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q-DOMINANT AND q-RECESSIVE MATRIX SOLUTIONS FOR LINEAR QUANTUM SYSTEMS

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ABSTRACT. In this study, linear second-order matrix q-difference equations are shown to be formally self-adjoint equations with respect to a certain inner product and the associated self-adjoint boundary conditions. A generalized Wronskian is introduced and a Lagrange identity and Abel's formula are established. Two reduction-of-order theorems are given. The analysis and characterization of q-dominant and q-recessive solutions at infinity are presented, emphasizing the case when the quantum system is disconjugate.

1. INTRODUCTION

Quantum calculus has been utilized since at least the time of Pierre de Fermat [10, Chapter B.5] to augment mathematical understanding gained from the more traditional continuous calculus and other branches of the discipline [3]. In this study we will analyze a second-order linear self-adjoint matrix q-difference system, especially in the case that admits q-dominant and q-recessive solutions at infinity. Historically, dominant and recessive solutions of linear matrix differential systems of the form

$$(PX')'(t) + Q(t)X(t) = 0$$

were introduced and extensively studied in a series of classic works by W. T. Reid [5, 6, 7, 8, 9], and in matrix difference systems of the form

$$\Delta\left(P(t)\Delta X(t-1)\right) + Q(t)X(t) = 0$$

by Ahlbrandt [1], Ahlbrandt and Peterson [2], and recently by Ma [4]; there the forward difference operator $\Delta X(t) := X(t+1) - X(t)$ was used. We introduce here an analysis of the quantum (q-difference) system

$$D^q\left(PD_qX\right)(t) + Q(t)X(t) = 0, \tag{1.1}$$

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where the real scalar $q > 1$ and the q-derivatives are given, respectively, by the difference quotients

$$(D_q y)(t) = \frac{y(qt) - y(t)}{(q-1)t} \quad \text{and} \quad (D^q y)(t) = \frac{y(t) - y(t/q)}{(1-1/q)t} = (D_q y)(t/q).$$

We will be particularly interested in the case where invertible solutions of (1.1) exist, and their characterization as q-dominant and/or q-recessive solutions at infinity.

The analysis of (1.1) and its solutions will unfold as follows. In Section 2 we explore (1.1), show how it is formally a self-adjoint equation, introduce a generalized Wronskian and establish a Lagrange identity and Abel's formula. Section 3 contains two reduction of order theorems, followed in Section 4 by the notion of a prepared basis. In the main section, Section 5, we give definitions of q-dominant and q-recessive solutions, a connection to disconjugacy, and the construction of q-recessive solutions. Finally, future directions are touched on in Section 6, where a Pólya factorization of (1.1) leads to a variation of parameters result.

2. Self-Adjoint Matrix Equations

Let $q > 1$ be a real scalar, and let P and Q be Hermitian $n \times n$-matrix-valued functions such that $P(t) > 0$ (positive definite) for all

$$t \in (0,\infty)_q := \{\ldots, q^{-2}, q^{-1}, 1, q, q^2, \ldots\}.$$

(A matrix M is *Hermitian* iff $M^* = M$, where * indicates conjugate transpose.) In this section we are concerned with the second-order matrix q-difference equation

$$LX = 0, \quad \text{where} \quad LX(t) := D^q\left(PD_qX\right)(t) + Q(t)X(t) = 0, \quad t \in (0,\infty)_q, \tag{2.1}$$

which will be shown to be (formally) self-adjoint.

Theorem 2.1. *Let $a \in (0,\infty)_q$ be fixed and X_a, X'_a be given constant $n \times n$ matrices. Then the initial value problem*

$$LX(t) = D^q\left(PD_qX\right)(t) + Q(t)X(t) = 0, \quad X(a) = X_a, \quad D_qX(a) = X'_a$$

has a unique solution.

Proof. For $a \in (0,\infty)_q$ fixed, expanding out (2.1) we obtain $LX(a) = 0$ in the form

$$LX(a) = \frac{q}{(q-1)^2a^2}\left[P(a)\left(X(qa) - X(a)\right) - qP(a/q)\left(X(a) - X(a/q)\right)\right] + Q(a)X(a);$$

since P is invertible and

$$X(a) = X_a, \quad X(qa) = X_a + a(q-1)X'_a,$$

the term $X(a/q)$ can be solved for uniquely and the unique solution X can be constructed to the left of $a \in (0,\infty)_q$. In the same way, $LX(qa) = 0$ given by

$$LX(qa) = \frac{1}{q(q-1)^2a^2}\left[P(qa)\left(X(q^2a) - X(qa)\right) - qP(a)\left(X(qa) - X(a)\right)\right] + Q(qa)X(qa),$$

and again the term $X(q^2a)$ can be solved for uniquely and the unique solution X can be constructed to the right of $a \in (0,\infty)_q$. $\square$

In view of the theorem just proven, the following definition is now possible.

Definition 2.2. *The unique solution of the initial value problem*

$$LX = 0, \quad X(a) = 0, \quad D_qX(a) = P^{-1}(a)$$

is called the principal solution of (2.1) *(at a), while the unique solution of the initial value problem*

$$LX = 0, \quad X(a) = -I, \quad D_qX(a) = 0$$

is called the associated (coprincipal) solution of (2.1) *(at a).*

Definition 2.3. *For matrix functions X and Y, the function $W(X,Y)$ given by*

$$W(X,Y)(t) = X^*(t)P(t)D_qY(t) - [P(t)D_qX(t)]^*Y(t), \quad t \in (0,\infty)_q$$

is the (generalized) Wronskian matrix *of X and Y.*

Lemma 2.4. *The product rule for D^q is given by*

$$D^q(XY)(t) = X(t/q)D^qY(t) + (D^qX)(t)Y(t) = X(t)D^qY(t) + (D^qX)(t)Y(t/q),$$

and for D_q is given by

$$D_q(XY)(t) = X(qt)D_qY(t) + (D_qX)(t)Y(t) = X(t)D_qY(t) + (D_qX)(t)Y(qt),$$

for matrix functions X and Y defined on $(0,\infty)_q$.

Proof. The proof is straightforward using the definitions of D^q and D_q and is omitted. $\square$

Theorem 2.5 (Lagrange Identity). *The Wronskian matrix $W(X,Y)$ satisfies*

$$D^qW(X,Y)(t) = X^*(t)(LY)(t) - (LX)^*(t)Y(t), \quad t \in (0,\infty)_q$$

for matrix functions X and Y defined on $(0,\infty)_q$.

Proof. For matrix functions X and Y, using the product rule for D^q derivatives we have

$$\begin{aligned} D^qW(X,Y)(t) &= D^q\left[X^*PD_qY-(PD_qX)^*Y\right](t) \\ &= X^*(t)D^q(PD_qY)(t)+(D^qX^*)(t)P(t/q)(D^qY)(t) \\ &\quad -(PD_qX)^*(t/q)(D^qY)(t)-D^q(PD_qX)^*(t)Y(t) \\ &= X^*(t)(LY-QY)(t)+(D^qX^*)(t)P(t/q)D^qY(t) \\ &\quad -(D^qX^*)(t)P(t/q)D^qY(t)-(LX-QX)^*(t)Y(t) \\ &= X^*(t)(LY)(t)-(LX)^*(t)Y(t) \end{aligned}$$

on $(0,\infty)_q$. □

Definition 2.6. *Let $a,b\in(0,\infty)_q$ with $a<b$. We define the q-inner product of $n\times n$ matrix functions M and N on $[a,b]_q$ to be*

$$\langle M,N\rangle=\left(1-\frac{1}{q}\right)\sum_{t\in(a,b]_q} tM^*(t)N(t),\quad a,b\in(0,\infty)_q. \tag{2.2}$$

Since $a=q^\alpha$, $b=q^\beta$, and $t=q^\tau$ for integers $\alpha\le\tau\le\beta$, the q-inner product is given by the expression

$$\langle M,N\rangle=(q-1)\sum_{\tau=\alpha+1}^{\beta} q^{\tau-1}M^*(q^\tau)N(q^\tau).$$

Corollary 2.7 (Self-Adjoint Operator)**.** *The operator L in* (2.1) *is formally self-adjoint with respect to the q-inner product* (2.2)*; that is, the identity*

$$\langle LX,Y\rangle=\langle X,LY\rangle$$

holds provided X,Y satisfy $W(X,Y)(t)\big|_a^b=0$, called the self-adjoint boundary conditions.

Proof. Let the matrix functions X and Y satisfy $W(X,Y)(t)\big|_a^b=0$. From Definition 2.3 and Theorem 2.5 we see that Green's formula holds, namely

$$\left(1-\frac{1}{q}\right)\sum_{t\in(a,b]_q} t\left(D^qW(X,Y)\right)(t)=W(X,Y)(t)\big|_a^b=\langle X,LY\rangle-\langle LX,Y\rangle,$$

and the proof is complete. □

Another immediate corollary of the Lagrange identity is Abel's matrix formula.

Corollary 2.8 (Abel's Formula). *If X, Y are solutions of* (2.1) *on* $(0, \infty)_q$, *then*

$$W(X, Y)(t) \equiv C, \quad t \in (0, \infty)_q,$$

where C is a constant matrix.

Corollary 2.9. *If X, Y are solutions of* (2.1) *on* $(0, \infty)_q$, *then either* $W(X, Y)(t) = 0$ *for all* $t \in (0, \infty)_q$, *or* $W(X, Y)(t) \neq 0$ *for all* $t \in (0, \infty)_q$.

From Abel's formula we get that if X is a solution of (2.1) on $(0, \infty)_q$, then

$$W(X, X)(t) \equiv C, \quad t \in (0, \infty)_q,$$

where C is a constant matrix. With this in mind we make the following definition.

Definition 2.10. *Let X and Y be matrix functions and $W(X, Y)$ be given as in* (2.3).

(i) *The matrix function X is a prepared (conjoined, isotropic) solution of* (2.1) *iff X is a solution of* (2.1) *and*

$$W(X, X)(t) \equiv 0, \quad t \in (0, \infty)_q.$$

(ii) *The matrix functions X and Y are normalized prepared bases of* (2.1) *iff X, Y are two prepared solutions of* (2.1) *with*

$$W(X, Y)(t) \equiv I, \quad t \in (0, \infty)_q.$$

Theorem 2.11. *Any two prepared solutions of* (2.1) *on* $(0, \infty)_q$ *are linearly independent iff their Wronskian is nonzero.*

Theorem 2.12. *Equation* (2.1) *on* $(0, \infty)_q$ *has two linearly independent solutions, and every solution of* (2.1) *on* $(0, \infty)_q$ *is a linear combination of these two solutions.*

Theorem 2.13 (Converse of Abel's Formula). *Assume X is a solution of* (2.1) *on* $(0, \infty)_q$ *such that X^{-1} exists on* $(0, \infty)_q$. *If Y satisfies $W(X, Y)(t) \equiv C$, where C is a constant matrix, then Y is also a solution of* (2.1).

Proof. Suppose that X is a solution of (2.1) such that X^{-1} exists on $(0, \infty)_q$, and assume Y satisfies $W(X, Y)(t) \equiv C$, where C is a constant matrix. By the Lagrange identity (Theorem 2.5) we have

$$0 \equiv D^q W(X, Y)(t) = X^*(t)(LY)(t) - (LX)(t)Y^*(t) = X^*(t)(LY)(t), \quad t \in (0, \infty)_q.$$

As $(X^*)^{-1}$ exists on $(0, \infty)_q$, $(LY)(t) = 0$ on $(0, \infty)_q$. Thus Y is also a solution of (2.1). □

Theorem 2.14. *Assume that X is a solution of* (2.1) *on* $(0,\infty)_q$. *Then the following are equivalent:*

(i) *X is a prepared solution;*

(ii) *$(X^* P D_q X)(t)$ is Hermitian for all $t \in (0,\infty)_q$;*

(iii) *$(X^* P D_q X)(a)$ is Hermitian for some $a \in (0,\infty)_q$.*

Proof. Use the Wronskian W and Abel's formula. □

Note that one can easily get prepared solutions of (2.1) by taking initial conditions at $a \in (0,\infty)_q$ so that $X^*(a)P(a)D_qX(a)$ is Hermitian.

In the Sturmian theory for equations of the form (2.1) the matrix function $X^*(t)P(t)X(qt)$ is important. We note the following result.

Lemma 2.15. *Let X be a solution of* (2.1)*. If X is prepared, then*

$$X^*(t)P(t)X(qt) \quad \text{is Hermitian for all} \quad t \in (0,\infty)_q.$$

Conversely, if there is an $a \in (0,\infty)_q$ such that $X^*(a)P(a)X(qa)$ is Hermitian, then X is a prepared solution of (2.1)*. Moreover, if X is an invertible prepared solution, then*

$$P(t)X(qt)X^{-1}(t), \; P(t)X(t)X^{-1}(qt), \; \text{and } Z(t) := \left(P(D_qX)X^{-1}\right)(t)$$

are all Hermitian for all $t \in (0,\infty)_q$.

Proof. Let X be a solution of (2.1). Then the relation

$$X^*(t)P(t)X(qt) = (X^*PX)(t) + (q-1)t(X^*PD_qX)(t) \tag{2.3}$$

proves the first two statements of this lemma. Now assume that X is an invertible prepared solution of (2.1). Then

$$X^*(t)P(t)X(qt) = X^*(qt)P(t)X(t) \quad \text{and} \quad (X^*PD_qX)(t) = \left((D_qX)^*PX\right)(t) \tag{2.4}$$

on $(0,\infty)_q$ by (2.3) and Theorem 2.14. We multiply the first equation in (2.4) from the left with $X^{*-1}(t)$ and from the right with $X^{-1}(t)$ to obtain that $P(t)X(qt)X^{-1}(t)$ is Hermitian. To see that $P(t)X(t)X^{-1}(qt)$ is Hermitian, we multiply the first equation in (2.4) with $X^{*-1}(qt)$ from the left and with $X^{-1}(qt)$ from the right. Multiplying the second equation in (2.4) with $X^{*-1}(t)$ from the left and with $X^{-1}(t)$ from the right shows that Z is Hermitian. □

Lemma 2.16. *Assume that X is a prepared solution of* (2.1) *on* $(0,\infty)_q$. *Then the following are equivalent:*

(i) $X^*(qt)P(t)X(t) = X^*(t)P(t)X(qt) > 0$ *on* $(0,\infty)_q$;
(ii) X *is invertible and* $P(t)X(qt)X^{-1}(t) > 0$ *on* $(0,\infty)_q$;
(iii) X *is invertible and* $P(t)X(t)X^{-1}(qt) > 0$ *on* $(0,\infty)_q$.

Proof. First note that $X^*(qt)P(t)X(t) > 0$ for $t \in (0,\infty)_q$ implies that $X(t)$ is invertible for $t \in (0,\infty)_q$. Since X is a prepared solution of (2.1), by Lemma 2.15 we have
(2.5)
$$P(t)X(qt)X^{-1}(t) = X^{*-1}(t)X^*(qt)P(t), \quad P(t)X(t)X^{-1}(qt) = X^{*-1}(qt)X^*(t)P(t)$$
for all $t \in (0,\infty)_q$. We multiply the right-hand side of the first equation in (2.5) from the right with $(XX^{-1})(t)$ to obtain the equivalence of (i) and (ii). For the equivalence of (i) and (iii), multiply the right-hand side of the second equation in (2.5) from the right with $(XX^{-1})(qt)$. The other implications are similar. □

3. Reduction of Order Theorems

In this section we establish two related reduction of order theorems; first, we need the following preparatory lemma, which allows us to q-differentiate an inverse matrix.

Lemma 3.1. *Let* $t \in (0,\infty)_q$, *and assume* X *is invertible on* $(0,\infty)_q$. *Then*
$$D^qX^{*-1}(t) = -X^{*-1}(t/q)(D^qX)^*(t)X^{*-1}(t) = -X^{*-1}(t)(D^qX)^*(t)X^{*-1}(t/q)$$
for $t \in (0,\infty)_q$.

Proof. Use the product rules given in Lemma 2.4 on the equation $XX^{-1} = I$. □

Remark 3.2. *Throughout this work it is to be understood that*
$$0 \equiv \sum_{s\in[a,a)_q} M(s) \equiv \sum_{s\in(a,a]_q} M(s), \quad a \in (0,\infty)_q$$
for any matrix function M *defined on* $(0,\infty)_q$.

Theorem 3.3 (Reduction of Order I)**.** *Let* $a \in (0,\infty)_q$, *and assume* X *is a prepared solution of* (2.1) *with* X *invertible on* $[a,\infty)_q$. *Then a second prepared solution* Y *of* (2.1) *is given by*
$$Y(t) := (q-1)X(t)\sum_{s\in[a,t)_q} s\left(X^*(s)P(s)X(qs)\right)^{-1}, \quad t \in [a,\infty)_q$$
such that X, Y *are normalized prepared bases of* (2.1).

Proof. For Y defined above, by the product rule in Lemma 2.4 for D_q we have

$$D_qY = P^{-1}X^{*-1} + (D_qX)X^{-1}Y.$$

For $W(X,Y)$ given in Definition 2.3,

$$\begin{aligned} W(X,Y) &= X^*PD_qY - (PD_qX)^*Y \\ &= X^*P(P^{-1}X^{*-1} + (D_qX)X^{-1}Y) - (PD_qX)^*Y \\ &= I + X^*P(D_qX)X^{-1}Y - (D_qX)^*PY \\ &= I + (X^*PD_qX - (D_qX)^*PX)X^{-1}Y = I \end{aligned}$$

since X^*PD_qX is Hermitian by Theorem 2.14 (ii). By Theorem 2.13, $W(X,Y) = I$ guarantees that Y is a solution of (2.1). To see that Y is prepared, note that

$$\begin{aligned} Y^*PD_qY &= Y^*P(P^{-1}X^{*-1} + (D_qX)X^{-1}Y) = Y^*X^{*-1} + Y^*\left(P(D_qX)X^{-1}\right)Y \\ &= (X^{-1}Y)^* + Y^*ZY = (q-1)\sum_{s\in[a,t)_q} s\left(X^*(s)P(s)X(qs)\right)^{-1} + Y^*ZY, \end{aligned}$$

which is Hermitian by Lemma 2.15 since X is prepared and Z is Hermitian. Consequently, X, Y are normalized prepared bases for (2.1). ☐

Lemma 3.4. *Assume X, Y are normalized prepared bases of* (2.1)*. Then $U := XE + YF$ is a prepared solution of* (2.1) *for constant $n \times n$ matrices E, F if and only if F^*E is Hermitian. If $F = I$, then X, U are normalized prepared bases of* (2.1) *if and only if E is a constant Hermitian matrix.*

Proof. Assume X, Y are normalized prepared bases of (2.1). Then by Theorem 2.14 and Definition 2.3,

$$X^*PD_qX = (D_qX)^*PX, \quad Y^*PD_qY = (D_qY)^*PY, \quad X^*PD_qY - (D_qX)^*PY = I.$$

By linearity $U := XE + YF$ is a solution of (2.1). Checking appropriate Wronskians,

$$\begin{aligned} W(U,U) &= U^*PD_qU - (D_qU)^*PU \\ &= (E^*X^* + F^*Y^*)P\left((D_qX)E + (D_qY)F\right) \\ &\quad - \left(E^*(D_qX)^* + F^*(D_qY)^*\right)P(XE + YF) \\ &= E^*\left(X^*PD_qX - (D_qX)^*PX\right)E + F^*\left(Y^*PD_qY - (D_qY)^*PY\right)F \\ &\quad + E^*\left(X^*PD_qY - (D_qX)^*PY\right)F + F^*\left(Y^*PD_qX - (D_qY)^*PX\right)E \\ &= 0 + 0 + E^*IF - F^*IE, \end{aligned}$$

and

$$\begin{aligned} W(X,U) &= X^* P D_q U - (D_q X)^* P U \\ &= X^* P\left[(D_q X)E + (D_q Y)F\right] - (D_q X)^* P[XE + YF] = F. \end{aligned}$$

Clearly the first claim holds. If $F = I$, then $W(X,U) = I$, and $U = XE + Y$ is a prepared solution of (2.1) if and only if E is a constant Hermitian matrix. $\square$

Theorem 3.5 (Reduction of Order II)**.** *Let $a \in (0,\infty)_q$, and assume X is a prepared solution of (2.1) with X invertible on $[a,\infty)_q$. Then U is a second $n \times n$ matrix solution of (2.1) iff U satisfies the first-order matrix equation*

$$D_q(X^{-1}U)(t) = (X^*(t)P(t)X(qt))^{-1}F, \quad t \in [a,\infty)_q, \tag{3.1}$$

for some constant $n \times n$ matrix F iff U is of the form

$$U(t) = X(t)E + (q-1)X(t)\left(\sum_{s\in[a,t)_q} s\left(X^*(s)P(s)X(qs)\right)^{-1}\right)F, \quad t \in [a,\infty)_q, \tag{3.2}$$

where E and F are constant $n \times n$ matrices. In the latter case,

$$E = X^{-1}(a)U(a), \qquad F = W(X,U)(a), \tag{3.3}$$

such that U is a prepared solution of (2.1) iff $F^*E = E^*F$.

Proof. Assume X is a prepared solution of (2.1) with X invertible on $[a,\infty)_q$. Let U be any $n \times n$ matrix solution of (2.1); we must show U is of the form (3.2). Using the Wronskian from Definition 2.3, set

$$F := W(X,U)(a) = \left(X^* P D_q U - (D_q X)^* P U\right)(a).$$

Since

$$D_q(X^{-1}U)(t) = -X^{-1}(qt)(D_qX)(t)X^{-1}(t)U(t) + X^{-1}(qt)D_qU(t)$$

and X is prepared we have that

$$\begin{aligned} \left(X^*(t)P(t)X(qt)\right)^{-1}F &= X^{-1}(qt)(D_qU)(t) - \left(X^*(t)P(t)X(qt)\right)^{-1}(D_qX)^*(t)P(t)U(t) \\ &= D_q(X^{-1}U)(t) + X^{-1}(qt)(D_qX)(t)X^{-1}(t)U(t) \\ &\quad - X^{-1}(qt)P^{-1}(t)X^{*-1}(t)(D_qX)^*(t)P(t)U(t) \\ &= D_q(X^{-1}U)(t) + X^{-1}(qt)P^{-1}(t)P(t)(D_qX)(t)X^{-1}(t)U(t) \\ &\quad - X^{-1}(qt)P^{-1}(t)\left(P(D_qX)X^{-1}\right)^* U \\ &= D_q(X^{-1}U)(t). \end{aligned}$$

Multiplying by the variable and summing both sides from a to t,

$$X^{-1}(t)U(t) - X^{-1}(a)U(a) = (q-1)\left(\sum_{s\in[a,t)_q} s\big(X^*(s)P(s)X(qs)\big)^{-1}\right)F;$$

recovering U yields

$$U(t) = X(t)E + (q-1)X(t)\left(\sum_{s\in[a,t)_q} s\big(X^*(s)P(s)X(qs)\big)^{-1}\right)F$$

provided $E = X^{-1}(a)U(a)$.

Conversely, assume U is given by (3.2). By Theorem 3.3 and linearity U is a solution of (2.1) on $[a,\infty)_q$. Setting $t=a$ in (3.2) leads to E in (3.3). By the constancy of the Wronskian, $W(X,U)(t) \equiv W(X,U)(a)$; suppressing the a, and using (3.2) and the fact that X is prepared,

$$\begin{aligned} W(X,U) &= X^*PD_qU - (D_qX)^*PU = X^*P\left[(D_qX)E + P^{-1}X^{*-1}F\right] - (D_qX)^*PU \\ &= X^*P(D_qX)E + F - (D_qX)^*PXE = F. \end{aligned}$$

From Lemma 3.4, U is a prepared solution of (2.1) iff F^*E is Hermitian. □

4. Prepared Bases

Let X be an $n\times p$ matrix function defined on $(0,\infty)_q$, and define the $2n\times p$ matrix $\mathcal{X}$ by

$$\mathcal{X}(t) = \begin{bmatrix} X(t) \\ \\ X(qt) \end{bmatrix}, \quad t\in(0,\infty)_q; \tag{4.1}$$

we also define the block matrix

$$\mathcal{P}(t) = \begin{bmatrix} 0 & P(t) \\ \\ -P(t) & 0 \end{bmatrix}, \quad t\in(0,\infty)_q.$$

It follows that

$$W(X,X)(t) = \frac{1}{(q-1)t}\left(\mathcal{X}^*\mathcal{P}\mathcal{X}\right)(t), \quad t\in(0,\infty)_q. \tag{4.2}$$

Theorem 4.1. *Assume X is an $n \times p$ matrix solution of* (2.1)*. Then $\mathcal{X}$ has constant rank on $(0,\infty)_q$. Furthermore, if X is a prepared solution of* (2.1) *and* rank $\mathcal{X} = p$*, then $p \le n$.*

Proof. Assume X is an $n \times p$ matrix solution of (2.1). Let $a \in (0,\infty)_q$, and suppose $\mathcal{X}(a)v = 0$ for some vector $v \in \mathbb{C}^p$. Then

$$X(a)v = 0, \quad X(qa)v = 0$$

by assumption; since X solves (2.1), as in the proof of Theorem 2.1 we have that

$$X(a/q)v = 0, \quad X(q^2a)v = 0$$

as well, so that

$$\mathcal{X}(a/q)v = 0, \quad \mathcal{X}(qa)v = 0.$$

Therefore $\mathcal{X}$ has constant rank on $(0,\infty)_q$. Now suppose X is an $n \times p$ prepared solution of (2.1) with rank $\mathcal{X} = p$. Since X is prepared, $W(X,X) \equiv 0$ on $(0,\infty)_q$. By (4.2),

$$\mathcal{X}^*\mathcal{P}\mathcal{X} \equiv 0 \quad \text{on} \quad (0,\infty)_q.$$

As $\mathcal{P}$ is invertible, rank $(\mathcal{P}\mathcal{X}) = p$, and it follows from the previous line that the nullity of $\mathcal{X}^*$ is at least p. Since

$$\text{rank } \mathcal{X}^* + \text{nullity } \mathcal{X}^* = 2n,$$

we have that

$$2p = p + p \le p + \text{nullity } \mathcal{X}^* = 2n,$$

putting $p \le n$. □

Definition 4.2. *An $n \times n$ solution X of* (2.1) *is a prepared basis for* (2.1) *iff X is a prepared solution of* (2.1) *and* rank $\mathcal{X} = n$ *on $(0,\infty)_q$, where $\mathcal{X}$ is given in* (4.1)*.*

Theorem 4.3. *Assume X, Y are $n \times n$ prepared solutions of* (2.1)*. If $W(X,Y)$ is invertible, then X and Y are both prepared bases of* (2.1)*.*

Proof. Assume X, Y are $n \times n$ prepared solutions of (2.1) with $W(X,Y)$ invertible. Note that by Abel's Formula and the definitions above,

$$\text{constant} \equiv W(X,Y)(t) = \frac{1}{(q-1)t}\left(\mathcal{X}^*\mathcal{P}\mathcal{Y}\right)(t), \quad t \in (0,\infty)_q.$$

Let $a \in (0,\infty)_q$, and suppose $\mathcal{Y}(a)v = 0$ for some vector $v \in \mathbb{C}^n$. Then $W(X,Y)v = 0$, so that by the assumption of invertibility $v = 0$. Hence rank $\mathcal{Y}(a) = n$ and due to constant rank by the theorem above, rank $\mathcal{Y} \equiv n$. Thus Y is a prepared basis. In the same manner $v^*\mathcal{X}^*(a) = 0$ implies $v^*W(X,Y) = 0$ implies $v = 0$, and rank $\mathcal{X}(a) =$ rank $\mathcal{X}(t) = n$ and X is a prepared basis as well. $\square$

5. q-Dominant and q-Recessive Solutions

In this main section we seek to introduce the notions of q-dominant and q-recessive solutions for the q-difference equation (2.1) when the equation has an invertible solution; in particular, we ultimately will be able to construct an (essentially) unique q-recessive solution for (2.1) in the event that it admits invertible solutions. Note that throughout the rest of the paper we assume $a \in (0,\infty)_q$.

Definition 5.1. *A solution V of (2.1) is q-dominant at infinity iff V is a prepared basis and there exists an a such that V is invertible on $[a,\infty)_q$ and*

$$\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s), \quad \Upsilon(s) := V^*(s)P(s)V(qs)$$

converges to a Hermitian matrix with finite entries.

Lemma 5.2. *Assume the self-adjoint equation $LX = 0$ has a q-dominant solution V at ∞. If X is any other $n\times n$ solution of (2.1), then*

$$\lim_{t\to\infty} V^{-1}(t)X(t) = K$$

for some $n\times n$ constant matrix K.

Proof. Since V is a q-dominant solution at ∞ of (2.1), there exists an a such that V is invertible on $[a,\infty)_q$. By the second reduction of order theorem, Theorem 3.5,

$$X(t) = V(t)V^{-1}(a)X(a) + (q-1)V(t)\left(\sum_{s\in[a,t)_q} s\Upsilon^{-1}(s)\right) W(V,X)(a).$$

Multiplying on the left by $V^{-1}(t)$ we have

$$V^{-1}(t)X(t) = V^{-1}(a)X(a) + (q-1)\left(\sum_{s\in[a,t)_q} s\Upsilon^{-1}(s)\right) W(V,X)(a).$$

Since V is q-dominant at ∞, the following limit exists:

$$\lim_{t\to\infty} V^{-1}(t)X(t) = K := V^{-1}(a)X(a) + (q-1)\left(\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s)\right) W(V,X)(a).$$

The proof is complete. $\square$

Definition 5.3. *A solution U of* (2.1) *is q-recessive at infinity iff U is a prepared basis and whenever X is any other $n \times n$ solution of* (2.1) *such that $W(X,U)$ is invertible, X is eventually invertible and*

$$\lim_{t\to\infty} X^{-1}(t)U(t) = 0.$$

Lemma 5.4. *If U is a solution of* (2.1) *which is q-recessive at ∞, then for any invertible constant matrix K, the solution UK of* (2.1) *is q-recessive at ∞ as well.*

Proof. The proof follows from the definition. □

Lemma 5.5. *If U is a solution of* (2.1) *which is q-recessive at ∞, and V is a prepared solution of* (2.1) *such that $W(V,U)$ is invertible, then V is q-dominant at ∞.*

Proof. Note that by the assumptions and Theorem 4.3, V is a prepared basis. By the definition of q-recessive, $W(V,U)$ invertible implies that V is invertible on $[a,\infty)_q$ for some $a \in (0,\infty)_q$, and

$$\lim_{t\to\infty} V^{-1}(t)U(t) = 0. \tag{5.1}$$

Let $K := W(V,U)$; by assumption K is invertible, and by Definition 2.3

$$K = \Upsilon(t)V^{-1}(qt)D_qU(t) - ((D_qV)^*PV)(t)V^{-1}(t)U(t)$$

for all $t \in [a,\infty)_q$. Since V is prepared,

$$\begin{aligned}\Upsilon^{-1}(t)K &= V^{-1}(qt)D_qU(t) - V^{-1}(qt)(D_qV)(t)V^{-1}(t)U(t)\\ &= D_q\left(V^{-1}U\right)(t).\end{aligned}$$

Multiply by s, sum both sides from a to ∞, and use (5.1) to see that

$$\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s) = \frac{-1}{(q-1)}V^{-1}(a)U(a)K^{-1}$$

converges to a Hermitian matrix, as Υ is Hermitian. Thus V is q-dominant at ∞. □

Theorem 5.6. *Assume* (2.1) *has a solution V which is q-dominant at ∞. Then*

$$U(t) := (q-1)V(t)\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1} = (q-1)V(t)\sum_{s\in[t,\infty)_q} s\Upsilon^{-1}(s)$$

is a solution of (2.1) *which is q-recessive at ∞ and $W(V,U) = -I$.*

Proof. Since V is q-dominant at ∞, U is a well-defined function and can be written as

$$U(t) = (q-1)V(t)\left[\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s) - \left(\sum_{s\in[a,t)_q} s\Upsilon^{-1}(s)\right) I\right], \quad t\in[a,\infty)_q;$$

by the second reduction of order theorem, Theorem 3.5, U is a solution of (2.1) of the form (3.2) with

$$E = (q-1)\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s), \qquad F=-I.$$

From (3.3), $W(V,U)=F=-I$. Since

$$E^*F = (1-q)\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s)$$

is Hermitian, U is a prepared solution of (2.1), and $W(-V,U)=I$ implies that U and $-V$ are normalized prepared bases. Let X be an $n\times n$ matrix solution of $LX=0$ such that $W(X,U)$ is invertible. By the second reduction of order theorem,

$$\begin{aligned} X(t) &= V(t)\left[V^{-1}(a)X(a) + (q-1)\sum_{s\in[a,t)_q} s\Upsilon^{-1}(s)W(V,X)\right] \\ &= V(t)C_1 + U(t)C_2, \end{aligned} \tag{5.2}$$

where

$$C_1 := V^{-1}(a)X(a) + (q-1)\sum_{s\in[a,\infty)_q} s\Upsilon^{-1}(s)W(V,X)$$

and

$$C_2 := -W(V,X).$$

Note that

$$W(X,U) = C_1^*W(V,U) + C_2^*W(U,U) = -C_1^*.$$

As $W(X,U)$ is invertible by assumption, C_1 is invertible. From (5.2),

$$\begin{aligned} \lim_{t\to\infty} V^{-1}(t)X(t) &= \lim_{t\to\infty}\left(C_1 + V^{-1}(t)U(t)C_2\right) \\ &= C_1 + (q-1)\lim_{t\to\infty}\sum_{s\in[t,\infty)_q} s\Upsilon^{-1}(s)C_2 = C_1 \end{aligned}$$

is likewise invertible. Consequently for large t, $X(t)$ is invertible. Lastly,

$$\begin{aligned}
\lim_{t\to\infty} X^{-1}(t)U(t) &= \lim_{t\to\infty} \left[V(t)C_1 + U(t)C_2\right]^{-1} U(t) \\
&= \lim_{t\to\infty} \left[C_1 + V^{-1}(t)U(t)C_2\right]^{-1} V^{-1}(t)U(t) \\
&= \left[C_1 + 0\right]^{-1} 0 = 0.
\end{aligned}$$

Therefore U is a q-recessive solution at ∞. □

Theorem 5.7. *Assume* (2.1) *has a solution U which is q-recessive at ∞, and $U(a)$ is invertible for some $a \in (0,\infty)_q$. Then U is uniquely determined by $U(a)$, and* (2.1) *has a solution V which is q-dominant at ∞.*

Proof. Assume $U(a)$ is invertible; let V be the unique solution of the intial value problem

$$LV = 0, \quad V(a) = 0, \quad D_qV(a) = I.$$

Then V is a prepared basis and

$$W(V,U) = W(V,U)(a) = (V^*PD_qU)(a) - (PD_qV)^*(a)U(a) = -P(a)U(a)$$

is invertible. It follows from Lemma 5.5 that V is q-dominant at ∞. Let Γ be an arbitrary but fixed $n \times n$ constant matrix. Let X solve the initial value problem

$$LX = 0, \quad X(a) = I, \quad D_qX(a) = \Gamma.$$

By Theorem 5.2,

$$\lim_{t\to\infty} V^{-1}(t)X(t) = K,$$

where K is an $n \times n$ constant matrix; note that K is independent of the q-recessive solution U. Using the initial conditions at a, by uniqueness of solutions it is easy to see that there exist constant $n \times n$ matrices C_1 and C_2 such that

$$U(t) = X(t)C_1 + V(t)C_2,$$

where $C_1 = U(a)$ is invertible. Consequently, using the q-recessive nature of U, we have

$$0 = \lim_{t\to\infty} V^{-1}(t)U(t) = \lim_{t\to\infty} \left(V^{-1}(t)X(t)U(a) + C_2\right) = KU(a) + C_2,$$

so that $C_2 = -KU(a)$. Thus the initial condition for D_qU is

$$D_qU(a) = (\Gamma - K)U(a),$$

and the q-recessive solution U is uniquely determined by its initial value $U(a)$. □

Theorem 5.8. *Assume* (2.1) *has a solution* U *which is* q*-recessive at* ∞ *and a solution* V *which is* q*-dominant at* ∞. *If* U *and* $\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}$ *are both invertible for large* $t \in (0,\infty)_q$, *then there exists an invertible constant matrix* K *such that*

$$U(t) = (q-1)V(t)\left(\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}\right)K$$

for large t. *In addition,* $W(U,V)$ *is invertible and*

$$\lim_{t\to\infty} V^{-1}(t)U(t) = 0.$$

Proof. For sufficiently large $t \in (0,\infty)_q$ define

$$Y(t) = (q-1)V(t)\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}.$$

By Theorem 5.6, Y is also a q-recessive solution of (2.1) at ∞ and $W(V,Y) = -I$. Because U and $\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}$ are both invertible for large $t \in (0,\infty)_q$, Y is likewise invertible for large t, and

$$\lim_{t\to\infty} V^{-1}(t)Y(t) = 0$$

by the q-recessive nature of Y. Choose $a \in (0,\infty)_q$ large enough to ensure that U and Y are invertible in $[a,\infty)_q$. By Lemma 5.4 the solution given by

$$X(t) := Y(t)Y^{-1}(a)U(a), \quad t \in [a,\infty)_q$$

is yet another q-recessive solution at ∞. Since U and X are q-recessive solutions at ∞ and $U(a) = X(a)$, we conclude from the uniqueness established in Theorem 5.7 that $X \equiv U$. Thus

$$\begin{aligned} U(t) &= Y(t)Y^{-1}(a)U(a), \quad t \in [a,\infty)_q \\ &= (q-1)V(t)\left(\sum_{s\in[t,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}\right)K, \end{aligned}$$

where $K := Y^{-1}(a)U(a)$ is an invertible constant matrix. □

The next result, when the domain is $\mathbb{Z}$ instead of $(0,\infty)_q$, relates the convergence of infinite series, the convergence of certain continued fractions, and the existence of recessive solutions; for more see [2] and the references therein.

Theorem 5.9 (Connection Theorem). *Let X and V be solutions of* (2.1) *determined by the initial conditions*

$$X(a) = I, \quad D_qX(a) = P^{-1}(a)K, \quad \textit{and} \quad V(a) = 0, \quad D_qV(a) = P^{-1}(a),$$

respectively, where $a \in (0,\infty)_q$ and K is a constant Hermitian matrix. Then X, V are normalized prepared bases of (2.1)*, and the following are equivalent:*

(i) *V is q-dominant at ∞;*
(ii) *V is invertible for large $t \in (0,\infty)_q$ and $\lim_{t\to\infty} V^{-1}(t)X(t)$ exists as a Hermitian matrix $\Omega(K)$ with finite entries;*
(iii) *there exists a solution U of* (2.1) *which is q-recessive at ∞, with $U(a)$ invertible.*

If (i), (ii), and (iii) hold then

$$(D_qU)(a)U^{-1}(a) = D_qX(a) - (D_qV)(a)\Omega(K) = -P^{-1}(a)\Omega(0).$$

Proof. Since $V(a) = 0$, V is a prepared solution of (2.1). Also,

$$W(X,X) = W(X,X)(a) = (X^*PD_qX - (D_qX)^*PX)(a) = IK - K^*I = 0$$

as K is Hermitian, making X a prepared solution of (2.1) as well. Checking

$$W(X,V) = W(X,V)(a) = (X^*PD_qV - (D_qX)^*PV)(a) = I - 0 = I,$$

we see that X, V are normalized prepared bases of (2.1). Now we show that (i) implies (ii). If V is a q-dominant solution of (2.1) at ∞, then there exists a $t_1 \in (a,\infty)_q$ such that $V(t)$ is invertible for $t \in [t_1,\infty)_q$, and the sum

$$\sum_{s\in[t_1,\infty)_q} s(V^*(s)P(s)V(qs))^{-1}$$

converges to a Hermitian matrix with finite entries. By the second reduction of order theorem,

$$X(t) = V(t)E + (q-1)V(t)\left(\sum_{s\in[t_1,t)_q} s\left(V^*(s)P(s)V(qs)\right)^{-1}\right)F, \tag{5.3}$$

where

$$E = V^{-1}(t_1)X(t_1), \qquad F = W(V,X)(t_1) = -W(X,V)^* = -I.$$

Since X is prepared, $E^*F = -E^*$ is Hermitian, whence E is Hermitian. As a result, by (5.3) we have

$$\lim_{t\to\infty} V^{-1}(t)X(t) = E - (q-1) \sum_{s\in[t_1,\infty)_q} s\left(V^*(s)P(s)V(qs)\right)^{-1}$$

converges to a Hermitian matrix with finite entries, and (ii) holds. Next we show that (ii) implies (iii). If V is invertible on $[t_1,\infty)_q$ and

$$\lim_{t\to\infty} V^{-1}(t)X(t) = \Omega \tag{5.4}$$

exists as a Hermitian matrix, then from (5.3) and (5.4) we have

$$\Omega = \lim_{t\to\infty} V^{-1}(t)X(t) = E - (q-1) \sum_{s\in[t_1,\infty)_q} s\left(V^*(s)P(s)V(qs)\right)^{-1};$$

in other words,

$$(q-1) \sum_{s\in[t_1,\infty)_q} s\left(V^*(s)P(s)V(qs)\right)^{-1} = E - \Omega.$$

Define

$$U(t) := X(t) - V(t)\Omega. \tag{5.5}$$

Then

$$\begin{aligned} W(U,U) &= W(X - V\Omega, X - V\Omega) \\ &= W(X,X) - W(X,V)\Omega - \Omega^*W(V,X) + \Omega^*W(V,V)\Omega \\ &= -\Omega + \Omega^* = 0, \end{aligned}$$

and $U(a) = X(a) = I$, making U a prepared basis for (2.1). If X_1 is an $n\times n$ matrix solution of $LX = 0$ such that $W(X_1,U)$ is invertible, then

$$X_1(t) = V(t)C_1 + U(t)C_2 \tag{5.6}$$

for some constant matrices C_1 and C_2 determined by the initial conditions at a. It follows that

$$\begin{aligned} W(X_1,U) &= W(VC_1 + UC_2, U) = C_1^*W(V,U) + C_2^*W(U,U) \\ &= C_1^*W(V,U) = C_1^*W(V,U)(a) = -C_1^* \end{aligned}$$

by (5.5), so that C_1 is invertible. From (5.4) and (5.5) we have that

$$\lim_{t\to\infty} V^{-1}(t)U(t) = \lim_{t\to\infty}\left[V^{-1}(t)X(t) - \Omega\right] = 0,$$

resulting in

$$\lim_{t\to\infty} V^{-1}(t)X_1(t) = \lim_{t\to\infty} \left[C_1 + V^{-1}(t)U(t)C_2\right] = C_1,$$

which is invertible. Thus $X_1(t)$ is invertible for large $t \in (0,\infty)_q$, and

$$\begin{aligned}\lim_{t\to\infty} X_1^{-1}(t)U(t) &= \lim_{t\to\infty}[V(t)C_1 + U(t)C_2]^{-1}U(t)\\ &= \lim_{t\to\infty}[C_1 + V^{-1}(t)U(t)C_2]^{-1}V^{-1}(t)U(t) = C_1^{-1}(0) = 0.\end{aligned}$$

Hence U is a q-recessive solution of (2.1) at ∞ and (iii) holds. Finally we show that (iii) implies (i). If U is a q-recessive solution of (2.1) at ∞ with $U(a)$ invertible, then

$$W(V,U) = W(V,U)(a) = -U(a)$$

is also invertible. Hence by Lemma 5.5, V is a q-dominant solution of (2.1) at ∞.

To complete the proof, assume (i), (ii), and (iii) hold. It can be shown via initial conditions at a that

$$U(t) = X(t)U(a) + V(t)C$$

for some suitable constant matrix C. By (ii),

$$\lim_{t\to\infty} V^{-1}(t)X(t) = \Omega(K),$$

and thus

$$V^{-1}(t)U(t) = V^{-1}(t)X(t)U(a) + C.$$

As U is a q-recessive solution at ∞ by (iii),

$$0 = \lim_{t\to\infty}\left(V^{-1}(t)X(t)U(a) + C\right) = \Omega(K)U(a) + C,$$

yielding

$$U(t) = [X(t) - V(t)\Omega(K)]\, U(a).$$

An application of the quantum derivative D_q at a yields

$$(D_qU)(a)U^{-1}(a) = D_qX(a) - (D_qV)(a)\Omega(K).$$

Now let Y be the unique solution of the initial value problem

$$LY = 0, \quad Y(a) = I, \quad D_qY(a) = 0.$$

Using the initial conditions at a we see that

$$X(t) = Y(t) + V(t)K.$$

Consequently,

$$\lim_{t\to\infty} V^{-1}(t)X(t) = \lim_{t\to\infty} V^{-1}(t)Y(t) + K$$

implies, by (ii) and the fact that $X = Y$ when $K = 0$, that

$$\Omega(K) = \Omega(0) + K.$$

Therefore

$$D_q X(a) - (D_q V)(a)\Omega(K) = -(D_q V)(a)\Omega(0) = -P^{-1}(a)\Omega(0).$$

Thus the proof is complete. $\square$

We will also be interested in analyzing the self-adjoint vector q-difference equation

$$Lx = 0, \quad \text{where} \quad Lx(t) := D^q \left(PD_q x\right)(t) + Q(t)x(t), \quad t \in [a, \infty)_q, \tag{5.7}$$

where x is an $n \times 1$ vector-valued function defined on $(0, \infty)_q$. We will see interesting relationships between the so-called unique two-point property (defined below) of the nonhomogeneous vector equation $Lx = h$, disconjugacy of $Lx = 0$, and the construction of q-recessive solutions at infinity to the matrix equation $LX = 0$. The following theorem can be proven by modifying the proof of Theorem 2.1.

Theorem 5.10. *Let h be an $n \times 1$ vector function defined on $[a, \infty)_q$. Then the nonhomogeneous vector initial value problem*

$$Ly = D^q(PD_q y) + Qy = h, \quad y(a) = y_a, \quad D_q y(a) = y_a' \tag{5.8}$$

has a unique solution.

Definition 5.11. *Assume h is an $n \times 1$ vector function defined on $[a, \infty)_q$. Then the vector dynamic equation $Lx = h$ has the unique two-point property on $[a, \infty)_q$ provided given any $a \le t_1 < t_2$ in $(0, \infty)_q$, if u and v are solutions of $Lx = h$ with $u(t_1) = v(t_1)$ and $u(t_2) = v(t_2)$, then $u \equiv v$ on $[a, \infty)_q$.*

Theorem 5.12. *If the homogeneous vector equation* (5.7) *has the unique two-point property on $[a, \infty)_q$, then the boundary value problem*

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta,$$

where $a \le t_1 < t_2$ in $(0, \infty)_q$ and $\alpha, \beta \in \mathbb{C}^n$, has a unique solution on $[a, \infty)_q$.

Proof. If $t_2 = qt_1$, then the boundary value problem is an initial value problem and the result holds by Theorem 5.10. Assume $t_2 > qt_1$. Let $X(t, t_1)$ and $Y(t, t_1)$ be the unique $n \times n$ matrix solutions of (2.1) determined by the initial conditions

$$X(t_1, t_1) = 0, \quad D_q X(t_1, t_1) = I, \quad \text{and} \quad Y(t_1, t_1) = I, \quad D_q Y(t_1, t_1) = 0;$$

then a general solution of (5.7) is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta, \tag{5.9}$$

for $\gamma, \delta \in \mathbb{C}^n$, as $x(t_1) = \delta$ and $D_q x(t_1) = \gamma$. By the unique two-point property the homogeneous boundary value problem

$$Lx = 0, \quad x(t_1) = 0, \quad x(t_2) = 0$$

has only the trivial solution. For x given by (5.9), the boundary condition at t_1 implies that $\delta = 0$, and the boundary condition at t_2 yields

$$X(t_2, t_1)\gamma = 0;$$

by uniqueness and the fact that x is trivial, $\gamma = 0$ is the unique solution, meaning $X(t_2, t_1)$ is invertible. Next let v be the solution of the initial value problem

$$Lv = h, \quad v(t_1) = 0, \quad D_q v(t_1) = 0.$$

Then the general solution of $Lx = h$ is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta + v(t).$$

We now show that the boundary value problem

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta$$

has a unique solution. The boundary condition at t_1 implies that $\delta = \alpha$. The condition at t_2 leads to the equation

$$\beta = X(t_2, t_1)\gamma + Y(t_2, t_1)\alpha + v(t_2);$$

since $X(t_2, t_1)$ is invertible, this can be solved uniquely for γ. □

Corollary 5.13. *If the homogeneous vector equation* (5.7) *has the unique two-point property on* $[a, \infty)_q$*, then the matrix boundary value problem*

$$LX = 0, \quad X(t_1) = M, \quad X(t_2) = N$$

has a unique solution, where M *and* N *are given constant* $n \times n$ *matrices.*

Proof. Modify the proof of Theorem 5.12 to get existence and uniqueness. □

Theorem 5.14. *Assume the homogeneous vector equation* (5.7) *has the unique two-point property on* $[a,\infty)_q$. *Further assume* U *is a solution of* (2.1) *which is* q*-recessive at* ∞ *with* $U(a)$ *invertible. For each fixed* $s \in (a,\infty)_q$, *let* $Y(t,s)$ *be the solution of the boundary value problem*

$$LY(t,s) = 0, \quad Y(a,s) = I, \quad Y(s,s) = 0.$$

Then the q*-recessive solution* $U(t)U^{-1}(a)$ *is uniquely determined by*

$$U(t)U^{-1}(a) = \lim_{s\to\infty} Y(t,s). \tag{5.10}$$

Proof. Assume U is a solution of (2.1) which is q-recessive at ∞ with $U(a)$ invertible. Let V be the unique solution of the initial value problem

$$LV = 0, \quad V(a) = 0, \quad D_qV(a) = P^{-1}(a).$$

By the connection theorem, Theorem 5.9, V is invertible for large t. By checking boundary conditions at a and s for s large, we get that

$$Y(t,s) = -V(t)V^{-1}(s)U(s)U^{-1}(a) + U(t)U^{-1}(a).$$

Then

$$W(V,U) = W(V,U)(a) = \left(V^*PD_qU - (D_qV)^*PU\right)(a) = -U(a)$$

is invertible, and by the q-recessive nature of U,

$$\lim_{t\to\infty} V^{-1}(t)U(t) = 0.$$

As a result,

$$\lim_{s\to\infty} Y(t,s) = 0 + U(t)U^{-1}(a),$$

and the proof is complete. □

Definition 5.15. *A prepared vector solution* x *of* (5.7) *has a generalized zero at* $a \in (0,\infty)_q$ *iff* $x(a/q) \neq 0$ *and* $x^*(a/q)P(a/q)x(a) \leq 0$. *Equation* (5.7) *is disconjugate on* $[a,\infty)_q$ *iff no nontrivial prepared vector solution of* (5.7) *has two generalized zeros in* $[a,\infty)_q$.

Definition 5.16. *A prepared basis* X *of* (2.1) *has a generalized zero at* $a \in (0,\infty)_q$ *iff* $X(a)$ *is noninvertible or* $X^*(a/q)P(a/q)X(a)$ *is invertible but* $X^*(a/q)P(a/q)X(a) \leq 0$.

Lemma 5.17. *If a prepared basis* X *of* (2.1) *has a generalized zero at* $a \in (0,\infty)_q$, *then there exists a vector* $\gamma \in \mathbb{C}^n$ *such that* $x = X\gamma$ *is a nontrivial prepared solution of* (5.7) *with a generalized zero at* a.

Proof. The proof follows from Definitions 5.15 and 5.16. □

Theorem 5.18. *If the vector equation* (5.7) *is disconjugate on* $[a,\infty)_q$*, then the matrix equation* (2.1) *has a solution* U *which is* q*-recessive at* ∞ *with* $U(t)$ *invertible for* $t \in [qa,\infty)_q$*.*

Proof. Let X be the solution of the initial value problem

$$LX = 0, \quad X(a) = 0, \quad D_q X(a) = I;$$

then X is a prepared solution of (2.1). If X is not invertible on $[qa,\infty)_q$, then there exists a $t_1 > a$ such that $X(t_1)$ is singular. But then there exists a nontrivial vector $\delta \in \mathbb{C}^n$ such that $X(t_1)\delta = 0$. If $x(t) := X(t)\delta$, then x is a nontrivial prepared solution of (5.7) with

$$x(a) = 0, \quad x(t_1) = 0,$$

a contradiction of disconjugacy. Hence X is invertible in $[qa,\infty)_q$. We next claim that

$$X^*(t)P(t)X(qt) > 0, \quad t \in [qa,\infty)_q; \tag{5.11}$$

if not, there exists $t_2 \in [qa,\infty)_q$ such that

$$X^*(t_2)P(t_2)X(qt_2) \not> 0.$$

It follows that there exists a nontrivial vector γ such that $x(t) := X(t)\gamma$ is a nontrivial prepared vector solution of $Lx = 0$ with a generalized zero at qt_2. Using the initial condition for X, however, we have $x(a) = 0$, another generalized zero, a contradiction of the assumption that the vector equation (5.7) is disconjugate on $[a,\infty)_q$. Thus (5.11) holds. Define the matrix function

$$V(t) := X(t)\left[I + (q-1)\sum_{s\in[qa,t)_q} s\left(X^*(s)P(s)X(qs)\right)^{-1}\right], \quad t \in [qa,\infty)_q.$$

By Theorem 3.5, V is a prepared solution of $LV = 0$ with $W(X,V) = I$. Note that V is also invertible on $[qa,\infty)_q$, so that by the reduction of order theorem again,

$$X(t) = V(t)\left[I - (q-1)\sum_{s\in[qa,t)_q} s\left(V^*(s)P(s)V(qs)\right)^{-1}\right], \quad t \in [qa,\infty)_q.$$

Consequently,

$$I = [V^{-1}X][X^{-1}V](t) = \left[I - (q-1)\sum_{s\in[qa,t)_q} s\Upsilon^{-1}(s)\right]\left[I + (q-1)\sum_{s\in[qa,t)_q} s\Xi^{-1}(s)\right],$$

where

$$\Upsilon(s) := V^*(s)P(s)V(qs), \quad \Xi(s) := X^*(s)P(s)X(qs) > 0.$$

Since the second factor is strictly increasing by (5.11) and bounded below by I, the first factor is positive definite and strictly decreasing, ensuring the existence of a limit, in other words, we have

$$0 \le I - (q-1)\sum_{s\in[qa,\infty)_q} s\Upsilon^{-1}(s) < I - (q-1)\sum_{s\in[qa,t)_q} s\Upsilon^{-1}(s) \le I.$$

It follows that

$$0 \le \sum_{s\in[qa,t)_q} s\Upsilon^{-1}(s) < \sum_{s\in[qa,\infty)_q} s\Upsilon^{-1}(s) \le I, \quad t\in[qa,\infty)_q,$$

and V is a q-dominant solution of (2.1) at ∞. Set

$$U(t) := (q-1)V(t)\sum_{s\in[t,\infty)_q} s\Upsilon^{-1}(s).$$

By Theorem 5.6, U is a q-recessive solution of (2.1) at ∞. Since

$$U(t) = (q-1)V(t)\left[\sum_{s\in[qa,\infty)_q} s\Upsilon^{-1}(s) - \sum_{s\in[qa,t)_q} s\Upsilon^{-1}(s)\right],$$

V is invertible on $[qa,\infty)_q$, and the difference in brackets is positive definite on $[qa,\infty)_q$, we get that U is invertible on $[qa,\infty)_q$ as well, and the conclusion of the theorem follows. □

Corollary 5.19. *Assume the vector equation* (5.7) *is disconjugate on* $[a,\infty)_q$, *and* K *is a constant Hermitian matrix. Let* U, V *be the matrix solutions of* $LX = 0$ *satisfying the initial conditions*

$$U(a) = I, \quad D_qU(a) = P^{-1}(a)K, \quad \text{and} \quad V(a) = 0, \quad D_qV(a) = P^{-1}(a).$$

Then V *is invertible in* $[qa,\infty)_q$, V *is a* q*-dominant solution of* (2.1) *at* ∞, *and*

$$\lim_{t\to\infty} V^{-1}(t)U(t)$$

exists as a Hermitian matrix.

Proof. By Theorem 5.18, the matrix equation (2.1) has a solution U which is q-recessive at ∞ with $U(t)$ invertible for $t \in [qa, \infty)_q$. Thus (iii) of the connection theorem, Theorem 5.9 holds; by (i), then, V is a q-dominant solution of (2.1) at ∞, and by (ii),

$$\lim_{t\to\infty} V^{-1}(t)U(t)$$

exists as a Hermitian matrix. Since $V(a) = 0$ and the vector equation (5.7) is disconjugate on $[a, \infty)_q$,

$$V^*(t)P(t)V(qt) > 0, \quad t \in [qa, \infty)_q.$$

In particular, V is invertible in $[qa, \infty)_q$. $\square$

Theorem 5.20. *If the vector equation* (5.7) *is disconjugate on* $[a, \infty)_q$*, then* $Lx(t) = h(t)$ *has the unique two-point property in* $[a, \infty)_q$*. In particular, every boundary value problem of the form*

$$Lx(t) = h(t), \qquad x(\tau_1) = \alpha, \qquad x(\tau_2) = \beta,$$

where $\tau_1, \tau_2 \in [a, \infty)_q$ *with* $\tau_1 < \tau_2$*, and where* α, β *are given* n*-vectors, has a unique solution.*

Proof. By Theorem 5.18, disconjugacy of (5.7) implies the existence of a prepared, invertible matrix solution of (2.1). Thus by Theorem 5.12, it suffices to show that (5.7) has the unique two-point property in $[a, \infty)_q$. To this end, assume u, v are solutions of $Lx = 0$, and there exist points $s_1, s_2 \in (0, \infty)_q$ such that $a \le s_1 < s_2$ and

$$u(s_1) = v(s_1), \qquad u(s_2) = v(s_2).$$

If $s_2 = qs_1$, then u and v satisfy the same initial conditions and $u \equiv v$ by uniqueness; hence we assume $s_2 > qs_1$. Setting $x = u - v$, we see that x solves the initial value problem

$$Lx = 0, \qquad x(\tau_1) = 0, \qquad x(\tau_2) = 0.$$

Since $Lx = 0$ is disconjugate and x is a prepared solution with two generalized zeros, it must be that $x \equiv 0$ in $[a, \infty)_q$. Consequently, $u = v$ and the two-point property holds. $\square$

Corollary 5.21 (Construction of the Recessive Solution)**.** *Assume the vector equation* (5.7) *is disconjugate on* $[a, \infty)_q$*. For each* $s \in (a, \infty)_q$*, let* $U(t, s)$ *be the solution of the boundary value problem*

$$LU(\cdot, s) = 0, \quad U(a, s) = I, \quad D_qU(s, s) = 0.$$

Then the solution U with $U(a) = I$ which is q-recessive at ∞ is given by

$$U(t) = \lim_{s\to\infty} U(t,s),$$

satisfying

$$U^*(t)P(t)U(qt) > 0, \quad t \in [a,\infty)_q. \tag{5.12}$$

Proof. By Theorem 5.18 and Theorem 5.20, $LX = 0$ has a q-recessive solution and $Lx = h$ has the unique two-point property. The conclusion then follows from Theorem 5.14, except for (5.12). From the initial condition $U(s,s) = 0$ and the fact that $Lx = 0$ is disconjugate, it follows that

$$U^*(t,s)P(t)U(qt,s) > 0$$

holds in $[a, s/q)_q$. Again from Theorem 5.14,

$$\lim_{s\to\infty} U(t,s) = U(t)U^{-1}(a) = U(t),$$

so that U invertible on $[a,\infty)_q$ and (5.12) holds. □

Remark 5.22. *In an analogous way we could analyze the related (formally) self-adjoint quantum (h-difference) system*

$$D^h(PD_hX)(t) + Q(t)X(t) = 0, \quad t \in (0,\infty)_h := \{h, 2h, 3h, \cdots\}, \tag{5.13}$$

where the real scalar $h > 0$ and the h-derivatives are given, respectively, by the difference quotients

$$(D_hy)(t) = \frac{y(t+h) - y(t)}{h} \quad \textit{and} \quad (D^hy)(t) = \frac{y(t) - y(t-h)}{h} = (D_hy)(t-h).$$

In the case where invertible solutions of (5.13) exist, their characterization as h-dominant and/or h-recessive solutions at infinity can be developed in parallel with the previous results on the q-equation (2.1). As q approaches 1 in the limit or h approaches zero in the limit, we can recover results from classical ordinary differential equations.

6. Future Directions

In this section we lay the groundwork for possible further exploration of the nonhomogeneous equation by introducing the Pólya factorization for the self-adjoint matrix q-difference operator L defined in (2.1), which in turn leads to a variation of parameters result.

Theorem 6.1 (Pólya Factorization). *If* (2.1) *has a prepared solution* $X > 0$ *(positive definite) on an interval* $\mathcal{I} \subset (0,\infty)_q$ *such that* $X^*(t)P(t)X(qt) > 0$ *for* $t \in \mathcal{I}$*, then for any matrix function* Y *defined on* $(0,\infty)_q$ *we have on the interval* $\mathcal{I}$ *a Pólya factorization*

$$LY = M_1^* D^q \left\{ M_2 D_q(M_1 Y) \right\}, \quad M_1(t) := X^{-1}(t) > 0, \quad M_2(t) := X^*(t)P(t)X(qt) > 0.$$

Proof. Assume $X > 0$ is a prepared solution of (2.1) on $\mathcal{I} \subset (0,\infty)_q$ such that $M_2 > 0$ on $\mathcal{I}$, and let Y be a matrix function defined on $(0,\infty)_q$. Then X is invertible and

$$\begin{aligned}
LY &\overset{\text{Thm 2.5}}{=} (X^*)^{-1} D^q W(X,Y) \\
&\overset{\text{Def 2.3}}{=} (X^*)^{-1} D^q \left\{ X^* P D_q Y - (D_q X)^* P Y \right\} \\
&= M_1^* D^q \left\{ X^* [P D_q Y - X^{*-1} (D_q X)^* P Y] \right\} \\
&\overset{\text{Thm 2.15}}{=} M_1^* D^q \left\{ X^* [P D_q Y - P (D_q X) X^{-1} Y] \right\} \\
&= M_1^* D^q \left\{ M_2 [X^{-1}(qt) D_q Y - X^{-1}(qt)(D_q X) X^{-1} Y] \right\} \\
&= M_1^* D^q \left\{ M_2 [X^{-1}(qt) D_q Y + (D_q X^{-1}) Y] \right\} \\
&= M_1^* D^q \left\{ M_2 D_q (X^{-1} Y) \right\} \\
&= M_1^* D^q \left\{ M_2 D_q (M_1 Y) \right\},
\end{aligned}$$

for M_1 and M_2 as defined in the statement of the theorem. $\square$

Theorem 6.2 (Variation of Parameters). *Let* H *be an* $n \times n$ *matrix function defined on* $[a,\infty)_q$*. If the homogeneous matrix equation* (2.1) *has a prepared solution* X *with* $X(t)$ *invertible for* $t \in [a,\infty)_q$*, then the nonhomogeneous equation* $LY = H$ *has a solution* Y *given by*

$$\begin{aligned}
Y(t) &= X(t)X^{-1}(a)Y(a) + (q-1)X(t) \sum_{s \in [a,t)_q} s \left(X^*(s)P(s)X(qs) \right)^{-1} W(X,Y)(a) \\
&\quad + \frac{(q-1)^2}{q} X(t) \sum_{s \in [a,t)_q} s \left((X^*(s)P(s)X(qs))^{-1} \sum_{\tau \in (a,s]_q} \tau X^*(\tau) H(\tau) \right).
\end{aligned}$$

Proof. Let Y be a matrix function defined on $(0,\infty)_q$, and assume X is a prepared solution of (2.1) invertible on $[a,\infty)_q$. As in Theorem 6.1, we factor LY to get

$$H(t) = LY(t) = X^{*-1}(t) D^q \left(X^*(t)P(t)X(qt) D_q (X^{-1}Y)(t) \right).$$

Multiplying by sX^* and summing over $(a,t]_q$ we arrive at

$$X^*(t)P(t)X(qt)D_q(X^{-1}Y)(t) - W(X,Y)(a) = \left(1-\frac{1}{q}\right)\sum_{s\in(a,t]_q} sX^*(s)H(s),$$

where

$$W(X,Y)(a) = X^*(a)P(a)X(qa)D_q(X^{-1}Y)(a)$$

since X is prepared. This leads to

$$D_q(X^{-1}Y)(t) = (X^*(t)P(t)X(qt))^{-1}\left(W(X,Y)(a) + \left(1-\frac{1}{q}\right)\sum_{s\in(a,t]_q} sX^*(s)H(s)\right),$$

which is then multiplied by $(q-1)s$ and summed over $[a,t)_q$ to obtain the form for Y given in the statement of the theorem. Clearly the right-hand side of the form of Y above reduces to $Y(a)$ at a, and since X is an invertible prepared solution, by Theorem 2.15 the quantum derivative reduces to $D_qY(a)$ at a. □

Corollary 6.3. *Let H be an $n\times n$ matrix function defined on $[a,\infty)_q$. If the homogeneous matrix equation* (2.1) *has a prepared solution X with $X(t)$ invertible for $t\in[a,\infty)_q$, then the nonhomogeneous initial value problem*

$$LY = D^q(PD_qY) + QY = H, \quad Y(a) = Y_a, \quad D_qY(a) = Y_a' \tag{6.1}$$

has a unique solution.

Proof. By Theorem 6.2, the nonhomogeneous initial value problem (6.1) has a solution. Suppose Y_1 and Y_2 both solve (6.1). Then $X = Y_1 - Y_2$ solves the homogeneous initial value problem

$$LX = 0, \quad X(a) = 0, \quad D_qX(a) = 0;$$

by Theorem 2.1, this has only the trivial solution $X = 0$. □

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