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VIABILITY PROBLEM WITH PERTURBATION IN HILBERT SPACE

MYELKEBIR AITALIOUBRAHIM AND SAID SAJID

University Hassan II-Mohammedia, U.F.R Mathematics and applications,
F.S.T, BP 146, Mohammedia, Morocco.
E-mail:saidsajid@hotmail.com and aitalifr@yahoo.fr

Abstract. This paper deals with the existence result of viable solutions of the differential inclusion

$$\dot{x}(t) \in f(t, x(t)) + F(x(t))$$
$$x(t) \in K \text{ on } [0, T],$$

where K is a locally compact subset in separable Hilbert space H, $(f(s,\cdot))_s$ is an equicontinuous family of measurable functions with respect to s and F is an upper semi-continuous set-valued mapping with compact values contained in the Clarke subdifferential $\partial_c V(x)$ of an uniformly regular function V.

Key words: Regularity, upper semi-continuous, equicontinuous perturbation, Clarke subdifferential.
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1. Introduction

Existence result of local solution for differential inclusion with upper semi-continuous and cyclically monotone right hand-side whose values in finite-dimensional space, was first established by Bressan, Cellina and Colombo (see [6]). The authors exploited rich properties of subdifferential of convex lower semi-continuous function; in order to overcome the weakly convergence of derivatives of approximate solutions, they used the basic relation (see [7])

$$\frac{d}{dt}V(x(t)) = \|\dot{x}(t)\|^2.$$

Later, Ancona, Cellina and Colombo (see [1]), under the same hypotheses as the above paper, extend this result to the perturbed problem

$$\dot{x}(t) \in f(t, x(t)) + F(x(t))$$

where $f(\cdot,\cdot)$ is a Carathéodory function.

This program of research was pursued by a series of works. In the first one (see [9]), Truong proved a viability result for similar problem, where the perturbation f is replaced by a globally continuous set-valued mapping G with values in finite-dimensional space. This result was extended by Bounkhel (see [4]) for a similar problem, where F is not cyclically monotone but contained in the Clarke subdifferential of locally Lipschitz uniformly regular function. However under very strong assumptions namely, the space of states is finite-dimensional and the following tangential condition

$$\Big(G(t,x)+F(x)\Big)\subset T_K(x)$$

where $T_K(x)$ is the contingent cone at x to K.

Recently, Morchadi and Sajid (see [8]) proved an exact viability version of the work of Ancona and Colombo assuming the same hypotheses and the following tangential condition

$\forall (t,x)\in\mathbb{R}\times K, \exists v\in F(x)$ such that

$$\lim_{h\mapsto 0^+}\inf\frac{1}{h}d_K\left(x+hv+\int_t^{t+h}f(s,x)ds\right)=0. \tag{1.1}$$

Remark that in all the above works, the convexity assumption of V and/or the finite-dimensional hypothesis of the space of states were widely used in the proof.

This paper is devoted to establish a local solution of the problem

$$\begin{gathered}\dot{x}(t)\in f(t,x(t))+F(x(t)),\quad F(x(t))\subset\partial_c V(x(t))\\ x(t)\in K\subset H,\end{gathered}$$

where K is a locally compact subset of a separable Hilbert space H, F is an upper semi-continuous multifunction, $\partial_c V$ denotes the Clarke subdifferential of a locally lipschitz function V and the set $\{f(s,.):s\in\mathbb{R}\}$ is equicontinuous, where for each $x\in K$, $s\mapsto f(s,x)$ is measurable and the same tangential condition (1.1). One case deserves mentioning: when f is globally continuous, the condition (1.1) is weaker than the following

$$\Big(f(t,x)+F(x)\Big)\cap T_K(x)\neq\emptyset.$$

To remove the convexity assumption of V and the finite-dimensional hypothesis of H, we rely on some properties of Clarke subdifferential of uniformly regular function and the local compactness of K.

2. Preliminaries and statement of the main result

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $<\cdot,\cdot>$. For $x\in H$ and $r>0$ let $B(x,r)$ be the open ball centered at x with radius r and $\bar{B}(x,r)$ be its closure and put $B=B(0,1)$.

Let us recall the definition of the Clarke subdifferential and the concept of regularity that will be used in the sequel.

Definition 2.1. *Let $V:H\to\mathbb{R}\cup\{+\infty\}$ be a lower semi-continuous function and x be any point where V is finite. The Clarke subdifferential of V at x is defined by*

$$\partial_c V(x):=\big\{y\in H\ :<y,h>\leq V^{\uparrow}(x,h),\ \textit{for all}\ h\in H\big\},$$

where $V^{\uparrow}(x,h)$ is the generalized Rockafellar directional derivative given by

$$V^{\uparrow}(x,h):=\limsup_{x'\to x,V(x')\to V(x),t\to 0}\ \inf_{h'\to h}\frac{V(x'+th')-V(x')}{t}.$$

Definition 2.2. *Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $U \subset DomV$ be a nonempty open subset. We will say that V is uniformly regular over U if there exists a positive number β such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has*

$$< \xi, x' - x > \leq V(x') - V(x) + \beta \|x' - x\|^2 \quad \textit{for all } x' \in U.$$

$\partial_p V(x)$ denotes the proximal subdifferential of V at x which is the set of all $y \in H$ for which there exist δ, $\sigma > 0$ such that for all $x' \in x + \delta \bar{B}$

$$< y, x' - x > \leq V(x') - V(x) + \sigma \parallel x' - x \parallel^2 .$$

We say that V is uniformly regular over closed set S if there exists an open set U containing S such that V is uniformly regular over U. For more details on the concept of regularity, we refer the reader to [4].

Proposition 2.3. [3, 4] *Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and S a nonempty closed set. If V is uniformly regular over S, then the following conditions holds:*

(a) *The proximal subdifferential of V is closed over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ with $\xi_n \in \partial_p V(x_n)$ one has $\xi \in \partial_p V(x)$.*
(b) *The proximal subdifferential of V coincides with the Clarke subdifferential of V for any point x.*
(c) *The proximal subdifferential of V is upper hemicontinuous over S, that is, the support function $x \mapsto \sigma(v, \partial_p V(x))$ is u.s.c. over S for every $v \in H$.*

Now let us state the main result.

Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and β-uniformly regular over $K \subset H$. Assume that

(H1) K is a nonempty locally compact subset in H;
(H2) $F : K \to 2^H$ is an upper semi-continuous set valued map with compact values satisfying

$$F(x) \subset \partial_c V(x) \ \text{ for all } \ x \in K;$$

(H3) $f : \mathbb{R} \times H \to H$ is a function with the following properties:
  (1) For all $x \in H$, $t \mapsto f(t, x)$ is measurable,
  (2) The family $\{f(s, .) : s \in \mathbb{R}\}$ is equicontinuous,
  (3) For all bounded subset S of H, there exists $M > 0$ such that

$$\parallel f(t, x) \parallel \leq M, \ \forall (t, x) \in \mathbb{R} \times S;$$

(H4) (Tangential condition) $\forall (t, x) \in \mathbb{R} \times K, \exists v \in F(x)$ such that

$$\lim_{h \mapsto 0^+} \inf \frac{1}{h} d_K \left(x + hv + \int_t^{t+h} f(s, x) ds \right) = 0.$$

For any $x_0 \in K$, consider the problem:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) + F(x(t)) \quad \text{a.e;} \\ x(0) = x_0; \\ x(t) \in K. \end{cases} \tag{2.1}$$

Theorem 2.4. *If assumptions (H1)-(H4) are satisfied, then there exists $T > 0$ such that the problem (2.1) admits a solution on $[0, T]$.*

3. Proof of the main result

Choose $r > 0$ such that $K_0 = K \cap (x_0 + r\bar{B})$ is compact and V is Lipschitz continuous on $x_0 + r\bar{B}$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(x) \subset \lambda \bar{B}$ for every $x \in K_0$. Let $M > 0$ such that

$$\| f(t, x) \| \leq M, \ \forall (t, x) \in \mathbb{R} \times (x_0 + r\bar{B}). \tag{3.1}$$

Set

$$T = \frac{r}{2(\lambda + 1 + M)}. \tag{3.2}$$

In the sequel, we will use the following important Lemma. It will play a crucial role in the proof of the main result.

Lemma 3.1. *If assumptions (H1)-(H4) are satisfied, then for all $0 < \varepsilon < inf(T, 1)$, there exists $\eta > 0$ ($\eta < \varepsilon$) such that*
$\forall (t, x) \in [0, T] \times K_0$, $\exists h_{t,x} \in [\eta, \varepsilon]$, $u \in F(x) + \frac{1}{h_{t,x}} \int_t^{t+h_{t,x}} f(s, x) ds + \frac{\varepsilon}{T} B$, $y_{t,x} \in K_0$ and $v \in F(y_{t,x})$ such that

$$\big(x + h_{t,x} u\big) \in K \cap B\Big(x + h_{t,x} v + \int_t^{t+h_{t,x}} f(s, x) ds, \lambda + M + 1\Big).$$

Proof. Let $(t, x) \in [0, T] \times K_0$, be fixed, let $0 < \varepsilon < \inf(T, 1)$. Since F is $u.s.c$ on x, then there exists $\delta_x > 0$ such that

$$F(y) \subset F(x) + \frac{\varepsilon}{2T} B, \ \text{ for all } y \in B(x, \delta_x).$$

Let $(s, y) \in [0, T] \times K_0$. By the tangential condition, there exists $v \in F(y)$ and $h_{s,y} \in]0, \varepsilon]$ such that

$$d_K \left(y + h_{s,y} v + \int_s^{s+h_{s,y}} f(\tau, y) d\tau \right) < h_{s,y} \frac{\varepsilon}{4T}.$$

Consider the subset

$$N(s, y) = \left\{ (t, z) \in \mathbb{R} \times H / d_K\big(z + h_{s,y} v + \int_t^{t+h_{s,y}} f(\tau, z) d\tau\big) < h_{s,y} \frac{\varepsilon}{4T} \right\}.$$

Since

$$\| f(\tau, z) \| \leq M, \ \forall (\tau, z) \in \mathbb{R} \times \bar{B}(x_0, r),$$

then the dominated convergence theorem applied to the sequence $(\chi_{[t,t+h_{s,y}]}f(\cdot,\cdot))_t$ of functions shows that the function

$$(l,z) \mapsto z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau,z)d\tau$$

is continuous. So that, the function

$$(l,z) \mapsto d_K\left(z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau,z)d\tau\right)$$

is continuous and consequently $N(s,y)$ is open. Moreover, since (s,y) belongs to $N(s,y)$, there exists a ball $B((s,y),\eta_{s,y})$ of radius $\eta_{s,y} < \delta_x$ contained in $N(s,y)$, therefore, the compact subset $[0,T]\times K_0$ can be covered by q such balls $B((s_i,y_i), \eta_{s_i,y_i})$. For simplicity, we set

$$h_{s_i,y_i} := h_i \ \text{ and } \eta_i := \eta_{s_i,y_i},\ i = 1,\dots,q.$$

Put $\eta = \min\{h_i/1 \le i \le q\}$ and let $i \in \{1,\dots,q\}$ such that $(t,x) \in B((s_i,y_i),\eta_i)$, hence $(t,x) \in N(s_i,y_i)$. Then there exists $v_i \in F(y_i)$ such that

$$d_K\left(x + h_iv_i + \int_t^{t+h_i} f(\tau,x)d\tau\right) < h_i\frac{\varepsilon}{4T}.$$

Let $x_i \in K$ such that

$$\frac{1}{h_i}\left\| x_i - \left(x + h_iv_i + \int_t^{t+h_i} f(\tau,x)d\tau\right)\right\| \le \frac{1}{h_i}d_K\left(x + h_iv_i + \int_t^{t+h_i} f(\tau,x)d\tau\right) + \frac{\varepsilon}{4T}.$$

Hence

$$\left\| \frac{x_i - x}{h_i} - \left(v_i + \frac{1}{h_i}\int_t^{t+h_i} f(\tau,x)d\tau\right)\right\| < \frac{\varepsilon}{2T}.$$

Set

$$u = \frac{x_i - x}{h_i},$$

then $x_i = x + h_iu \in K$ and

$$u \in \left(\frac{1}{h_i}\int_t^{t+h_i} f(\tau,x)d\tau + F(y_i) + \frac{\varepsilon}{2T}B\right).$$

Since $\| x - y_i \| < \eta_i < \delta_x$ we have

$$F(y_i) \subset F(x) + \frac{\varepsilon}{2T}B,$$

then

$$u \in \left(\frac{1}{h_i}\int_t^{t+h_i} f(\tau,x)d\tau + F(x) + \frac{\varepsilon}{T}B\right).$$

On the other hand, since $x \in K$, we have

$$\left\| x_i - \left(x + h_iv_i + \int_t^{t+h_i} f(\tau,x)d\tau\right)\right\| \ \le \ d_K\left(x + h_iv_i + \int_t^{t+h_i} f(\tau,x)d\tau\right) + \frac{\varepsilon}{4T}$$

$$\leq \left\| h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right\| + \frac{\varepsilon}{4T}$$
$$\leq h_i(\lambda + M) + 1 < \lambda + M + 1.$$

Thus $x_i \in B\left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau, \lambda + M + 1\right)$. $\square$

Now, we are able to prove the main result. Our approach consists of constructing, in a first step, a sequence of approximate solutions and deduce, in a second step, from available estimates that a subsequence converges to a solution of (2.1).

Step 1. Approximate solutions. Let $x_0 \in K_0$ and $0 < \varepsilon < inf(T, 1)$. By Lemma 3.1, there exist $\eta > 0$, $h_0 \in [\eta, \varepsilon]$, $u_0 \in \left(\frac{1}{h_0} \int_0^{h_0} f(s, x_0) ds + F(x_0) + \frac{\varepsilon}{T} B\right)$, $y_0 \in K_0$ and $v_0 \in F(y_0)$ such that

$$x_1 = x_0 + h_0 u_0 \in K \cap B\left(x_0 + h_0 v_0 + \int_0^{h_0} f(s, x_0) ds, \lambda + M + 1\right).$$

Then by (H2), (3.1) and (3.2), we have

$$\| x_1 - x_0 \| = \| h_0 u_0 \| \leq (\lambda + 1 + M)T < r$$

and thus $x_1 \in K_0$. Set $h_{-1} = 0$. By induction, for $q \geq 2$ and for every $p = 1, \ldots, q-1$, we construct the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $\big((x_p)_p, (y_p)_p\big) \subset K_0 \times K_0$ and $\big((u_p)_p, (v_p)_p\big) \subset H \times H$ such that $\sum_{p=1}^{q-1} h_p \leq T$ and

$$\begin{cases} x_p = x_{p-1} + h_{p-1} u_{p-1}; \\ x_p \in K \cap B\left(x_{p-1} + h_{p-1} v_{p-1} + \displaystyle\int_{\sum_{i=0}^{p-2} h_i}^{\sum_{i=0}^{p-1} h_i} f(s, x_{p-1}) ds, \lambda + M + 1\right); \\ u_p \in \left(\frac{1}{h_p} \displaystyle\int_{\sum_{i=0}^{p-1} h_i}^{\sum_{i=0}^{p} h_i} f(s, x_p) ds + F(x_p) + \frac{\varepsilon}{T} B\right); \\ v_p \in F(y_p). \end{cases}$$

Since $h_i \geq \eta > 0$ there exists an integer s such that

$$\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i.$$

Then we have constructed the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $\big((x_p)_p, (y_p)_p\big) \subset K_0 \times K_0$ and $\big((u_p)_p, (v_p)_p\big) \subset H \times H$ such that for every $p = 1, \ldots, s$, we have

(i) $x_p = x_{p-1} + h_{p-1} u_{p-1}$;

(ii) $x_p \in K \cap B\left(x_{p-1} + h_{p-1} v_{p-1} + \displaystyle\int_{\sum_{i=0}^{p-2} h_i}^{\sum_{i=0}^{p-1} h_i} f(s, x_{p-1}) ds, \lambda + M + 1\right)$;

(iii) $u_p \in F(x_p) + \frac{1}{h_p} \int\limits_{\sum_{i=0}^{p-1} h_i}^{\sum_{i=0}^{p} h_i} f(s, x_p)ds + \frac{\varepsilon}{T} B$;

(iv) $v_p \in F(y_p)$.

By induction, for all $p = 1, \ldots, s$ we have

$$x_p = x_0 + \sum_{i=0}^{p-1} h_i u_i.$$

Moreover by (iii), (H2), (3.1), (3.2) and because $\sum_{i=0}^{p-1} h_i < T$, we have

$$\|x_p - x_0\| = \left\| \sum_{i=0}^{p-1} h_i u_i \right\| \leq \sum_{i=0}^{p-1} h_i \|u_i\| \leq \sum_{i=0}^{p-1} h_i (\lambda + 1 + M) < r, \tag{3.3}$$

hence $x_p \in K_0$.

For any nonzero integ k and for every integer $q = 0, \ldots, s-1$ denote by h_q^k a real associated to $\varepsilon = \frac{1}{k}$ and $x = x_q$ given by Lemma 3.1. Consider the sequence $(\tau_k^q)_k$ defined as the following

$$\begin{cases} \tau_k^0 = 0, \tau_k^{s+1} = T; \\ \tau_k^q = h_0^k + \ldots + h_{q-1}^k \quad \text{if } 1 \leq q \leq s, \end{cases}$$

and define on $[0, T]$ the sequence of functions $(x_k(.))_k$ by

$$\begin{cases} x_k(t) = x_{q-1} + \left(t - \tau_k^{q-1}\right) u_{q-1}, \quad \forall t \in [\tau_k^{q-1}, \tau_k^q]; \\ x_k(0) = x_0. \end{cases}$$

Step 2. Convergence of approximate solutions. By definition of $x_k(.)$, for all $t \in [\tau_k^{q-1}, \tau_k^q]$ we have $\dot{x}_k(t) = u_{q-1}$. By (iii), (H2), (3.1), for a. e. $t \in [0, T]$, we have

$$\| \dot{x}_k(t) \| \quad \leq \quad \lambda + 1 + M.$$

On the other hand, by (ii), (iv), (H2), (3.1) and (3.3) we have

$$\begin{aligned} \|x_q\| \quad &\leq \quad \left\| x_q - (x_{q-1} + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1})ds) \right\| \\ &\quad + \left\| x_{q-1} + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1})ds \right\| \\ &\leq \quad \lambda + M + 1 + \left\| x_0 - (x_0 - x_{q-1}) + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1})ds \right\| \\ &\leq \quad \lambda + M + 1 + \|x_0\| + \|x_0 - x_{q-1}\| + h_{q-1}^k \|v_{q-1}\| + h_{q-1}^k M \\ &\leq \quad \lambda + M + 1 + \|x_0\| + r + \lambda + M \\ &< \quad 2(\lambda + M + 1) + \|x_0\| + r = R. \end{aligned}$$

Then $x_q \in K_0 \cap \bar{B}(0, R) = K_1$. By construction, for all $t \in [\tau_k^{q-1}, \tau_k^q]$ we have

$$x_k(t) = x_{q-1} + (t - \tau_k^{q-1}) u_{q-1} = x_{q-1} + \frac{(t - \tau_k^{q-1})}{h_{q-1}^k} (x_q - x_{q-1}).$$

Also since $0 \le t - \tau_k^{q-1} \le \tau_k^q - \tau_k^{q-1} = h_{q-1}^k$, we have

$$0 \le \frac{(t - \tau_k^{q-1})}{h_{q-1}^k} \le 1.$$

Then

$$\frac{(t - \tau_k^{q-1})}{h_{q-1}^k}(x_q - x_{q-1}) \in \bar{co}\{\{0\} \cup (K_1 - K_0)\},$$

hence $x_k(t) \in K_0 + \bar{co}\{\{0\} \cup (K_1 - K0)\}$ which is compact. Therefore, we can select a subsequence, again denoted by $(x_k(.))_k$ which converges uniformly to an absolutely continuous function $x(.)$ on $[0, T]$, moreover $\dot{x}_k(.)$ converges weakly to $\dot{x}(.)$ in $L^2([0, T], H)$. The family of approximate solution $x_k(.)$ satisfies the following property.

Proposition 3.2. *For every $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that*

$$\lim_{k \to +\infty} d_{grF}\left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) = 0.$$

Proof. Let $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \to +\infty} \tau_k^{q-1} = t$. Since

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \in F(x_{q-1}) + \frac{1}{kT} B, \tag{3.4}$$

we have

$$d_{grF}\left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) \le \|x_k(t) - x_k(\tau_k^{q-1})\| + \frac{1}{kT},$$

hence

$$\lim_{k \to +\infty} d_{grF}\left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) = 0.$$

□

Claim 3.3.

$$\lim_{k \to +\infty} \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds = f(t, x(t)).$$

Proof. Fix any $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$, $\lim_{k \to +\infty} \tau_k^{q-1} = \lim_{k \to +\infty} \tau_k^q = t$ and $\lim_{k \to +\infty} x_k(\tau_k^{q-1}) = x(t)$. Put

$$G(t, y) = \int_0^t f(s, y) ds.$$

Note that the function G is differentiable on t and

$$\frac{dG}{dt}(t, y) = f(t, y).$$

We have

$$
\begin{aligned}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\
\leq\ & \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\
&+ \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\
=\ & \left\| \frac{\tau_k^q - t}{\tau_k^q - \tau_k^{q-1}} \left(\frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} \right) \right\| \\
\leq\ & \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x) \right\| \\
&+ \left\| \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} - f(t, x(t)) \right\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\
&\leq \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x(t)) \right\| \\
&+2 \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|.
\end{aligned}
$$

As

$$
\lim_{k \to +\infty} \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t))
$$

and

$$
\lim_{k \to +\infty} \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t)),
$$

we have

$$
\lim_{k \to +\infty} \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} = f(t, x(t)). \tag{3.5}
$$

Put

$$
\rho_k = \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\|.
$$

On the other hand we have

$$
\begin{aligned}
&\left\|\frac{1}{h^k_{q-1}}\int_{\tau_k^{q-1}}^{\tau_k^q} f(s,x_k(\tau_k^{q-1}))ds - f(t,x(t))\right\| \\
= &\left\|\frac{G(\tau_k^q,x_k(\tau_k^{q-1}))-G(\tau_k^{q-1},x_k(\tau_k^{q-1}))}{\tau_k^q-\tau_k^{q-1}} - f(t,x(t))\right\| \\
\leq &\left\|\frac{G(\tau_k^q,x_k(\tau_k^{q-1}))-G(\tau_k^{q-1},x_k(\tau_k^{q-1}))}{\tau_k^q-\tau_k^{q-1}} - \frac{G(\tau_k^q,x(t))-G(\tau_k^{q-1},x(t))}{\tau_k^q-\tau_k^{q-1}}\right\| + \rho_k \\
= &\left\|\frac{1}{\tau_k^q-\tau_k^{q-1}}\int_{\tau_k^{q-1}}^{\tau_k^q}(f(s,x_k(\tau_k^{q-1}))-f(s,x(t)))ds\right\| + \rho_k.
\end{aligned}
$$

Since the family $\{f(s,\cdot) : s\in\mathbb{R}\}$ is equicontinuous, then there exists k_0 such that

$$\|f(s,x_k(\tau_k^{q-1}))-f(s,x(t))\| \leq \frac{1}{k} \text{ for all } k\geq k_0 \text{ and for all } s\in\mathbb{R},$$

consequently we have for $k\geq k_0$

$$\left\|\frac{G(\tau_k^q,x_k(\tau_k^{q-1}))-G(\tau_k^{q-1},x_k(\tau_k^{q-1}))}{\tau_k^q-\tau_k^{q-1}} - f(t,x(t))\right\| \leq \frac{1}{k}+\rho_k.$$

By (3.5), the last term converges to 0. This completes the proof of the Claim. □

The function $x(.)$ has the following property

Proposition 3.4. *For all $t\in[0,T]$, we have $\dot{x}(t)-f(t,x(t))\in\partial_c V(x(t))$.*

Proof. The weak convergence of $\dot{x}_k(.)$ to $\dot{x}(.)$ in $L^2([0,T],H)$ and the Mazur's Lemma entail

$$\dot{x}(t)\in\bigcap_k \bar{co}\{\dot{x}_m(t) : m\geq k\}, \text{ for a.e. on } [0,T].$$

Fix any $t\in[0,T]$, there exists $q\in\{1,\ldots,s+1\}$ such that $t\in[\tau_k^{q-1},\tau_k^q]$ and $\lim_{k\to+\infty}\tau_k^{q-1}=t$. Then for all $y\in H$

$$< y,\dot{x}(t) > \leq \inf_m \sup_{k\geq m} < y,\dot{x}_k(t) > .$$

Since $F(x)\subset\partial_c V(x)$, then by (3.4), one has

$$\dot{x}_k(t)\in\partial_c V(x_{q-1}) + \frac{1}{h^k_{q-1}}\int_{\tau_k^{q-1}}^{\tau_k^q} f(s,x_{q-1})ds + \frac{1}{kT}B.$$

Thus for all m

$$< y,\dot{x}(t) > \leq \sup_{k\geq m}\sigma\left(y,\partial_c V(x_{q-1}) + \frac{1}{h^k_{q-1}}\int_{\tau_k^{q-1}}^{\tau_k^q} f(s,x_{q-1})ds + \frac{1}{kT}B\right),$$

from which we deduce that

$$< y,\dot{x}(t) > \leq \limsup_{k\to+\infty}\sigma\left(y,\partial_c V(x_{q-1}) + \frac{1}{h^k_{q-1}}\int_{\tau_k^{q-1}}^{\tau_k^q} f(s,x_{q-1})ds + \frac{1}{kT}B\right).$$

By Proposition 2.3, the function $x \mapsto \sigma(y, \partial_c V(x))$ is $u.s.c$ and hence we get

$$< y, \dot{x}(t) > \leq \sigma(y, \partial_c V(x(t)) + f(t, x(t))).$$

So, the convexity and the closedness of the set $\partial_c V(x(t))$ ensure

$$\dot{x}(t) - f(t, x(t)) \in \partial_c V(x(t)).$$

□

Proposition 3.5. *The application $x(.)$ is a solution of the problem (2.1).*

Proof. As $x(.)$ is an absolutely continuous function and V is uniformly regular locally Lipschitz function over K (hence directionally regular over K (see [5])), by Theorem 2 in Valadier [10, 11] and by Proposition 3.4, we obtain

$$\frac{d}{dt}V(x(t)) = < \dot{x}(t), \dot{x}(t) - f(t, x(t)) > \text{ a. e. on } [0, T],$$

therefore,

$$V(x(T)) - V(x_0) = \int_0^T \| \dot{x}(s) \|^2 \, ds - \int_0^T < \dot{x}(s), f(s, x(s)) > ds. \tag{3.6}$$

On the other hand, by construction, for all $q = 1, \dots, s+1$, we have

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \in \partial_c V(x_{q-1}) + \frac{1}{kT} B.$$

Let b_q such that

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q \in \partial_c V(x_{q-1}).$$

Since V is $\beta-$uniformly regular over K, we have

$$\begin{aligned}
V(x_k(\tau_k^q)) - V(x_k(\tau_k^{q-1})) \quad &\geq \quad < x_k(\tau_k^q) - x_k(\tau_k^{q-1}), \dot{x}_k(t) \\
&\quad - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q > \\
&\quad - \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2 \\
&= \quad < \int_{\tau_k^{q-1}}^{\tau_k^q} \dot{x}_k(s) ds, \dot{x}_k(t) \\
&\quad - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q > \\
&\quad - \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2 \\
&= \quad \int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), \dot{x}_k(s) > ds \\
&\quad - \int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau > ds
\end{aligned}$$

$$+\frac{1}{kT}\int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), b_q > ds$$
$$-\beta\left\|x_k(\tau_k^q)-x_k(\tau_k^{q-1})\right\|^2.$$

By adding, we obtain

$$\begin{array}{ll} V(x_k(T))-V(x_0) & \geq \int_0^T \parallel \dot{x}_k(s) \parallel^2 ds \\ & -\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1})d\tau > ds \\ & +\frac{1}{kT}\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), b_q > ds \\ & -\sum_{q=1}^{s+1}\beta \parallel x_k(\tau_k^q)-x_k(\tau_k^{q-1}) \parallel^2 . \end{array} \tag{3.7}$$

Claim 3.6.

$$\lim_{k\to+\infty}\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau,x_{q-1})d\tau > ds = \int_0^T < \dot{x}(s), f(s,x(s)) > ds.$$

Proof. We have

$$\left\|\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau,x_{q-1})d\tau > ds - \int_0^T < \dot{x}(s), f(s,x(s)) > ds\right\|$$
$$= \left\|\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} (< \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau,x_{q-1})d\tau > - < \dot{x}(s), f(s,x(s)) >)ds\right\|$$
$$\leq \left\|\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} (< \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau,x_{q-1})d\tau > - < \dot{x}_k(s), f(s,x(s)) >)ds\right\|$$
$$+\left\|\sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} (< \dot{x}_k(s), f(s,x(s)) > - < \dot{x}(s), f(s,x(s)) >)ds\right\|$$
$$\leq \sum_{q=1}^{s+1}\int_{\tau_k^{q-1}}^{\tau_k^q} \| < \dot{x}_k(s), \frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau,x_{q-1})d\tau > - < \dot{x}_k(s), f(s,x(s)) > \| ds$$
$$+\left\|\int_0^T (< \dot{x}_k(s), f(s,x(s)) > - < \dot{x}(s), f(s,x(s)) >)ds\right\|.$$

Since

$$\|\dot{x}_k(t)\| \leq \lambda+M+1, \lim_{k\to+\infty}\frac{1}{h_{q-1}^k}\int_{\tau_k^{q-1}}^{\tau_k^q} f(s,x_{q-1})ds = f(t,x(t))$$

and $\dot{x}_k(.)$ converges weakly to $\dot{x}(.)$, the last term converges to 0. This completes the proof of the Claim. □

Claim 3.7.

$$\lim_{k\to+\infty}\sum_{q=1}^{s+1}\beta \parallel x_k(\tau_k^q)-x_k(\tau_k^{q-1}) \parallel^2 = 0.$$

Proof. By construction we have

$$\begin{aligned}\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \| &= \| (\tau_k^q - \tau_k^{q-1}) u_{q-1} \| \\ &\leq (\tau_k^q - \tau_k^{q-1}) \| u_{q-1} \| \\ &\leq (\tau_k^q - \tau_k^{q-1})(\lambda + 1 + M).\end{aligned}$$

Hence

$$\begin{aligned}\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 &\leq (\tau_k^q - \tau_k^{q-1})^2 (\lambda + 1 + M)^2 \\ &\leq (\tau_k^q - \tau_k^{q-1}) h_{q-1}^k (\lambda + 1 + M)^2 \\ &\leq (\tau_k^q - \tau_k^{q-1}) \frac{1}{k} (\lambda + 1 + M)^2.\end{aligned}$$

Then

$$\sum_{q=1}^{s+1} \beta \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 \leq \frac{\beta T (\lambda + 1 + M)^2}{k},$$

hence

$$\lim_{k \to +\infty} \sum_{q=1}^{s+1} \beta \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 = 0.$$

□

Note that

$$\lim_{k \to +\infty} \frac{1}{kT} \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), b_q > ds = 0.$$

By passing to the limit for $k \to \infty$ in (3.7) and using the continuity of the function V on the ball $B(x_0, r)$, we obtain

$$V(x(T)) - V(x_0) \geq \limsup_{k \to +\infty} \int_0^T \| \dot{x}_k(s) \|^2 \, ds - \int_0^T < \dot{x}(s), f(s, x(s)) > ds.$$

Moreover, by (3.6), we have

$$\| \dot{x} \|_2^2 \geq \limsup_{k \to +\infty} \| \dot{x}_k \|_2^2$$

and by the weak $l.s.c$ of the norm ensures

$$\| \dot{x} \|_2^2 \leq \liminf_{k \to +\infty} \| \dot{x}_k \|_2^2 \, .$$

Hence we get

$$\| \dot{x} \|_2^2 = \lim_{k \to +\infty} \| \dot{x}_k \|_2^2 \, .$$

Finally, there exists a subsequence of $(\dot{x}_k(.))_k$ (still denoted $(\dot{x}_k(.))_k$) converges pointwisely to $\dot{x}(.)$. In view of Proposition (3.2), we conclude that

$$d_{grF}((x(t), \dot{x}(t) - f(t, x(t)))) = 0$$

and as F has a closed graph, we obtain

$$\dot{x}(t) \in f(t, x(t)) + F(x(t)) \text{ a.e on } [0, T].$$

Now, let $t \in [0, T]$, there exists $q \in \{1, \dots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k\to+\infty} \tau_k^{q-1} = t$. Since

$$\lim_{k\to+\infty} \| x(t) - x_k(\tau_k^{q-1}) \| = 0,$$

$x_k(\tau_k^{q-1}) \in K_0$ and K_0 is closed we obtain $x(t) \in K_0 \subset K$. The proof is complete.

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