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Global Exponential Stability of Impulsive Dynamical Systems with Distributed Delays

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Abstract: In this paper, the global exponential stability of dynamical systems with distributed delays and impulsive effect is investigated. By establishing an impulsive differential-integro inequality, we obtain some sufficient conditions ensuring the global exponential stability of the dynamical system. Three examples are given to illustrate the effectiveness of our theoretical results.
Key Words and Phrases: Global Exponential Stability, Impulsive Differential-integro Inequality, Distributed Delays.
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1 Introduction

Recently, dynamical system structure has played an important role in real life, so the stability of it has been extensively studied due to its important role in designs and applications [1-11]. Most of those widely used dynamical system today are classified into two groups: continuous and discrete dynamical systems. However, there are still many dynamical systems existing in nature which display some kind of dynamics between the two groups. These include, for example, frequency-modulated signal processing systems, optimal control models in economics, flying object motions and many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology. All these systems are characterized by the fact that at certain moments of time they experience abrupt changes of states [12,13]. Moreover, impulsive phenomena can also be found in other fields of electronics, automatic control systems, and information science. Many sudden and sharp changes occur instantaneously, in the form of impulse , which cannot be well described by using pure continuous or pure discrete models. Therefore, the study of stability to impulsive systems has attracted considerable attention [14-18].

As is well known, the use of constant fixed delays or time-varying delays in models of delayed feedback provides a good approximation in simple circuits consisting of a

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small number of cells, therefore, in papers [15,16], time-varying delay models with impulsive effects are considered. However, dynamical systems usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances, the signal propagation is not instantaneous and cannot be modelled with discrete delays. A more appropriate way is to incorporate continuously distributed delays. To the best of the authors' knowledge, there are few authors who have studied the global exponential stability of the dynamical system with distributed delays and impulsive effect [19,20]. The goal of this paper is to provide such a study. By establishing an impulsive differential-integro inequality, we obtain some sufficient conditions ensuring the global exponential stability of impulsive dynamical system with distributed delays.

In this paper, on the basis of the structure of recurrent neural networks with distributed delays, we consider a class of general dynamical system(s) with distributed delays.

$$\begin{cases} \dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n}\{a_{ij} f_j(x_j(t)) + c_{ij}\int_{-\infty}^{t} k(t-s) g_j(x_j(s))ds\} + I_i, t \neq t_k, \\ x_i(t_k) = \sum_{j=1}^{n}\{w_{ij}^k x_j(t_k^-) + e_{ij}^k \int_{-\infty}^{t_k} k(t_k - s) n_{jk}(x_j(s))ds\} + J_{ik}, \qquad t = t_k, \end{cases} \tag{1}$$

where $i = 1, \cdots, n, t \geq t_0$, the fixed times t_k satisfy $t_1 < t_2 < \cdots, \lim\limits_{k\to\infty} t_k = \infty, k = 1, 2, \cdots$.The first part (called the continuous part) of model (1) describes the continuous evolution processes of the dynamical system, $b_i > 0, a_{ij}, c_{ij}, I_i$ are constants, $f_j(x_j), g_j(x_j)$ are continuous functions, $k(s)$ are delay kernel functions, and satisfy

$$\int_0^{+\infty} k(s)ds = 1, \int_0^{+\infty} k(s)e^{\delta_0 s}ds < \infty,$$

where δ_0 is a small positive constant. The second part (called the discrete part) of model (1) describes that the evolution processes experience abrupt change of states at the moments of time t_k (called impulsive moments), $n_{jk}(x_j(t_k))$ are also continuous functions, $w_{ij}^k, e_{ij}^k, J_{ik}$ are constants which have nothing to do with t. If the second part of (1) is replaced by $x_i(t_k) = x_i(t_k^-)$ and the state variable represents a neuron, then model (1) becomes a continuous recurrent neural networks model.

The paper is organized as follows. In the following section we discuss some notations, definitions and lemmas. In section 3, we consider the global exponential stability of the equilibrium of (1), two theorems and a corollary are given. In section 4, three examples are given to illustrate the effectiveness of our theoretical results.

2 Preliminaries

To begin with, we introduce some notations and recall some basic definitions.

Let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ denote the set of $m\times n$ real matrices. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B (A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality $\geq (\leq, >, <)$. Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

For $\psi : R \to R$, denote $[\psi(t)]_\infty = \sup\limits_{-\infty<s\leq 0} \{\psi(t+s)\}, \psi(t^+) = \lim\limits_{s\to 0^+} \psi(t+s), \psi(t^-) = \lim\limits_{s\to 0^-} \psi(t+s)$.

For $x \in R^n, A \in R^{m\times n}$, we denote $|x| = (|x_1|, \cdots, |x_n|)^T, |A| = (|a_{ij}|)_{n\times n}, \|x\| = \sum_{i=1}^n |x_i|, \|A\| = \max\limits_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$.

$PC := \{\phi|\phi : R \to R^n$ is a function of bounded variation and is right-hand continuous on any subinterval $(-\infty, t]\}$. Denote $\|\phi(t)\|_\infty = \sup\limits_{-\infty<s\leq 0} \|\phi(t + s)\|$.

Definition 1 For any given $t_0 \in R, \phi \in PC$, a function $x(t) \in PC[(-\infty, +\infty), R^n]$ is called a solution of (1) through (t_0, ϕ), if $x(t)$ satisfies the initial conditions in the form

$$x(t_0 + s) = \phi(s),\ \ s \in (-\infty, t_0], \tag{2}$$

and satisfies (1) for $t \geq t_0$, denoted by $x(t, t_0, \phi)$. Especially, a point $x^* \in R^n$ is called an equilibrium of (1), if $x(t) = x^*$ is a solution of (1).

For any $\phi \in PC$, we assume that there exists at least one solution of (1) with the initial condition (2). Let x^* be an equilibrium point of (1), $x(t)$ be any solution of (1) and $y(t) = x(t) - x^*$. Substituting them into (1), we get

$$\begin{cases} \dot{y}_i(t) = -b_i y_i(t) + \sum_{j=1}^n \{a_{ij} F_j(y_j(t)) + c_{ij} \int_{-\infty}^t k(t-s) G_j(y_j(s))ds\}, & t \neq t_k, \\ y_i(t_k) = \sum_{j=1}^n \{w_{ij}^k y_j(t_k^-) + e_{ij}^k \int_{-\infty}^{t_k} k(t_k - s) N_{jk}(y_j(s))ds\}, & t = t_k, \end{cases} \tag{3}$$

where $F_j(y_j(t)) = f_j(y_j(t)+x_j^*)-f_j(x_j^*), G_j(y_j(t)) = g_j(y_j(t)+x_j^*)-g_j(x_j^*), N_{jk}(y_j(t_k)) = n_{jk}(y_j(t_k) + x_j^*) - n_{jk}(x_j^*)$.

Definition 2 The zero solution of (3) is said to be globally exponentially stable if for any solution $x(t, t_0, \phi)$ with the initial condition $\phi \in PC$, there exist constant $\alpha > 0$ and $K > 1$ such that

$$\|x(t, t_0, \phi)\| \leq K\|\phi\| e^{-\alpha(t-t_0)}, t \geq t_0. \tag{4}$$

For convenience,we shall rewrite (3) in the vector form:

$$\begin{cases} \dot{y}(t) = -By(t) + AF(y(t)) + C \int_{-\infty}^t k(t-s) G(y(s))ds, & t \neq t_k, \\ y(t_k) = W_k y(t_k^-) + E_k \int_{-\infty}^{t_k} k(t_k - s) N_k(y(s))ds, & t = t_k, \end{cases} \tag{5}$$

where $F(y(t)) = (F_1(y_1(t)), \cdots, F_n(y_n(t)))^T, G(y(t)) = (G_1(y_1(t)), \cdots, G_n(y_n(t)))^T$, $N_k(y(t)) = (N_{k1}(y_1(t)), \cdots, N_{kn}(y_n(t)))^T, A = (a_{ij})_{n\times n}, C = (c_{ij})_{n\times n}, W_k = (w_{ij}^k)_{n\times n}$, and
$E_k = (e_{ij}^k)_{n\times n}, y(t) = (y_1(t), \cdots, y_n(t))^T$.

As we all know, the stability of the zero solution of (3) or (5) is equivalent to the stability of the equilibrium point x^* of (1). So we mainly discuss the stability of the zero solution of (3) or (5) in section 3.

Lemma 1 Suppose $r > l \geq 0$ and $p(t)$ satisfies scalar impulsive differential-integro inequality

$$\begin{cases} D^+p(t) \leq -rp(t) + l\int_0^{+\infty} k(s)p(t-s)ds, & t \neq t_k,\ t \geq t_0, \\ p(t_k) \leq p_k p(t_k^-) + q_k \int_0^{+\infty} k(s)p(t_k - s)ds, & k \in N, \\ p(t_0 + s) = \phi(s), & s \in (-\infty, 0], \end{cases} \tag{6}$$

where $p(t)$ is continuous at $t \neq t_k, t \geq t_0, p(t_k^+) = p(t_k)$, and $p(t_k^-)$ exists, $\phi \in PC$ with $n = 1, \int_0^{+\infty} k(s)ds = 1, \Delta(\lambda_0) \triangleq \int_0^{+\infty} k(s)e^{\lambda_0 s}ds < \infty$ for a given positive constant λ_0. Then

$$p(s) \leq \|\phi(t_0)\|_\infty e^{-\lambda(s-t_0)},\ -\infty < s \leq t_0, \tag{7}$$

implies

$$p(t) \leq \prod_{t_0 < t_k \leq t} \delta_k \|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \geq t_0, \tag{8}$$

where $\delta_k := \max\{1, |p_k| + |q_k| \int_0^{+\infty} k(s)e^{\lambda s}ds\}$ and $\lambda \in (0, \lambda_0)$ is a solution of inequality

$$\lambda - r + l\int_0^{+\infty} k(s)e^{\lambda s}ds \leq 0. \tag{9}$$

Proof. Since $r > l \geq 0$ and function $\Delta(\lambda)$ is continuous and $\Delta(0) = 1$, there exists at least a solution $\lambda \in (0, \lambda_0)$ satisfying (9). We shall prove that (7) implies

$$p(t) \leq \|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \in [t_0, t_1). \tag{10}$$

We consider two possible cases as follows:

One case is $l = 0$.

From (6) and (7), we have

$$D^+p(t) \leq -rp(t),\ p(t_0) \leq \|\phi(t_0)\|_\infty,\ t \in [t_0, t_1).$$

Then , from (9) and $l = 0$, we have $r \geq \lambda$, and

$$p(t) \leq \|\phi(t_0)\|_\infty e^{-r(t-t_0)} \leq \|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \in [t_0, t_1).$$

Another case is $l > 0$.

Next, for any constant $z > \|\phi(t_0)\|_\infty \geq 0$, we claim that

$$p(t) < ze^{-\lambda(t-t_0)} \equiv m(t),\ t \in [t_0, t_1). \tag{11}$$

If (11) is not true, then from (7) and the continuity of $p(t)$, for $t \in [t_0, t_1)$, then there must exist a $t^* \in [t_0, t_1)$ such that

$$p(t^*) = m(t^*),\ D^+p(t^*) \geq m'(t^*),\ p(t) < m(t),\ t < t^*. \tag{12}$$

By using (6),(9),(12) and $l > 0$, we obtain that

$$\begin{aligned}
D^+p(t^*) &\leq -rp(t^*) + l\int_0^{+\infty} k(s)p(t^* - s)ds \\
&< -rm(t^*) + l\int_0^{+\infty} k(s)m(t^* - s)ds \\
&= -rze^{-\lambda(t^*-t_0)} + l\int_0^{+\infty} k(s)ze^{-\lambda(t^*-t_0-s)}ds \\
&= (-r + l\int_0^{+\infty} k(s)e^{\lambda s}ds)ze^{-\lambda(t^*-t_0)} \\
&\leq -\lambda ze^{-\lambda(t^*-t_0)} \\
&= m'(t^*), \qquad (13)
\end{aligned}$$

which contradicts the inequality in (12). Therefore, (11) holds for any $z > \|\phi(t_0)\|_\infty$. Letting $z \to \|\phi(t_0)\|_\infty$, we obtain (10).

Using (6), (7) and (10), we can get

$$\begin{aligned}
p(t_1) &\leq p_1p(t_1^-) + q_1\int_0^{+\infty} k(s)p(t_1 - s)ds \\
&\leq |p_1|\|\phi(t_0)\|_\infty e^{-\lambda(t_1-t_0)} + |q_1|\int_0^{+\infty} k(s)\|\phi(t_0)\|_\infty e^{-\lambda(t_1-s-t_0)}ds \\
&= (|p_1| + |q_1|\int_0^{+\infty} k(s)e^{\lambda s}ds)\|\phi(t_0)\|_\infty e^{-\lambda(t_1-t_0)} \\
&\leq \delta_1\|\phi(t_0)\|_\infty e^{-\lambda(t_1-t_0)}.
\end{aligned}$$

Therefore

$$p(t) \leq \delta_1\|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \in (-\infty, t_1]. \qquad (14)$$

In a similar way as the proof of (10), we can prove that (14) implies

$$p(t) \leq \delta_1\|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \in [t_1, t_2). \qquad (15)$$

By a simple induction, we can obtain for any $k \in N$, there is

$$p(t) \leq \delta_1 \cdots \delta_{k-1}\|\phi(t_0)\|_\infty e^{-\lambda(t-t_0)},\ t \in [t_{k-1}, t_k).$$

The proof is completed.

Lemma 2[21] Let $A \in R^{n\times n}$, then

1). $\rho(A) \leq \|A\|$,where $\rho(\cdot)$ denotes the spectral radius;

2). $\|(E - A)^{-1}\| \leq (1 - \|A\|)^{-1}$ if $\|A\| < 1$;

3). $(E - A)^{-1}$ exists and $(E - A)^{-1} \geq 0$ if $\rho(A) < 1$ and $A \geq 0$;

4). $\lambda_m(A)x^Tx \leq x^TAx \leq \lambda_M(A)x^Tx$ for any $x \in R^n$ if A is a symmetric matrix, where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum eigenvalue of the matrix and the maximum one, respectively.

Lemma 3 For any constant $\epsilon > 0$, we have

$$2x^TAy \leq \epsilon x^TPP^Tx + \frac{1}{\epsilon}y^T(P^{-1}A)^T(P^{-1}A)y, \tag{16}$$

where A is a real matrix and P is a inversive real matrix.

Proof. We have

$$\begin{aligned}
0 \leq |\sqrt{\epsilon}P^Tx - \frac{1}{\sqrt{\epsilon}}P^{-1}Ay|^2 &= (\sqrt{\epsilon}P^Tx - \frac{1}{\sqrt{\epsilon}}P^{-1}Ay)^T(\sqrt{\epsilon}P^Tx - \frac{1}{\sqrt{\epsilon}}P^{-1}Ay) \\
&= (\sqrt{\epsilon}x^TP - \frac{1}{\sqrt{\epsilon}}y^TA^T(P^{-1})^T)(\sqrt{\epsilon}P^Tx - \frac{1}{\sqrt{\epsilon}}P^{-1}Ay) \\
&= \epsilon x^TPP^Tx - x^TAy - y^TA^Tx + \frac{1}{\epsilon}y^T(P^{-1}A)^T(P^{-1}A)y \\
&= \epsilon x^TPP^Tx - 2x^TAy + \frac{1}{\epsilon}y^T(P^{-1}A)^T(P^{-1}A)y,
\end{aligned}$$

so (16) follows.

3 Main Results

Theorem 1 For some positive constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$, the following conditions are satisfied for $k \in N$

(A_1). There exist symmetric nonnegative definite matrices D_1, D_2, H_k such that

$F^T(y)F(y) \leq y^TD_1y$, $G^T(y)G(y) \leq y^TD_2y$, $N_k^T(y)N_k(y) \leq y^TH_ky$;

(A_2). The Riccati equation $P^{-\frac{1}{2}}(PB+BP-\alpha PAA^TP-\frac{1}{\alpha}D_1-\beta PCC^TP)P^{-\frac{1}{2}} = Q$ for some symmetric positive solution P, where Q is symmetric positive matrix;

(A_3). $\lambda_m(Q) > \lambda_M(R)$, where $R = \frac{P^{-\frac{1}{2}}D_2P^{-\frac{1}{2}}}{\beta}$;

(A_4). Let $\lambda \in (0, \delta_0]$ satisfy $\lambda - \lambda_m(Q) + \lambda_M(R)\int_0^{+\infty}k(s)e^{\lambda s}ds \leq 0$;

(A_5). $\theta < \lambda$, where $\theta := \sup\{\frac{ln\theta_k}{t_k-t_{k-1}}\}$, $\theta_k := \max\{1, \xi_k + \zeta_k\int_0^{+\infty}k(s)e^{\lambda s}ds\}$, $\zeta_k = (\frac{1}{\gamma}+\lambda_M(E_k^TPE_k))\cdot\lambda_M(P^{-\frac{1}{2}}H_kP^{-\frac{1}{2}})$, $\xi_k = \lambda_M(P^{-\frac{1}{2}}(W_k^TPW_k+\gamma W_k^TPE_kE_k^TPW_k)P^{-\frac{1}{2}})$.

Then the zero solution of (3) is globally exponentially stable.

Proof. From (A_3), the inequality $\lambda - \lambda_m(Q) + \lambda_M(R)\int_0^{+\infty}k(s)e^{\lambda s}ds \leq 0$ has at least one solution $\lambda > 0$. Let $y(t)$ be a solution of (3) through $(t, \phi), \phi \in PC$ and

$v(t) := y^T(t)Py(t)$. From (5), for $t \neq t_k$, we can get

$$\begin{aligned} D^+v(t) &= 2y^T(t)P\dot{y}(t) \\ &= 2y^T(t)P(-By(t) + AF(y(t)) + C\int_{-\infty}^{t} k(t-s)G(y(s))ds) \\ &= -2y^T(t)PBy(t) + 2y^T(t)PAF(y(t)) + 2y^T(t)C\int_{-\infty}^{t} k(t-s)G(y(s))ds \end{aligned}$$

By using Lemma 2 and Lemma 3, there are positive constants α and β such that

$$\begin{aligned} D^+v(t) &\leq -y^T(t)(PB+BP)y(t) + \alpha y^T(t)PAA^TPy(t) + \frac{1}{\alpha}F^T(y(t))F(y(t)) \\ &\quad +\beta y^T(t)PCC^TPy(t) + \frac{1}{\beta}\int_{-\infty}^{t} k(t-s)G^T(y(s))G(y(s))ds \\ &\leq -y^T(t)P^{\frac{1}{2}}(P^{-\frac{1}{2}}(PB+BP-\alpha PAA^TP - \frac{1}{\alpha}D_1 - \beta PCC^TP)P^{-\frac{1}{2}})P^{\frac{1}{2}}y(t) \\ &\quad +\int_{-\infty}^{t} k(t-s)y^T(s)P^{\frac{1}{2}}(\frac{P^{-\frac{1}{2}}D_2P^{-\frac{1}{2}}}{\beta})P^{\frac{1}{2}}y(s)ds \\ &\leq -\lambda_m(Q)v(t) + \lambda_M(R)\int_0^{+\infty} k(s)v(t-s)ds, \qquad (t \neq t_k). \end{aligned} \tag{17}$$

On the other hand, from (5), (A_1), Lemma 2 and Lemma 3, we can get

$$\begin{aligned} v(t_k) &= y(t_k)^TPy(t_k) \\ &= (W_ky(t_k^-) + E_k\int_{-\infty}^{t_k} k(t_k-s)N_k(y(s))ds)^TP \\ &\quad \times(W_ky(t_k^-) + E_k\int_{-\infty}^{t_k} k(t_k-s)N_k(y(s))ds) \\ &= y^T(t_k^-)W_k^TPW_ky(t_k^-) + 2y^T(t_k^-)W_k^TPE_k\int_0^{+\infty} k(s)N_k(y(t_k-s))ds \\ &\quad +\int_0^{+\infty} k(s)N_k^T(y(t_k-s))dsE_k^TPE_k\int_0^{+\infty} k(s)N_k(y(t_k-s))ds \\ &\leq y^T(t_k^-)(W_k^TPW_k + \gamma W_k^TPE_kE_k^TPW_k)y(t_k^-) \\ &\quad +(\frac{1}{\gamma} + \lambda_M(E_k^TPE_k))\int_0^{+\infty} k(s)N_k^T(y(t_k-s))ds \cdot \int_0^{+\infty} k(s)N_k(y(t_k-s))ds \\ &\leq y^T(t_k^-)P^{\frac{1}{2}}(P^{-\frac{1}{2}}(W_k^TPW_k + \gamma W_k^TPE_kE_k^TPW_k)P^{-\frac{1}{2}})P^{\frac{1}{2}}y(t_k^-) \\ &\quad +(\frac{1}{\gamma} + \lambda_M(E_k^TPE_k))\int_0^{+\infty} k(s)N_k^T(y(t_k-s))N_k(y(t_k-s))ds \\ &\leq y^T(t_k^-)P^{\frac{1}{2}}(P^{-\frac{1}{2}}(W_k^TPW_k + \gamma W_k^TPE_kE_k^TPW_k)P^{-\frac{1}{2}})P^{\frac{1}{2}}y(t_k^-) \\ &\quad +(\frac{1}{\gamma} + \lambda_M(E_k^TPE_k))\int_0^{+\infty} k(s)y^T(t_k-s)P^{\frac{1}{2}}(P^{-\frac{1}{2}}H_kP^{-\frac{1}{2}})P^{\frac{1}{2}}y(t_k-s)ds \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_M(P^{-\frac{1}{2}}(W_k^T P W_k + \gamma W_k^T P E_k E_k^T P W_k) P^{-\frac{1}{2}}) v(t_k^-) \\
&\quad + (\frac{1}{\gamma} + \lambda_M(E_k^T P E_k)) \cdot \lambda_M(P^{-\frac{1}{2}} H_k P^{-\frac{1}{2}}) \int_0^{+\infty} k(s) v(t_k - s) ds \\
&= \xi_k v(t_k^-) + \zeta_k \int_0^{+\infty} k(s) v(t_k - s) ds. \qquad (18)
\end{aligned}
$$

Employing Lemma 1, from (17) (18) (A_3) and (A_4) we have

$$
\begin{aligned}
v(t) &\leq \theta_1 \cdots \theta_{k-1} e^{-\lambda(t-t_0)} \|v(t_0)\|_\infty \\
&\leq e^{\theta(t_1-t_0)} \cdots e^{\theta(t_{k-1}-t_{k-2})} e^{-\lambda(t-t_0)} \|v(t_0)\|_\infty \\
&\leq e^{\theta(t-t_0)} e^{-\lambda(t-t_0)} \|v(t_0)\|_\infty \\
&= e^{-(\lambda-\theta)(t-t_0)} \|v(t_0)\|_\infty, \quad t_{k-1} \leq t < t_k, k \in N,
\end{aligned}
$$

and so the conclusion holds. The proof is completed.

Remark If $W_k = E$ (unit matrix), $E_k = 0$ for all $k = 1, 2, \cdots$ in the (5), then the equation (5) becomes a dynamical system without impulses in vector form

$$
\dot{y}(t) = -By(t) + AF(y(t)) + C \int_{-\infty}^{t} k(t-s) G(y(s)) ds, \qquad (19)
$$

which contains many popular models such as Hopfield neural networks, cellular neural networks and recurrent neural networks, etc.. By using of Theorem 1, we can easily get the following corollary.

Corollary Assume that the conditions $(A_1), (A_2), (A_3), (A_4)$ in the theorem 1 are all satisfied. Then the zero solution of (19) is globally exponentially stable with exponential convergent rate λ.

Theorem 2 Assume that the following conditions are satisfied for $k \in N$

(A_1'). There exist k_j, l_j, n_{jk}, $j = 1, \cdots, n$ such that

$|f_j(x) - f_j(y)| \leq k_j|x - y|$, $|g_j(x) - g_j(y)| \leq l_j|x - y|$,

$|n_{jk}(x) - n_{jk}(y)| \leq n_{jk}|x - y|$;

(A_2'). $\nu < h$, where $h = \min\limits_{1 \leq j \leq n} (b_j - \sum_{i=1}^n |a_{ij}| k_j)$, $\nu = \|CL\|$, and $L = diag(l_1, \cdots, l_n)$;

(A_3'). Let $\lambda \in (0, \delta_0]$ be a solution of $\lambda - h + \nu \int_0^{+\infty} k(s) e^{\lambda s} ds \leq 0$;

(A_4'). $\eta < \lambda$, where $\eta := \sup\{\frac{ln \eta_k}{t_k - t_{k-1}}\}$, $\eta_k := \max\{1, \|W_k\| + \|E_k N_k'\| \int_0^{+\infty} k(s) e^{\lambda s} ds\}$, and $N_k' = diag(n_{1k}, \cdots, n_{nk})$.

Then the zero solution of (3) is globally exponentially stable.

Proof. Since $\nu < h$, the inequality $\lambda - h + \nu \int_0^{+\infty} k(s) e^{\lambda s} ds \leq 0$ has at least one solution $\lambda > 0$. Let $y(t)$ be a solution of (3) through $(t, \phi), \phi \in PC$ and $v(t) =$

$\sum_{i=1}^{n}|y_i(t)| = \|y(t)\|$. From (A_1) we have

$$\begin{aligned}
D^+v(t) &= \sum_{i=1}^{n} sgn(y_i(t))y_i'(t) \\
&= \sum_{i=1}^{n} sgn(y_i(t))(-b_iy_i(t) + \sum_{j=1}^{n}\{a_{ij}F_j(y_j(t)) + c_{ij}\int_{-\infty}^{t} k(t-s)G_j(y_j(s))ds\}) \\
&\leq -\sum_{i=1}^{n} b_i|y_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}|k_j|y_j(t)| + |c_{ij}|l_j\int_{0}^{+\infty} k(s)|y_j(t-s)|ds) \\
&\leq -\sum_{j=1}^{n}(b_j - \sum_{i=1}^{n}|a_{ij}|k_j)|y_j(t)| + \|CL\|\int_{0}^{+\infty} k(s)v(t-s)ds \\
&\leq -hv(t) + \nu\int_{0}^{+\infty} k(s)v(t-s)ds, \qquad (t \neq t_k). \qquad (20)
\end{aligned}$$

On the other hand, from (5), we can get

$$v(t_k) = \|y(t_k)\| \leq \|W_k\|v(t_k^-) + \|E_kN_k'\|\int_{0}^{+\infty} k(s)v(t_k-s)ds. \qquad (21)$$

From (20) (21) (A_2') (A_3') and Lemma 1, we can get

$$v(t) \leq e^{-(\lambda-\eta)(t-t_0)}\|v(t_0)\|_\infty, \quad t \geq t_0.$$

So the conclusion holds and the proof is completed.

4 Examples

Example 1: Consider the following impulsive dynamical system with distributed delays:

$$\begin{cases} \dot{y}_i(t) = -b_iy_i(t) + \sum_{j=1}^{2}\{a_{ij}F_j(y_j(t)) + c_{ij}\int_{-\infty}^{t} k(t-s)G_j(y_j(s))ds\}, & t \neq t_k, \\ y_i(t_k) = \sum_{j=1}^{2}\{w_{ij}^ky_j(t_k^-) + e_{ij}^k\int_{-\infty}^{t_k} k(t_k-s)N_{jk}(y_j(s))ds\}, & t = t_k, \end{cases} \qquad (22)$$

where $b_1 = 4, b_2 = 3, a_{11} = a_{12} = a_{22} = 1, a_{21} = -1, c_{11} = c_{22} = 1, c_{21} = \frac{1}{2}, c_{12} = -\frac{1}{2}, w_{12}^k = w_{21}^k = 0, w_{11}^k = 0.3e^{0.01k}, w_{22}^k = 0.2e^{0.01k}, e_{12}^k = e_{21}^k = 0, e_{11}^k = 0.2e^{0.01k}, e_{22}^k = 0.15e^{0.01k}, F_j(s) = \frac{|s+1|-|s-1|}{2}, G_j(s) = N_{jk}(s) = s, k(s) = e^{-s}, t_k = t_{k-1}+1.66k, k \in N$.

It is easy to see $D_1 = D_2 = H_k = E$. We choose $P = E$ i.e. $v(t) = x^T(t)x(t)$ and $\alpha = \beta = \gamma = 1$. By simple computation, we can get $\lambda_m(Q) = \frac{7}{4}, \lambda_M(R) = 1, t_k - t_{k-1} = 1.66k$,

$$\xi_k = \lambda_M(W_k^TW_k + W_k^TE_kE_k^TW_k) \doteq 0.09e^{0.02k} + 0.0036e^{0.04k},$$

$$\zeta_k = (1+\lambda_M(E_k^T E_k))\cdot\lambda_M(H_k) \doteq 1+0.04e^{0.02k},$$

$$\theta_k := \max\{1, \xi_k+\zeta_k\int_0^{+\infty} k(s)e^{\lambda s}ds\} \doteq 1.443+0.14772e^{0.02k}+0.0036e^{0.04k}.$$

Therefore we have

$$\frac{\ln\theta_k}{t_k-t_{k-1}} \le 0.30509 < \lambda = \frac{11-\sqrt{73}}{8} \doteq 0.30700,$$

where λ is an unique solution of equation:$\lambda-\lambda_m(Q)+\lambda_M(R)\int_0^{+\infty}k(s)e^{\lambda s}ds = 0$.

According to Theorem 1, we know the equilibrium point $(0,0)^T$ of (22) is globally exponentially stable with approximate exponential convergent rate 0.00191.

Example 2: Consider the following 2-dimensional neural network with distributed delays

$$\dot{y}_i(t) = -b_i y_i(t) + \sum_{j=1}^{2} c_{ij}\int_{-\infty}^{t} k(t-s)G_j(y_j(s))ds, \quad (i=1,2), \tag{23}$$

where $b_1 = 2$, $b_2 = 1$, $c_{11} = -1$, $c_{12} = c_{21} = 0.3$, $c_{22} = 0.5$, $G_j(s) = tanh(s) = \frac{e^s-e^{-s}}{e^s+e^{-s}}$, $k(s) = e^{-s}$.

It is easy to see $D_2 = E$. We choose $P = E$; i.e., $v(t) = x^T(t)x(t)$. By simple computation, we know when $\beta \in (0,\ 3.669724771)$, the matrix $2B-\beta CC^T$ is positive matrix; when $\beta \in (0.5532290950,\ 3.06367700)$, and we get $\lambda_m(Q) = -0.715\beta + 3 - 0.025\sqrt{261\beta^2-1200\beta+1600} > \lambda_M(R) = \frac{1}{\beta}$. So, by using the Corollary, when $\beta \in (0.5532290950,\ 3.06367700)$, the zero solution of (23) is exponential stable.

Moreover, when $\beta = 1.78$, by using our results, the maximum exponential convergent rate of (23) is $\lambda_{max} = 0.3859545295$. However, if we use the results in paper [1], the maximum exponential convergent rate of (23) is only $\lambda'_{max} = 0.2878679656$.

Example 3: Consider the following 2-dimensional impulsive neural network with distributed delays:

$$\begin{cases} \dot{y}_i(t) = -b_i y_i(t) + \sum_{j=1}^{2}\{a_{ij}F_j(y_j(t)) + c_{ij}\int_{-\infty}^{t} k(t-s)G_j(y_j(s))ds\}, t \neq t_k, \\ y_i(t_k) = \sum_{j=1}^{2} w_{ij}^k y_j(t_k^-), \qquad t = t_k, \end{cases} \tag{24}$$

with the initial conditions $y_1(s) = \cos(s), y_2(s) = \sin(s), -\infty < s \le 0$, where $b_1 = 4, b_2 = 6, a_{11} = a_{21} = a_{22} = 1, a_{12} = -1, c_{11} = 1, c_{21} = c_{22} = \frac{1}{2}, c_{12} = -\frac{1}{2}, w_{12}^k = -0.072e^{0.2k}, w_{21}^k = 0.092e^{0.2k}, w_{11}^k = 0.921e^{0.2k}, w_{22}^k = -0.727e^{0.2k}, F_j(s) = \frac{|s+1|-|s-1|}{2}, G_j(s) = s, k(s) = e^{-s}, t_k = t_{k-1}+1.3k, k \in N$.

By simple computation, we can get $k_j = l_j = 1, h = 2, \nu = \frac{3}{2}, \|E_k N'_k\| = 0, \|W_k\| = 1.013e^{0.2k}, t_k - t_{k-1} = 1.3k$,

$$\eta_k = \max\{1, \|W_k\| + \|E_k N'_k\|\int_0^{+\infty} k(s)e^{\lambda s}ds\} = \|W_k\| = 1.013e^{0.2k}.$$

Therefore we have

$$\frac{\ln \eta_k}{t_k - t_{k-1}} \leq 0.1641 < \lambda = \frac{3-\sqrt{7}}{2} \doteq 0.1771,$$

where λ is a unique solution of equation:$\lambda - h + \nu \int_0^{+\infty} k(s)e^{\lambda s}ds = 0$.

According to Theorem 2, we know the equilibrium point $(0,0)^T$ of (24) is globally exponentially stable with approximate exponential convergent rate 0.013.

Next, by utilizing a standard Runge-Kutta method, the simulation result of Example 3 above is illustrated in Fig.1.

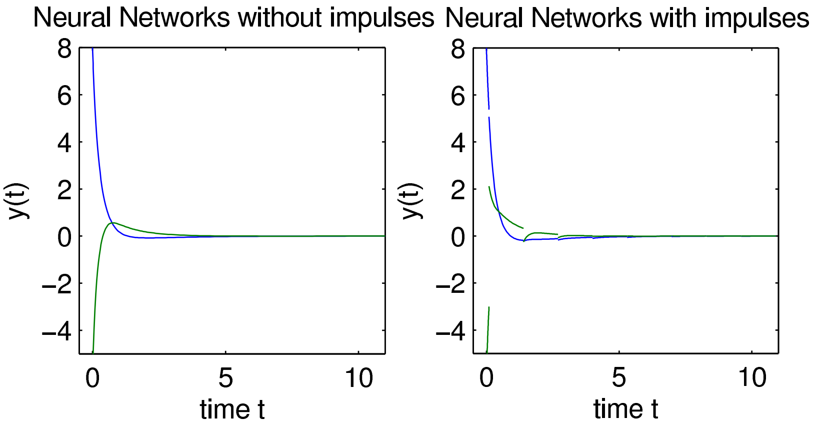

Figure 1: Stability for neural network without impulses or with impulses.

5 Conclusions

In this letter, the impulsive dynamical system with distributed delays is investigated. For the model (see (3)), by the established impulsive differential-integro inequality (see Lemma 1), we have obtained some sufficient conditions of global exponential stability for the equilibrium point. To the best of our knowledge, the results presented here have been not appeared in the related literature. When model (3) is a continuous dynamical system (see (19)), we obtained the sufficient conditions ensuring the global exponential stability of such model. In the example 2, we point out our result can get the larger exponential convergent rate than the results in paper [1] can do.

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