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On the existence of solutions for a higher order differential inclusion without convexity

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Abstract

We prove a Filippov type existence theorem for solutions of a higher order differential inclusion in Banach spaces with nonconvex valued right hand side by applying the contraction principle in the space of the derivatives of solutions instead of the space of solutions.

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1 Introduction

In this note we study the n-th order differential inclusion

$$x^{(n)} - \lambda x \in F(t,x), \quad a.e.\ (I) \tag{1.1}$$

with boundary conditions of the form

$$x^{(i)}(0) - x^{(i)}(T) = \mu_i, \quad i = 0,1,\ldots,n-1, \tag{1.2}$$

where $\lambda \in \mathbf{R}$, E is real separable Banach space, $I = [0,T]$, $F(\cdot,\cdot) : I \times E \to \mathcal{P}(E)$ and $\mu_i \in E$, $i = 0,1,\ldots,n-1$. When $\lambda \neq 0$ the linear part in equation

(1.1) is invertible and, in this case, the problem (1.1)-(1.2) is well known as a nonresonance problem. Moreover, if $\mu_i = 0$, $i = 0, 1, \ldots, n-1$ then the conditions (1.2) are periodic boundary conditions.

The present note is motivated by a recent paper of Benchohra, Graef, Henderson and Ntouyas ([1]) in which several existence results concerning nonresonance impulsive higher order differential inclusions are obtained via fixed point techniques. The aim of our paper is to provide a Filippov type result concerning the existence of solutions to problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution.

Our approach is different from the one in [1] and consists in applying the contraction principle in the space of derivatives of solutions instead of the space of solutions. In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. The idea of applying the set-valued contraction principle due to Covitz and Nadler ([6]) in the space of derivatives of the solutions belongs to Kannai and Tallos ([7]) and it was already used for other results concerning differential inclusions ([3,4,5] etc.).

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space and consider a set valued map T on X with nonempty closed values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X,$$

where $d_H(\cdot, \cdot)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

If X is complete, then every set valued contraction has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$ ([6]).

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. ([8]) *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in X} d(T_1(z), T_2(z)).$$

In what follows E is a real separable Banach space with norm $|\cdot|$, $C(I, E)$ is the Banach space of all continuous functions from I to E with the norm $\|x(\cdot)\|_C = \sup_{t \in I} |x(t)|$, $AC^i(I, E)$ is the space of i-times differentiable functions $x : I \to E$ whose i-th derivative $x^i(\cdot)$ is absolutely continuous and $L^1(I, E)$ is the Banach space of (Bochner) integrable functions $u(\cdot) : I \to E$ endowed with the norm $\|u(\cdot)\|_1 = \int_0^1 |u(t)| dt$.

A function $x(\cdot) \in AC^{n-1}(I, E)$ is called a solution of problem (1.1)-(1.2) if there exists a function $v(\cdot) \in L^1(I, E)$ with $v(t) \in F(t, x(t))$, *a.e.* (I) such that $x^{(n)}(t) - \lambda x(t) = v(t)$, *a.e.* (I) and $x(\cdot)$ satisfies conditions (1.2).

In what follows we consider the Green function $G(\cdot, \cdot) : I \times I \to E$ associated to the periodic boundary problem

$$x^{(n)} - \lambda x = 0, \quad x^{(i)}(0) - x^{(i)}(T) = 0, \quad i = 0, 1, \ldots, n-1.$$

For the properties of $G(\cdot, \cdot)$ we refer to [1].

The next result is well known (e.g. [1]).

Lemma 2.2. ([1]) *If $v(\cdot) : [0, T] \to E$ is an integrable function then the problem*

$$\begin{gathered} x^{(n)}(t) - \lambda x(t) = v(t) \quad a.e.\ (I) \\ x^{(i)}(0) - x^{(i)}(T) = \mu_i, \quad i = 0, 1, \ldots, n-1. \end{gathered}$$

has a unique solution $x(\cdot) \in AC^{n-1}(I, E)$ given by

$$x(t) = P_\mu(t) + \int_0^T G(t, s) v(s) ds,$$

where

$$P_\mu(t) = \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, 0) \mu_{n-1-i}. \tag{2.1}$$

In the sequel we assume the following conditions on F.

Hypothesis 2.3. (i) *$F(\cdot,\cdot) : I \times E \to \mathcal{P}(E)$ has nonempty closed values and for every $x \in E$ $F(\cdot,x)$ is measurable.*

(ii) *There exists $L(\cdot) \in L^1(I,E)$ such that for almost all $t \in I, F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x),F(t,y)) \le L(t)|x-y| \quad \forall \ x,y \in E$$

and $d(0,F(t,0)) \le L(t)$ a.e. (I).

Denote $L_0 := \int_0^T L(s)ds$ and $M_0 := \sup_{(t,s)\in I\times I} |G(t,s)|$.

3 The main result

We are able now to prove our main result.

Theorem 3.1. *Assume that Hypothesis* 2.3 *is satisfied and $M_0L_0 < 1$. Let $y(\cdot) \in AC^{n-1}(I,E)$ be such that there exists $q(.) \in L^1(I,E)$ with $d(y^{(n)}(t) - \lambda y(t), F(t,y(t))) \le q(t)$, a.e. (I). Denote $\tilde{\mu}_i = y^{(i)}(0) - y^{(i)}(T)$, $i = 0,1,\ldots,n-1$.*

Then for every $\varepsilon > 0$ there exists $x(\cdot)$ a solution of (1.1)-(1.2) *satisfying for all $t \in I$*

$$|x(t)-y(t)| \le \frac{1}{1-M_0L_0}\sup_{t\in I}|P_\mu(t) - P_{\tilde{\mu}}(t)| + \frac{M_0}{1-M_0L_0}\int_0^T q(t)dt + \varepsilon,$$

where $P_\mu(t)$ is defined in (2.1).

Proof. For $u(\cdot) \in L^1(I,E)$ define the following set valued maps:

$$M_u(t) = F\left(t, P_\mu(t) + \int_0^T G(t,s)u(s)ds\right), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1(I,E); \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from the definition and Lemma 2.2 that $x(\cdot)$ is a solution of (1.1)-(1.2) if and only if $x^{(n)}(\cdot) - \lambda x(\cdot)$ is a fixed point of $T(\cdot)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I,E)$. The fact that the set valued map $M_u(\cdot)$ is measurable is well known. For example the map $t \to P_\mu(t) + \int_0^T G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [2]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [2]) we infer that $M_u(\cdot)$ admits a measurable selection ϕ. One has

$$\begin{aligned} |\phi(t)| &\leq d(0, F(t,0)) + d_H\left(F(t,0), F\left(t, P_\mu(t) + \int_0^T G(t,s)u(s)ds\right)\right) \\ &\leq L(t)\left(1 + |P_\mu(t)| + \sup_{t,s\in I}|G(t,s)|\int_0^T |u(s)|ds\right), \end{aligned}$$

which shows that $\phi \in L^1(I, E)$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $\|\phi_n - \phi\|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(\cdot)$ is a contraction on $L^1(I, E)$.

Let $u, v \in L^1(I, E)$ be given, $\phi \in T(u)$ and let $\delta > 0$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \left\{x \in E;\ |\phi(t) - x| \leq L(t)\left|\int_0^T G(t,s)(u(s) - v(s))ds\right| + \delta\right\}.$$

From Proposition III.4 in [2], $H(\cdot)$ is measurable and from Hypothesis 2.3 ii) $H(\cdot)$ has nonempty closed values. Therefore, there exists $\psi(\cdot)$ a measurable selection of $H(\cdot)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$\begin{aligned} \|\phi - \psi\|_1 &= \int_0^T |\phi(t) - \psi(t)|dt \\ &\leq \int_0^T L(t)\left(\int_0^T |G(t,s)||u(s) - v(s)|ds\right)dt + \int_0^T \delta dt \\ &= \int_0^T \left(\int_0^T L(t)|G(t,s)|dt\right)|u(s) - v(s)|ds + T\delta \\ &\leq M_0 L_0 \|u - v\|_1 + T\delta. \end{aligned}$$

Since $\delta > 0$ was chosen arbitrary, we deduce that

$$d(\phi, T(v)) \leq M_0 L_0 \|u - v\|_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq M_0 L_0 \|u - v\|_1,$$

thus $T(\cdot)$ is a contraction on $L^1(I,E)$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)B, \quad (t,x) \in I \times E,$$

$$M_u^1(t) = F_1\left(t, P_{\tilde{\mu}}(t) + \int_0^T G(t,s)u(s)ds\right), \quad t \in I, \quad u(\cdot) \in L^1(I,E),$$

$$T_1(u) = \{\psi(\cdot) \in L^1(I,E); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\},$$

where B denotes the closed unit ball in E. Obviously, $F_1(\cdot,\cdot)$ satisfies Hypothesis 2.3.

Repeating the previous step of the proof we obtain that T_1 is also a M_0L_0-contraction on $L^1(I,E)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \sup_{t\in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| L_0 + \int_0^T q(t)dt. \tag{3.1}$$

Let $\phi \in T(u), \delta > 0$ and define

$$H_1(t) = M_u^1(t) \cap \left\{z \in E; \quad |\phi(t) - z| \leq L(t)|P_\mu(t) - P_{\tilde{\mu}}(t)| + q(t) + \frac{\delta}{T}\right\}.$$

With the same arguments used for the set valued map $H(\cdot)$, we deduce that $H_1(\cdot)$ is measurable with nonempty closed values. Hence let $\psi(\cdot)$ be a measurable selection of $H_1(\cdot)$. It follows that $\psi \in T_1(u)$ and one has

$$\begin{aligned}
\|\phi - \psi\|_1 &= \int_0^T |\phi(t) - \psi(t)|dt \\
&\leq \int_0^T \left[L(t)|P_\mu(t) - P_{\tilde{\mu}}(t)| + q(t) + \frac{\delta}{T}\right] dt \\
&\leq \int_0^T L(t)|P_\mu(t) - P_{\tilde{\mu}}(t)|dt + \int_0^T q(t) + \delta \\
&\leq L_0 \sup_{t\in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \int_0^T q(t)dt + \delta.
\end{aligned}$$

Since δ is arbitrary, as above we obtain (3.1).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{L_0}{1 - M_0L_0} \sup_{t\in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \frac{1}{1 - M_0L_0} \int_0^T q(t)dt.$$

Since $v(\cdot) = y^{(n)}(\cdot) - \lambda y(\cdot) \in Fix(T_1)$ it follows that there exists $u(\cdot) \in Fix(T)$ such that for any $\varepsilon > 0$

$$\|v - u\|_1 \leq \frac{L_0}{1 - M_0 L_0} \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \frac{1}{1 - M_0 L_0} \int_0^T q(t)dt + \frac{\varepsilon}{M_0}.$$

We define $x(t) = P_\mu(t) + \int_0^T G(t,s)u(s)ds$, $t \in I$ and we have

$$\begin{aligned}
|x(t) - y(t)| &\leq |P_\mu(t) - P_{\tilde{\mu}}(t)| + \int_0^T |G(t,s)||u(s) - v(s)|ds \\
&\leq \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| \frac{M_0 L_0}{1 - M_0 L_0} \\
&\quad + \frac{1}{1 - M_0 L_0} M_0 \int_0^T q(t)dt + \varepsilon \\
&\leq \frac{1}{1 - M_0 L_0} \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \frac{M_0}{1 - M_0 L_0} \int_0^T q(t)dt + \varepsilon,
\end{aligned}$$

which completes the proof.

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