - u_2) \le \epsilon \quad for \quad \forall t \in [0,1].$$

From here, for $t \in [0,1)$

$$\begin{aligned}
&\epsilon \ge \Big|\Big|\tfrac{\partial}{\partial r}\tilde{u}\Big|\Big|^2_{L^2([0,\infty))} = \\
&= \textstyle\int_0^{r_1} \Big(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s))\Big) ds d\tau\Big)^2 dr \ge \\
&\ge \textstyle\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s))\Big) ds d\tau\Big)^2 dr \ge
\end{aligned}$$

here we use that for $s \in \Big[\frac{1}{\alpha}, r_1\Big]$ and for $t \in [0,1]$ we have that $\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s)) \ge 0$(see [7, p. 305-306]) and $\frac{1}{r^2 - Kr + Q^2} \ge 0$ for $r \in [0, r_1]$,

$$\begin{aligned}
&\ge \textstyle\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s))\Big) ds d\tau\Big)^2 dr \ge \\
&\ge \textstyle\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s))\Big) ds d\tau\Big)^2 dr \ge
\end{aligned}$$

from $(H3)$ we have that $\min_{t \in [0,1]} \frac{v''(t)}{v(t)} \ge \frac{a}{2A^4}$

$$\ge \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} \tilde{u}(t,s) - s^2 f(\tilde{u}(t,s))\Big) ds d\tau\Big)^2 dr \ge$$

here we use that $f(0)=0,\ f(\tilde{u})\leq \frac{b}{2}\tilde{u}^2$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{a}{2A^4}\tilde{u}(t,s)-s^2\frac{b}{2}\tilde{u}^2(t,s)\Big)dsd\tau\Big)^2dr\geq$$

here we use that $\tilde{u}(t,s)\geq \frac{1}{A^2}$ for $t\in[0,1]$ and $s\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$, $\tilde{u}^2(t,s)\leq \frac{4}{A^2B^2}$ for $t\in[0,1]$ and $s\in[0,r_1]$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{a}{2A^4}\frac{1}{A^2}-\frac{1}{\beta^2}\frac{b}{2}\frac{4}{A^2B^2}\Big)dsd\tau\Big)^2dr\geq$$

here we use that $\frac{s^2}{s^2-Ks+Q^2}$ is increase function for $s\in[0,r_1]$. Therefore, for $s\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ we have $\frac{s^4}{s^2-Ks+Q^2}\geq \frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\frac{a}{2A^6}-\frac{2b}{\beta^2A^2B^2}\Big)dsd\tau\Big)^2dr\geq$$

here we use that for $r\in[0,r_1]$ the function $\frac{1}{r^2-Kr+Q^2}$ is increase function. Therefore for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ we have $\frac{1}{r^2-Kr+Q^2}\geq\frac{\alpha^2}{1-\alpha K+\alpha^2Q^2}$

$$\geq\Big(\frac{1}{\beta}-\frac{1}{\alpha}\Big)^3\Big(\frac{\alpha^2}{1-\alpha K+\alpha^2Q^2}\Big(\frac{a}{2A^6\alpha^2(1-\alpha L+\alpha^2Q^2)}-\frac{2b}{\beta^2A^2B^2}\Big)\Big)^2$$

which is contradiction with (4). •

3. Proof of Theorem 1.2.

3.1. Local existence of nontrivial solutions for nonhomogenious Cauchy problem (1), (2)

Let $v(t)$ is fixed function which satisfies the conditions (H1), (H2) and (H4), where

$$(H4)\quad \min_{t\in[0,1]}\frac{v''(t)}{v(t)}\geq\frac{a}{4A^4},\quad \max_{t\in[0,1]}\frac{v''(t)}{v(t)}\leq\frac{2a}{A^2}.$$

For instance, the function

$$v(t)=\frac{(t-1)^2+\frac{8A^4}{a}-1}{\frac{A^3}{a}},$$

satisfies the hypothesis (H1), (H2) and (H4).

Let us consider the equation

$(\star')$

$$u(t,r)=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u(t,s))-s^2g(s)\Big)dsd\tau, 0\leq r\leq r_1,\\ 0\quad for\quad r\geq r_1,\end{cases}$$

$t \in [0,1]$, where $u(t,r) = v(t)\omega(r)$.

Theorem 3.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1), (H2), (H4). Let also $p > 1$, $q \in [1,\infty)$ and $\gamma \in (0,1)$ are fixed and the positive constants A, a, b, B, Q,K, $\alpha > \beta > 1$ satisfy the conditions i1)-i7) and $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \le f'(u) \le b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \ge 0$ for $\forall r \in \mathcal{R}^+$, $g(r) = 0$ for $r \ge r_1$. Then the equation $(\star')$ has unique nontrivial solution $u(t,r) = v(t)\omega(r)$ for which $w \in \mathcal{C}^2[0,r_1]$, $u(t,r) = u_r(t,r) = u_{rr}(t,r) = 0$ for $r \ge r_1$, $u(t,r) \in \mathcal{C}((0,1]\dot{H}^1[0,r_1])$, for $r \in \left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ and $t \in [0,1]$ $u(t,r) \ge \frac{1}{A^2}$, for $r \in \left[0,r_1\right]$ and $t \in [0,1]$ $|u(t,r)| \le \frac{2}{AB}$.*

Proof. In [7, p. 313-316,theorem 4.1] is proved that the equation $(\star')$ has solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ for which

$$\begin{aligned}
&u(t,r) \in \mathcal{C}([0,1]\times[0,r_1]);\\
&u(t,r) = u_r(t,r) = u_{rr}(t,r) = 0 \quad for \quad r \ge r_1 \quad and \quad t \in [0,1],\\
&u(t,r) \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p.q}[0,r_1]);\\
&for \quad r \in \left[\tfrac{1}{\alpha},\tfrac{1}{\beta}\right] \quad and \quad t \in [0,1] \quad u(t,r) \ge \tfrac{1}{A^2};\\
&u(t,r) \ge 0 \quad for \quad t \in [0,1] \quad and \quad r \in \left[\tfrac{1}{\alpha},r_1\right];\\
&for \quad r \in \left[0,r_1\right] \quad and \quad t \in [0,1] \quad |u(t,r)| \le \tfrac{2}{AB}.
\end{aligned}$$

Let $t \in [0,1]$ is fixed. Then

$$\begin{aligned}
&\left|\left|\tfrac{\partial}{\partial r}u\right|\right|^2_{L^2([0,\infty))} =\\
&= \int_0^{r_1}\Big(\tfrac{1}{r^2-Kr+Q^2}\int_r^{r_1}\Big(\tfrac{s^4}{s^2-Ks+Q^2}\tfrac{v''(t)}{v(t)}u(t,s) - s^2(f(u(t,s)) + g(s))\Big)dsd\tau\Big)^2 dr \le
\end{aligned}$$

here we use that from i5) we have that $r_1 < 1$, $\frac{s^4}{s^2-Ks+Q^2} \le \frac{1}{1-K+Q^2}$ for $s \in [0,r_1]$, $\frac{1}{r^2-Kr+Q^2} \le \frac{1}{1-K+Q^2}$ for $r \in [0,r_1]$(see [7, Remark, p.300])

$$\le \int_0^{r_1}\Big(\frac{1}{1-K+Q^2}\int_r^{r_1}\Big(\frac{1}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u(t,s)| + s^2(|f(u(t,s))| + g(s))\Big)dsd\tau\Big)^2 dr \le$$

here we use that $f(0) = 0$, $|f(u)| \le \frac{b}{2}|u|^2$,

$$\le \int_0^{r_1}\Big(\frac{1}{1-K+Q^2}\int_r^{r_1}\Big(\frac{1}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u(t,s)| + s^2(\frac{b}{2}|u(t,s)|^2 + g(s))\Big)dsd\tau\Big)^2 dr \le$$

here we use that $|u(t,r)| \le \frac{2}{AB}$ for $r \in [0,r_1]$, $t \in [0,1]$, from i7) we have $\max_{r\in[0,r_1]} g(r) \le \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)^2 \frac{1}{A^4}$

$$\le \int_0^{r_1}\Big(\frac{1}{1-K+Q^2}\int_r^{r_1}\Big(\frac{1}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}\frac{2}{AB} + r_1^2\frac{b}{2}\frac{4}{A^2B^2} + r_1^2\Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)^2\frac{1}{A^4}\Big)dsd\tau\Big)^2 dr \le$$

here we use that $\max_{t\in[0,1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}$,

$$\leq \int_0^{r_1}\Big(\tfrac{1}{1-K+Q^2}\int_r^{r_1}\Big(\tfrac{1}{1-K+Q^2}\tfrac{2a}{A^2}\tfrac{2}{AB}+r_1^2\tfrac{b}{2}\tfrac{4}{A^2B^2}+r_1^2\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)^2\tfrac{1}{A^4}\Big)dsd\tau\Big)^2dr \leq$$
$$\leq r_1^3\Big(\tfrac{1}{1-K+Q^2}\Big(\tfrac{1}{1-K+Q^2}\tfrac{4a}{A^3B}+\tfrac{2r_1^2b}{A^2B^2}+r_1^2\Big(\tfrac{1}{\beta}-\tfrac{1}{\alpha}\Big)^2\tfrac{1}{A^4}\Big)\Big)^2<\infty.$$

From here

$$\Big|\Big|\frac{\partial}{\partial r}u\Big|\Big|_{L^2([0,\infty))}<\infty$$

for every fixed $t\in(0,1]$. Therefore $u(t,r)\in\mathcal{C}((0,1]\dot{H}^1([0,\infty)))$. •

Let $\bar{u}$ is the solution from the theorem 3. 1. , i.e $\bar{u}$ is the solution to the equation $(\star')$. From proposition 2.3[7] we have that $\bar{u}$ satisfies the equation (1). Then $\bar{u}$ is solution to the Cauchy problem (1), (2) with initial data

$$\bar{\bar{u}}_0=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v''(1)\omega(s)-s^2f(v(1)\omega(s))-s^2g(s)\Big)dsd\tau \quad for \quad r\leq r_1,\\ 0 \quad for \quad r\geq r_1,\end{cases}$$

$$\bar{\bar{u}}_1=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v'''(1)\omega(s)-s^2f'(u)v'(1)\omega(s)\Big)dsd\tau\equiv 0\\ for \quad r\leq r_1,\\ 0 \quad for \quad r\geq r_1,\end{cases}$$

$\bar{\bar{u}}_0\in\dot{H}^1(\mathcal{R}^+)$, $\bar{\bar{u}}_1\in L^2(\mathcal{R}^+)$, $\bar{u}\in\mathcal{C}((0,1]\dot{H}^1[0,r_1])$.

3.2. Uniformly continuity of the solution map for the nonhomogeneous Cauchy problem (1), (2)

Let $v(t)$ is same function as in Theorem 3.1.

Theorem 3.2. *Let $p>1$, $q\geq 1$ and $\gamma\in(0,1)$ are fixed and the positive constants a, b, A, B, Q, K, $1<\beta<\alpha$ satisfy the conditions i1)-i7). Let $f\in\mathcal{C}^1(\mathcal{R}^1)$, $f(0)=0$, $a|u|\leq f'(u)\leq b|u|$, $g\in\mathcal{C}(\mathcal{R}^+)$, $g(r)\geq 0$ for $r\geq 0$, $g(r)=0$ for $r\geq r_1$. Then there exists $t_\circ\in[0,1)$ for which there exists constant $\epsilon>0$ such that for every positive constants δ and R exist solutions u_1, u_2 of (1), (2) with right hands $g=g_1$, $g=g_2$ of (1), so that*

$$E(1,u_1-u_2)\leq\delta,\quad ||g_1||_{L^2(\mathcal{R}^+)}\leq R,\quad ||g_2||_{L^2(\mathcal{R}^+)}\leq R,\quad ||g_1-g_2||_{L^2(\mathcal{R}^+)}\leq\delta,$$

and

$$E(t_\circ,u_1-u_2)\geq\epsilon.$$

Proof. Let us suppose that the solution map $(u_\circ,u_1,g)\longrightarrow u(t,r)$ is uniformly continuous. Let

(5)

$$0<\epsilon<\Big(\frac{1}{\beta}-\frac{1}{\alpha}\Big)^3\Big(\frac{\alpha^2}{1-\alpha K+\alpha^2Q^2}\Big(\frac{a}{4A^6\alpha^2(1-\alpha K+\alpha^2Q^2)}-\frac{2b}{\beta^2A^2B^2}-r_1^2\Big(\frac{1}{\beta}-\frac{1}{\alpha}\Big)^2\frac{1}{A^4}\Big)\Big)^2.$$

Let also

$$u_1 = \bar{u}, \quad u_2 = 0, \quad g_2 \equiv 0, \quad g_1 \equiv g,$$

where g is the function from theorem 3.1. Then there exist positive constants δ and R such that

$$||g_1||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_2||_{L^2(\mathcal{R}^+)} \leq R, \quad ||g_1 - g_2||_{L^2(\mathcal{R}^+)} \leq \delta,$$
$$E(1, u_1 - u_2) \leq \delta,$$

and

$$E(t, u_1 - u_2) \leq \epsilon \quad for \quad \forall t \in [0,1].$$

From here, for $t \in [0,1)$

$$\begin{aligned}
&\epsilon \geq \left|\left|\frac{\partial}{\partial r}\bar{u}\right|\right|^2_{L^2([0,\infty))} = \\
&= \int_0^{r_1}\left(\frac{1}{r^2-Kr+Q^2}\int_r^{r_1}\left(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\bar{u}(t,s) - s^2(f(\bar{u}(t,s)) + g(s))\right)dsd\tau\right)^2 dr \geq \\
&\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\left(\frac{1}{r^2-Kr+Q^2}\int_r^{r_1}\left(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\bar{u}(t,s) - s^2(f(\bar{u}(t,s)) + g(s))\right)dsd\tau\right)^2 dr \geq
\end{aligned}$$

here we use that for $r \in \left[\frac{1}{\alpha}, r_1\right]$ we have

$$\begin{aligned}
&\tfrac{s^4}{s^2-Ks+Q^2}\tfrac{v''(t)}{v(t)}\bar{u}(t,s) - s^2 f(\bar{u}(t,s)) - s^2 g(s) \geq \\
&\geq \tfrac{a}{4A^6\alpha^2(1-\alpha K+\alpha^2Q^2)} - \tfrac{2br_1^2}{A^2B^2} - r_1^2 \max_{r\in[0,r_1]} g(r) \geq 0
\end{aligned}$$

(see i5)) and for $r \in [0, r_1]$ we have $\frac{1}{r^2-Kr+Q^2} > 0$

$$\begin{aligned}
&\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\left(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\left(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\bar{u}(t,s) - s^2(f(\bar{u}(t,s)) + g(s))\right)dsd\tau\right)^2 dr \geq \\
&\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\left(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\left(\frac{s^4}{s^2-Ks+Q^2}\min_{t\in[0,1]}\frac{v''(t)}{v(t)}\bar{u}(t,s) - s^2(f(\bar{u}(t,s)) + g(s))\right)dsd\tau\right)^2 dr \geq
\end{aligned}$$

from $(H4)$ we have that $\min_{t\in[0,1]}\frac{v''(t)}{v(t)} \geq \frac{a}{4A^4}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{a}{4A^4}\bar{u}(t,s) - s^2(f(\bar{u}(t,s)) + g(s))\Big)dsd\tau\Big)^2 dr \geq$$

here we use that $f(0) = 0$, $f(\bar{u}) \leq \frac{b}{2}\bar{u}^2$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{a}{4A^4}\bar{u}(t,s) - s^2\frac{b}{2}\bar{u}^2(t,s) - s^2 g(s)\Big)dsd\tau\Big)^2 dr \geq$$

here we use that $\bar{u}(t,s) \geq \frac{1}{A^2}$ for $t \in [0,1]$ and $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\bar{u}^2(t,s) \leq \frac{4}{A^2B^2}$ for $t \in [0,1]$ and $s \in [0, r_1]$, $\max_{r\in[0,r_1]} g(r) \leq \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)^2 \frac{1}{A^4}$,

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{a}{4A^4}\frac{1}{A^2} - \frac{1}{\beta^2}\frac{b}{2}\frac{4}{A^2B^2} - r_1^2\Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)^2\frac{1}{A^4}\Big)dsd\tau\Big)^2 dr \geq$$

here we use that $\frac{s^2}{s^2-Ks+Q^2}$ is increase function for $s \in [0, r_1]$. Therefore, for $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{s^4}{s^2-Ks+Q^2} \geq \frac{1}{\alpha^2(1-\alpha K+\alpha^2 Q^2)}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{1}{\alpha^2(1 - \alpha K + \alpha^2 Q^2)} \frac{a}{4A^6} - \frac{2b}{\beta^2 A^2 B^2} - r_1^2 \Big(\frac{1}{\beta} - \frac{1}{\alpha} \Big)^2 \frac{1}{A^4} \Big) ds d\tau \Big)^2 dr \geq$$

here we use that for $r \in [0, r_1]$ the function $\frac{1}{r^2-Kr+Q^2}$ is increase function. Therefore for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{1}{r^2-Kr+Q^2} \geq \frac{\alpha^2}{1-\alpha K+\alpha^2 Q^2}$

$$\geq \Big(\frac{1}{\beta} - \frac{1}{\alpha} \Big)^3 \Big(\frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2} \Big(\frac{a}{4A^6 \alpha^2 (1 - \alpha K + \alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2} - r_1^2 \Big(\frac{1}{\beta} - \frac{1}{\alpha} \Big)^2 \frac{1}{A^4} \Big) \Big)^2$$

which is contradiction with (5). •

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