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On the stability of some fractional-order non-autonomous systems

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Abstract

The fractional calculus (integration and differentiation of fractional-order) is a one of the singular integral and integro-differential operators. In this work a class of fractional-order non-autonomous systems will be considered. The stability (and some other properties concerning the existence and uniqueness) of the solution will be proved.

Key words: Fractional calculus, fractional-order non-autonomous systems, stability, asymptotic stability.

1 Introduction

Let $L_1[a, b]$ denotes the space of all Lebesgue integrable functions on the interval $[a, b]$, $0 \leq a < b < \infty$.

Definition 1.1 The fractional (arbitrary) order integral of the function $f \in L_1[a, b]$ of order $\beta \in R^+$ is defined by (see [2] and [4] - [6])

$$I_a^{\beta} f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [2] and [4] - [6])

$$_*D_a^{\alpha} f(t) = \frac{d}{dt} I_a^{1 - \alpha} f(t), \quad t \in [a, b].$$

Definition 1.3 The (Caputo) fractional-order derivative D^{α} of order $\alpha \in (0, 1]$ of the function $g(t)$ is defined as (see [4] - [6])

$$D_a^{\alpha} g(t) = I_a^{1 - \alpha} \frac{d}{dt} g(t), \quad t \in [a, b].$$

Now consider the non-autonomous linear system:

$$x'(t) \;=\; A(t)\; x(t), \tag{1}$$

with the initial condition

$$x(t_0) \;=\; x^0, \quad t \;\geq\; t_0,$$

where $A(t)$ is a continuous n by n matrix on the half-axis $t \geq 0$. We know that (see [1]) the solution of system (1) is given by:

$$x(t, t_0, x^0) \;=\; X(t)\; X^{-1}(t_0)\; x^0, \quad t \;\geq\; 0,$$

where $X(t)$ is an arbitrary fundamental matrix of the system (1) defined on the whole half-axis $t \geq 0$.
Now we shall present the main definitions (see [1]) related to the concepts of stability of the solution $x = 0$ of (1).
Definition 1.4 The solution $x = 0$ of (1) will be called stable if to any $\varepsilon > 0,\ t_0 \geq 0$ there corresponds $\delta(\varepsilon, t_0) > 0$ such that $||x(t, t_0, x^0)|| < \varepsilon$ for $t \geq t_0$ as soon as $||x^0|| < \delta$.
Definition 1.5 The solution $x = 0$ of (1) will be called uniformly stable if $\delta(\varepsilon, t_0)$ from definition 1.4 can be chosen independent of $t_0 :\ \delta(\varepsilon, t_0) \equiv \delta(\varepsilon)$.
Definition 1.6 The solution $x = 0$ of (1) will be called asymptotically stable if it is stable in the sense of definition 1.4 and there exists $\gamma(t_0) > 0$ such that $\lim_{t\to\infty} ||x(t, t_0, x^0)|| = 0$ for every $x(t, t_0, x^0)$ with $||x^0|| < \gamma$.
Definition 1.7 The solution $x = 0$ of (1) will be called uniformly asymptotically stable if it is uniformly stable in the sense of definition 1.5 and moreover, for any $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $||x(t, t_0, x^0)|| < \varepsilon$ for every $t \geq t_0 + T(\varepsilon)$ and all x^0 with $||x^0|| < \gamma_0$, where γ_0 is independent of t_0.
In other words, $x = 0$ of (1) will be called uniformly asymptotically stable if it is uniformly stable and $lim||x(t, t_0, x^0)|| = 0$ as $t - t_0 \to +\infty$ uniformly with respect to $(t_0, x^0), t_0 \geq 0, ||x^0|| < \gamma_0$.

Theorem 1.1

Let $X(t)$ be a fundamental matrix of the system (1). A necessary and sufficient condition for the stability of the solution $x = 0$ is the boundedness of $X(t)$ on $t \geq 0$:

$$||X(t)|| \;\leq\; M, \quad t \;\geq\; 0.$$

A necessary and sufficient condition for the asymptotic stability of the solution $x = 0$ is

$$\lim_{t\to\infty} ||X(t)|| \;=\; 0.$$

Theorem 1.2

Let $X(t)$ be a fundamental matrix of the system (1). A necessary and sufficient condition for the uniform stability of the solution $x = 0$ is the existence of a number $M > 0$ such that:

$$||X(t)\; X^{-1}(t_0)|| \;\leq\; M, \quad t \;\geq\; t_0 \;\geq\; 0.$$

A necessary and sufficient condition for the uniform asymptotic stability of the solution $x = 0$ is the existence of two positive numbers M and η such that:

$$||X(t)\ X^{-1}(t_0)||\ \leq\ M\ \exp[-\eta\ (t\ -\ t_0)],\quad t\ \geq\ t_0\ \geq\ 0.$$

Here in this work, we study the stability (and some other properties concerning the existence and uniqueness) of the solutions of the non-autonomous linear systems:

$$D^{\alpha}_{t_0}\ x(t)\ =\ A(t)\ x(t)\ +\ f(t),\qquad \alpha\ \in\ (0,1]$$

and

$$x'(t)\ =\ A(t)\ \frac{d}{dt}\ I^{\alpha}_{t_0}\ x(t)\ +\ f(t),\qquad \alpha\ \in\ (0,1],$$

with the initial condition

$$x(t_0)\ =\ x^0.$$

Also the special cases:

$$D^{\alpha}_{t_0}\ x(t)\ =\ A\ x(t),\qquad \alpha\ \in\ (0,1],\quad x(t_0)\ =\ x^0$$

and

$$x'(t)\ =\ A\ \frac{d}{dt}\ I^{\alpha}_{t_0}\ x(t),\qquad \alpha\ \in\ (0,1],\quad x(t_0)\ =\ x^0$$

will be studied.

2 Existence of solution

Here the space $B[t_0,T]$ denotes the space of all n vector functions y such that $e^{-Nt}|y_i(t)| \in L_1[t_0,T],\ T < \infty,\ N > 0$, and the space $C^*[t_0,T]$ denotes the space of all n vector functions x such that $e^{-Nt}|x_i(t)| \in C[t_0,T],\ T < \infty,\ N > 0$, while the space $AC^*[t_0,T]$ denotes the space of all n vector absolutely continuous functions, in addition the norm on $B[t_0,T]$ will be denoted by $||.||_1$, that is, for $y \in B[t_0,T], ||y||_1 = \sum_{i=1}^{n} ||y_i||_1 = \sum_{i=1}^{n} \int_{t_0}^{t} e^{-Ns}|y_i(s)|\ ds$, while the norm on $C^*[t_0,T]$ will be denoted by $||.||_2$, that is, for $x \in C^*[t_0,T], ||x||_2 = \sum_{i=1}^{n} ||x_i||_2 = \sum_{i=1}^{n} \sup_t e^{-Nt}|x_i(t)|$. Throughout this paper we define an $n \times n$ matrix function $A(t) = (a_{ij}(t)),\ \ i,j = 1,2,...,n$ such that $A : [t_0,T] \to R,\ T < \infty$, also define

$$||A^*||\ :=\ ||a^*_{ij}||\ =\ \sup_t |a_{ij}(t)|\quad \text{and}\quad ||\widetilde{A}||\ :=\ ||\widetilde{a}_{ij}||\ =\ \sup_t |a'_{ij}(t)|$$

Consider firstly the problem:

$$D^{\alpha}_{t_0}\ x(t)\ =\ A(t)\ x(t)\ +\ f(t),\qquad \alpha\ \in\ (0,1],\ \ x(t_0)\ =\ x^0. \tag{2}$$

Theorem 2.1

Let $f(t) \in AC^*[t_0, T]$. If $A(t) \in AC^*[t_0, T]$, then there exists a unique solution of problem (2).

Proof. Problem (2) is equivalent to the equation

$$y(t) \;=\; x^0 \, \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \; I \; y(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, f(t). \tag{3}$$

Indeed: let $x(t)$ be a solution of (2) and take $y(t) = x'(t) \Rightarrow x(t) = x^0 + I \; y(t)$, then

$$I_{t_0}^{1\,-\,\alpha} \, y(t) \;=\; A(t) \; (x^0 \;+\; I \; y(t)) \;+\; f(t)$$

Operating by $I_{t_0}^{\alpha}$ on both sides of the last equation, we obtain

$$I \; y(t) \;=\; x^0 \; I_{t_0}^{\alpha} \; A(t) \;+\; I_{t_0}^{\alpha} \; A(t) \; I \; y(t) \;+\; I_{t_0}^{\alpha} \; f(t),$$

differentiating both sides, we get (3). Conversely Let $y(t)$ be a solution of (3), take $y(t) = x'(t) \Rightarrow x(t) = x^0 + I \; y(t)$ and $x(t_0) = x^0$, then

$$\begin{aligned} x'(t) \;&=\; x^0 \, \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \; (x(t) \;-\; x^0) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, f(t) \\ &=\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \; x(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, f(t). \end{aligned}$$

Operating by $I_{t_0}^{1\,-\,\alpha}$ on both sides of the last equation, we obtain

$$\begin{aligned} I_{t_0}^{1\,-\,\alpha} \; x'(t) \;&=\; I_{t_0}^{1\,-\,\alpha} \, \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \; x(t) \;+\; I_{t_0}^{1\,-\,\alpha} \, \frac{d}{dt} \, I_{t_0}^{\alpha} \, f(t) \\ &=\; I_{t_0}^{1\,-\,\alpha} \left(A(t_0) \; x^0 \, \frac{(t \;-\; t_0)^{\alpha - 1}}{\Gamma(\alpha)} \;+\; I_{t_0}^{\alpha} \; (A(t) \; x(t))' \right) \;+\; \frac{d}{dt} \, I_{t_0}^{1\,-\,\alpha} \, I_{t_0}^{\alpha} \, f(t) \\ &=\; A(t_0) \; x^0 \;+\; A(t) \; x(t) \;-\; A(t_0) \; x^0 \;+\; f(t), \end{aligned}$$

then

$$D_{t_0}^{\alpha} \; x(t) \;=\; A(t) \; x(t) \;+\; f(t).$$

Which proves the equivalence.

Now define the operator $F : B \to B$ by

$$Fy(t) \;=\; x^0 \, \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, A(t) \; I \; y(t) \;+\; \frac{d}{dt} \, I_{t_0}^{\alpha} \, f(t). \tag{4}$$

Let $y_i, \; z_i \; \in \; B$, then

$$\begin{aligned} Fy_i(t) - Fz_i(t) \;&=\; \frac{d}{dt} \, I_{t_0}^{\alpha} \; a_{ij}(t) \; I \; (y_j(t) \;-\; z_j(t)) \\ &=\; I_{t_0}^{\alpha} \; a'_{ij}(t) \; I \; (y_j(t) \;-\; z_j(t)) \;+\; I_{t_0}^{\alpha} \; a_{ij}(t) \; (y_j(t) \;-\; z_j(t)), \\ e^{-Nt} \; |Fy_i(t) - Fz_i(t)| \;&\leq\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha \,-\, 1}}{\Gamma(\alpha)} \; |a'_{ij}(s)| \int_{t_0}^{s} |y_j(\theta) \;-\; z_j(\theta)| \; d\theta \; ds \\ &+\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha \,-\, 1}}{\Gamma(\alpha)} \; |a_{ij}(s)| \; |y_j(s) \;-\; z_j(s)| \; ds \end{aligned}$$

$$
\begin{aligned}
&\leq \; ||\widetilde{a}_{ij}||\; e^{-Nt} \int_{t_0}^{t} |y_j(\theta) \;-\; z_j(\theta)| \int_{\theta}^{t} \frac{(t \;-\; s)^{\alpha \;-\; 1}}{\Gamma(\alpha)}\; ds\; d\theta \\
&+\; ||a^*_{ij}||\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha \;-\; 1}}{\Gamma(\alpha)}\; |y_j(s) \;-\; z_j(s)|\; ds \\
&\leq \; ||\widetilde{a}_{ij}||\; e^{-Nt} \int_{t_0}^{t} |y_j(\theta) \;-\; z_j(\theta)|\; \frac{(t \;-\; \theta)^{\alpha}}{\Gamma(1+\alpha)}\; d\theta \\
&+\; ||a^*_{ij}||\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha \;-\; 1}}{\Gamma(\alpha)}\; |y_j(s) \;-\; z_j(s)|\; ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t_0}^{t} e^{-Ns}\; |Fy_i(s) - Fz_i(s)|\; ds \;&\leq\; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-Ns} \int_{t_0}^{s} |y_j(\theta) \;-\; z_j(\theta)|\; \frac{(s \;-\; \theta)^{\alpha}}{\Gamma(1+\alpha)}\; d\theta\; ds \\
&+\; ||a^*_{ij}|| \int_{t_0}^{t} e^{-Ns} \int_{t_0}^{s} \frac{(s \;-\; \theta)^{\alpha \;-\; 1}}{\Gamma(\alpha)}\; |y_j(\theta) \;-\; z_j(\theta)|\; d\theta\; ds \\
&\leq\; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-N\theta}\; |y_j(\theta) - z_j(\theta)| \int_{\theta}^{t} e^{-N(s-\theta)}\; \frac{(s \;-\; \theta)^{\alpha}}{\Gamma(1+\alpha)}\; ds\; d\theta \\
&+\; ||a^*_{ij}|| \int_{t_0}^{t} e^{-N\theta}\; |y_j(\theta) - z_j(\theta)| \int_{\theta}^{t} e^{-N(s-\theta)}\; \frac{(s - \theta)^{\alpha-1}}{\Gamma(\alpha)}\; ds\; d\theta \\
&\leq\; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-N\theta}\; |y_j(\theta) - z_j(\theta)| \int_{0}^{N(t-\theta)} e^{-u} \frac{u^{\alpha}}{N^{\alpha}\; \Gamma(1+\alpha)}\; \frac{du}{N}\; d\theta \\
&+\; ||a^*_{ij}|| \int_{t_0}^{t} e^{-N\theta}\; |y_j(\theta) - z_j(\theta)| \int_{0}^{N(t-\theta)} e^{-u}\; \frac{u^{\alpha-1}}{N^{\alpha-1}\; \Gamma(\alpha)}\; \frac{du}{N}\; d\theta \\
&\leq\; \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\; ||y_j \;-\; z_j||_1 \;+\; \frac{||a^*_{ij}||}{N^{\alpha}}\; ||y_j \;-\; z_j||_1 \\
&\leq\; \left(\frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \;+\; \frac{||a^*_{ij}||}{N^{\alpha}} \right) ||y_j \;-\; z_j||_1,
\end{aligned}
$$

therefore

$$
\begin{aligned}
||Fy_i - Fz_i||_1 \;&\leq\; \left(\frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \;+\; \frac{||a^*_{ij}||}{N^{\alpha}} \right) ||y_j \;-\; z_j||_1, \\
\sum_{i=1}^{n} ||Fy_i \;-\; Fz_i||_1 \;&\leq\; \sum_{i=1}^{n} \left(\frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \;+\; \frac{||a^*_{ij}||}{N^{\alpha}} \right) ||y_j \;-\; z_j||_1, \\
||Fy \;-\; Fz||_1 \;&\leq\; \left(\frac{||\widetilde{A}||}{N^{1+\alpha}} \;+\; \frac{||A^*||}{N^{\alpha}} \right) ||y \;-\; z||_1.
\end{aligned}
$$

Now choose N large enough such that $\frac{||\widetilde{A}||}{N^{1+\alpha}} \;+\; \frac{||A^*||}{N^{\alpha}} < 1$, then we get

$$
||Fy \;-\; Fz||_1 \;<\; ||y \;-\; z||_1,
$$

therefore the map $F : B \to B$ is contraction and (4) has a unique fixed point $y \in B[t_0, T]$, therefore we deduce that the problem (2) has a unique solution $x \in AC^*[t_0, T]$.

Consider secondly the problem

$$x'(t) \;=\; A(t)\,\frac{d}{dt}\,I^{\alpha}_{t_0}\,x(t) \;+\; f(t), \qquad \alpha \;\in\; (0,1], \;\; x(t_0) \;=\; x^0. \tag{5}$$

Theorem 2.2

Let $f(t) \in B[t_0, T]$. If $A(t)$ be an $n \times n$ matrix function which is bounded and measurable, then there exists a unique solution of problem (5).

Proof. Problem (5) is equivalent to the equation

$$y(t) \;=\; x^0\,\frac{(t\;-\;t_0)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,A(t) \;+\; A(t)\,I^{\alpha}_{t_0}\,y(t) \;+\; f(t), \tag{6}$$

Indeed: let $x(t)$ be a solution of (5) and take $y(t) = x'(t) \Rightarrow x(t) = x^0 + I\;y(t)$, then

$$\begin{aligned}
y(t) \;&=\; A(t)\,\frac{d}{dt}\,I^{\alpha}_{t_0}\,(x^0 \;+\; I\;y(t)) \;+\; f(t) \\
&=\; x^0\,\frac{(t\;-\;t_0)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,A(t) \;+\; A(t)\,\frac{d}{dt}\,I\,I^{\alpha}_{t_0}\,y(t) \;+\; f(t) \\
&=\; x^0\,\frac{(t\;-\;t_0)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,A(t) \;+\; A(t)\,I^{\alpha}_{t_0}\,y(t) \;+\; f(t).
\end{aligned}$$

Conversely Let $y(t)$ be a solution of (6) and take $y(t) = x'(t) \Rightarrow x(t) = x^0 + I\;y(t)$ and $x(t_0) = x^0$, then

$$\begin{aligned}
x'(t) \;&=\; x^0\,\frac{(t\;-\;t_0)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,A(t) \;+\; A(t)\,I^{\alpha}_{t_0}\,x'(t) \;+\; f(t) \\
&=\; A(t)\,\frac{d}{dt}\,I^{\alpha}_{t_0}\,x(t) \;+\; f(t).
\end{aligned}$$

Which proves the equivalence.

Now define the operator $F : B \to B$ by

$$Fy(t) \;=\; x^0\,\frac{(t\;-\;t_0)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,A(t) \;+\; A(t)\,I^{\alpha}_{t_0}\,y(t) \;+\; f(t). \tag{7}$$

Let $y_i,\; z_i \;\in\; B$, then

$$\begin{aligned}
Fy_i(t) - Fz_i(t) \;&=\; a_{ij}(t)\,I^{\alpha}_{t_0}\,(y_j(t) \;-\; z_j(t)), \\
e^{-Nt}\,|Fy_i(t) - Fz_i(t)| \;&\leq\; e^{-Nt}\,|a_{ij}(t)|\,\int_{t_0}^{t}\frac{(t\;-\;s)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,|y_j(s) \;-\; z_j(s)|\,ds, \\
\int_{t_0}^{t} e^{-Ns}|Fy_i(s) - Fz_i(s)|\,ds \;&\leq\; ||a^*_{ij}||\int_{t_0}^{t} e^{-Ns}\int_{t_0}^{s}\frac{(s\;-\;\theta)^{\alpha\;-\;1}}{\Gamma(\alpha)}\,|y_j(\theta) \;-\; z_j(\theta)|\,d\theta\,ds \\
&\leq\; ||a^*_{ij}||\int_{t_0}^{t} e^{-N\theta}|y_j(\theta) - z_j(\theta)|\int_{\theta}^{t} e^{-N(s-\theta)}\,\frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,d\theta \\
&\leq\; ||a^*_{ij}||\int_{t_0}^{t} e^{-N\theta}|y_j(\theta) - z_j(\theta)|\int_{0}^{N(t-\theta)} e^{-u}\frac{u^{\alpha-1}}{N^{\alpha-1}\Gamma(\alpha)}\frac{du}{N}\,d\theta \\
&\leq\; \frac{||a^*_{ij}||}{N^{\alpha}}\,||y_j \;-\; z_j||_1,
\end{aligned}$$

therefore

$$\begin{aligned}
||Fy_i - Fz_i||_1 \quad &\leq \quad \frac{||a^*_{ij}||}{N^\alpha} \; ||y_j \; - \; z_j||_1, \\
\sum_{i=1}^{n} \; ||Fy_i \; - \; Fz_i||_1 \quad &\leq \quad \sum_{i=1}^{n} \frac{||a^*_{ij}||}{N^\alpha} \; ||y_j \; - \; z_j||_1, \\
||Fy \; - \; Fz||_1 \quad &\leq \quad \frac{||A^*||}{N^\alpha} \; ||y \; - \; z||_1.
\end{aligned}$$

Now choose N large enough such that $||A^*|| < N^\alpha$, then we get

$$||Fy \; - \; Fz||_1 \; < \; ||y \; - \; z||_1,$$

therefore the map $F : B \to B$ is contraction and (7) has a unique fixed point $y \in B[t_0, T]$, therefore we deduce that the problem (5) has a unique solution $x \in AC^*[t_0, T]$.

3 Stability of non-autonomous systems

In this section we study the stability of the solution of the initial-value problems (2) and (5).

Theorem 3.1

The solution of the initial-value problem (2) is uniformly stable
Proof. Let $y(t)$ be a solution of

$$\begin{aligned}
y(t) \quad &= \quad x^0 \, \frac{d}{dt} \, I^\alpha_{t_0} \, A(t) \; + \; \frac{d}{dt} \, I^\alpha_{t_0} \, A(t) \; I \; y(t) \; + \; \frac{d}{dt} \, I^\alpha_{t_0} \; f(t) \\
&= \quad A(t_0) \; x^0 \; \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} \; + \; x^0 \; I^\alpha_{t_0} \; A'(t) \; + \; I^\alpha_{t_0} \; A'(t) \; I \; y(t) \; + \; I^\alpha_{t_0} \; A(t) \; y(t) \; + \; \frac{d}{dt} \; I^\alpha_{t_0} \; f(t),
\end{aligned}$$

and let $\widetilde{y}(t)$ be a solution of the above linear system such that $\widetilde{y}(t_0) = \widetilde{x}^0$, then

$$\begin{aligned}
y_i(t) \; - \; \widetilde{y}_i(t) \quad &= \quad (x^0_j \; - \; \widetilde{x}^0_j) \; a_{ij}(t_0) \; \frac{(t \; - \; t_0)^{\alpha - 1}}{\Gamma(\alpha)} \; + \; (x^0_j \; - \; \widetilde{x}^0_j) \; I^\alpha_{t_0} \; a'_{ij}(t) \\
&+ \quad I^\alpha_{t_0} \; a'_{ij}(t) \; I \; (y_j(t) \; - \; \widetilde{y}_j(t)) \; + \; I^\alpha_{t_0} \; a_{ij}(t) \; (y_j(t) \; - \; \widetilde{y}_j(t)), \\
e^{-Nt} \; |y_i(t) - \widetilde{y}_i(t)| \quad &\leq \quad e^{-Nt} \; |x^0_j \; - \; \widetilde{x}^0_j| \; |a_{ij}(t_0)| \; \frac{(t \; - \; t_0)^{\alpha - 1}}{\Gamma(\alpha)} \\
&+ \quad e^{-Nt} \; |x^0_j \; - \; \widetilde{x}^0_j| \; \int_{t_0}^{t} \frac{(t \; - \; s)^{\alpha - 1}}{\Gamma(\alpha)} \; |a'_{ij}(s)| \; ds \\
&+ \quad e^{-Nt} \int_{t_0}^{t} \frac{(t \; - \; s)^{\alpha - 1}}{\Gamma(\alpha)} \; |a'_{ij}(s)| \; \int_{t_0}^{s} \; |y_j(\theta) \; - \; \widetilde{y}_j(\theta)| \; d\theta \; ds \\
&+ \quad e^{-Nt} \int_{t_0}^{t} \frac{(t \; - \; s)^{\alpha - 1}}{\Gamma(\alpha)} \; |a_{ij}(s)| \; |y_j(s) \; - \; \widetilde{y}_j(s)| \; ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \; ||a^*_{ij}||\; e^{-Nt}\; |x^0_j - \widetilde{x}^0_j|\; \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \;+\; ||\widetilde{a}_{ij}||\; e^{-Nt}\; |x^0_j - \widetilde{x}^0_j|\; \frac{(t-t_0)^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \; ||\widetilde{a}_{ij}||\; e^{-Nt} \int_{t_0}^{t} |y_j(\theta) \;-\; \widetilde{y}_j(\theta)| \int_{\theta}^{t} \frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, ds\, d\theta \\
&+ \; ||a^*_{ij}||\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, |y_j(s) \;-\; \widetilde{y}_j(s)|\, ds \\
&\leq \; ||a^*_{ij}||\; e^{-Nt}\; |x^0_j - \widetilde{x}^0_j|\; \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \;+\; ||\widetilde{a}_{ij}||\; e^{-Nt}\; |x^0_j - \widetilde{x}^0_j|\; \frac{(t-t_0)^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \; ||\widetilde{a}_{ij}||\; e^{-Nt} \int_{t_0}^{t} |y_j(\theta) \;-\; \widetilde{y}_j(\theta)|\, \frac{(t \;-\; \theta)^{\alpha}}{\Gamma(1+\alpha)}\, d\theta \\
&+ \; ||a^*_{ij}||\; e^{-Nt} \int_{t_0}^{t} \frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, |y_j(s) \;-\; \widetilde{y}_j(s)|\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t_0}^{t} e^{-Ns}\, |y_i(s) - \widetilde{y}_i(s)|\, ds \;&\leq\; ||a^*_{ij}||\; e^{-Nt_0}\; |x^0_j \;-\; \widetilde{x}^0_j| \int_{t_0}^{t} e^{-N(s-t_0)}\, \frac{(s \;-\; t_0)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&+ \; ||\widetilde{a}_{ij}||\; e^{-Nt_0}\; |x^0_j \;-\; \widetilde{x}^0_j| \int_{t_0}^{t} e^{-N(s-t_0)}\, \frac{(s \;-\; t_0)^{\alpha}}{\Gamma(1+\alpha)}\, ds \\
&+ \; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-Ns} \int_{t_0}^{s} |y_j(\theta) \;-\; \widetilde{y}_j(\theta)|\, \frac{(s \;-\; \theta)^{\alpha}}{\Gamma(1+\alpha)}\, d\theta\, ds \\
&+ \; ||a^*_{ij}|| \int_{t_0}^{t} e^{-Ns} \int_{t_0}^{s} \frac{(s \;-\; \theta)^{\alpha-1}}{\Gamma(\alpha)}\, |y_j(\theta) \;-\; \widetilde{y}_j(\theta)|\, d\theta\, ds \\
&\leq \; ||a^*_{ij}||\; ||x^0_j \;-\; \widetilde{x}^0_j||_2 \int_{0}^{N(t-t_0)} e^{-u}\, \frac{u^{\alpha-1}}{N^{\alpha-1}\;\Gamma(\alpha)}\, \frac{du}{N} \\
&+ \; ||\widetilde{a}_{ij}||\; ||x^0_j \;-\; \widetilde{x}^0_j||_2 \int_{0}^{N(t-t_0)} e^{-u}\, \frac{u^{\alpha}}{N^{\alpha}\;\Gamma(1+\alpha)}\, \frac{du}{N} \\
&+ \; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-N\theta}\, |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{\theta}^{t} e^{-N(s-\theta)} \frac{(s-\theta)^{\alpha}}{\Gamma(1+\alpha)}\, ds\, d\theta \\
&+ \; ||a^*_{ij}|| \int_{t_0}^{t} e^{-N\theta}\, |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{\theta}^{t} e^{-N(s-\theta)} \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}\, ds\, d\theta \\
&\leq \; \frac{||a^*_{ij}||}{N^{\alpha}}\; ||x^0_j \;-\; \widetilde{x}^0_j||_2 \;+\; \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\; ||x^0_j \;-\; \widetilde{x}^0_j||_2 \\
&+ \; ||\widetilde{a}_{ij}|| \int_{t_0}^{t} e^{-N\theta}\, |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{0}^{N(t-\theta)} e^{-u} \frac{u^{\alpha}}{N^{\alpha}\;\Gamma(1+\alpha)} \frac{du}{N}\, d\theta \\
&+ \; ||a^*_{ij}|| \int_{t_0}^{t} e^{-N\theta}\, |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{0}^{N(t-\theta)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1}\;\Gamma(\alpha)} \frac{du}{N}\, d\theta,
\end{aligned}
$$

then

$$
||y_i \;-\; \widetilde{y}_i||_1 \;\leq\; \frac{||a^*_{ij}||}{N^{\alpha}}\; ||x^0_j \;-\; \widetilde{x}^0_j||_2 \;+\; \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\; ||x^0_j \;-\; \widetilde{x}^0_j||_2
$$

$$
\begin{aligned}
&+ \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \, ||y_j \ - \ \widetilde{y}_j||_1 \ + \ \frac{||a^*_{ij}||}{N^{\alpha}} \, ||y_j \ - \ \widetilde{y}_j||_1, \\
\left(1 - \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} - \frac{||a^*_{ij}||}{N^{\alpha}}\right) ||y_j \ - \ \widetilde{y}_j||_1 \ &\leq \ \left(\frac{||a^*_{ij}||}{N^{\alpha}} \ + \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\right) ||x^0_j \ - \ \widetilde{x}^0_j||_2 \\
\Rightarrow \ ||y_i \ - \ \widetilde{y}_i||_1 \ &\leq \ \left(1 - \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} - \frac{||a^*_{ij}||}{N^{\alpha}}\right)^{-1} \left(\frac{||a^*_{ij}||}{N^{\alpha}} + \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\right) ||x^0_j \ - \ \widetilde{x}^0_j||_2.
\end{aligned}
$$

Now, since

$$
x(t) \ = \ x^0 \ + \ I \ y(t),
$$

then

$$
\begin{aligned}
x_i(t) \ - \ \widetilde{x}_i(t) \ &= \ x^0_i \ - \ \widetilde{x}^0_i \ + \ \int_{t_0}^{t} (y_i(s) \ - \ \widetilde{y}_i(s)) \, ds, \\
e^{-Nt}|x_i(t) \ - \ \widetilde{x}_i(t)| \ &\leq \ e^{-Nt} \, |x^0_i \ - \ \widetilde{x}^0_i| \ + \ e^{-Nt} \int_{t_0}^{t} |y_i(s) \ - \ \widetilde{y}_i(s)| \, ds \\
&\leq \ e^{-Nt_0} \, |x^0_i \ - \ \widetilde{x}^0_i| \ + \ e^{-Nt} \int_{t_0}^{t} e^{Ns} \, e^{-Ns} \, |y_i(s) \ - \ \widetilde{y}_i(s)| \, ds, \\
||x_i \ - \ \widetilde{x}_i||_2 \ &\leq \ ||x^0_i \ - \ \widetilde{x}^0_i||_2 \ + \ ||y_i \ - \ \widetilde{y}_i||_1 \\
&\leq \ ||x^0_i \ - \ \widetilde{x}^0_i||_2 \\
&+ \ \left(1 \ - \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \ - \ \frac{||a^*_{ij}||}{N^{\alpha}}\right)^{-1} \left(\frac{||a^*_{ij}||}{N^{\alpha}} \ + \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}}\right) ||x^0_j \ - \ \widetilde{x}^0_j||_2 \\
&\leq \ \left(1 \ - \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \ - \ \frac{||a^*_{ij}||}{N^{\alpha}}\right)^{-1} ||x^0_j \ - \ \widetilde{x}^0_j||_2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{i=1}^{n} ||x_i \ - \ \widetilde{x}_i||_2 \ &\leq \ \sum_{i=1}^{n} \left(1 \ - \ \frac{||\widetilde{a}_{ij}||}{N^{1+\alpha}} \ - \ \frac{||a^*_{ij}||}{N^{\alpha}}\right)^{-1} ||x^0_j \ - \ \widetilde{x}^0_j||_2, \\
||x \ - \ \widetilde{x}||_2 \ &\leq \ \left(1 \ - \ \frac{||\widetilde{A}||}{N^{1+\alpha}} \ - \ \frac{||A^*||}{N^{\alpha}}\right)^{-1} ||x^0 \ - \ \widetilde{x}^0||_2
\end{aligned}
$$

Therefore, if $||x^0 \ - \ \widetilde{x}^0||_2 \ < \ \delta(\varepsilon)$, then $||x \ - \ \widetilde{x}||_2 \ < \ \varepsilon$, which complete the proof of the theorem.

Theorem 3.2

The solution of the initial-value problem (5) is uniformly stable.
Proof. Let $y(t)$ be a solution of

$$
y(t) \ = \ x^0 \, \frac{(t \ - \ t_0)^{\alpha \ - \ 1}}{\Gamma(\alpha)} \, A(t) \ + \ A(t) \, I^{\alpha}_{t_0} \, y(t) \ + \ f(t)
$$

and let $\widetilde{y}(t)$ be a solution of the above linear system such that $\widetilde{y}(t_0) = \widetilde{x}^0$, then

$$\begin{aligned}
y_i(t) - \widetilde{y}_i(t) &= (x_j^0 - \widetilde{x}_j^0) \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} a_{ij}(t) + a_{ij}(t) I_{t_0}^{\alpha} (y_j(t) - \widetilde{y}_j(t)), \\
e^{-Nt} |y_i(t) - \widetilde{y}_i(t)| &\leq e^{-Nt} |x_j^0 - \widetilde{x}_j^0| \frac{(t - t_0)^{\alpha-1}}{\Gamma(\alpha)} | |a_{ij}(t)| \\
&+ |a_{ij}(t)| e^{-Nt} \int_{t_0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - \widetilde{y}_j(s)| ds \\
&\leq ||a_{ij}^*|| e^{-Nt} |x_j^0 - \widetilde{x}_j^0| \frac{(t - t_0)^{\alpha-1}}{\Gamma(\alpha)} \\
&+ ||a_{ij}^*|| e^{-Nt} \int_{t_0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - \widetilde{y}_j(s)| ds, \\
\int_{t_0}^{t} e^{-Ns} |y_j(s) - \widetilde{y}_j(s)| ds &\leq ||a_{ij}^*|| e^{-Nt_0} |x_j^0 - \widetilde{x}_j^0| \int_{t_0}^{t} e^{-N(s-t_0)} \frac{(s - t_0)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ ||a_{ij}^*|| \int_{t_0}^{t} e^{-Ns} \int_{t_0}^{s} \frac{(s - \theta)^{\alpha-1}}{\Gamma(\alpha)} |y_j(\theta) - \widetilde{y}_j(\theta)| d\theta\, ds \\
&\leq ||a_{ij}^*|| ||x_j^0 - \widetilde{x}_j^0||_2 \int_{0}^{N(t-t_0)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} \\
&+ ||a_{ij}^*|| \int_{t_0}^{t} e^{-N\theta} |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{\theta}^{t} e^{-N(s-\theta)} \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} ds\, d\theta \\
&\leq \frac{||a_{ij}^*||}{N^{\alpha}} ||x_j^0 - \widetilde{x}_j^0||_2 \\
&+ ||a_{ij}^*|| \int_{t_0}^{t} e^{-N\theta} |y_j(\theta) - \widetilde{y}_j(\theta)| \int_{0}^{N(t-\theta)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} d\theta \\
&\leq \frac{||a_{ij}^*||}{N^{\alpha}} ||x_j^0 - \widetilde{x}_j^0||_2 + \frac{||a_{ij}^*||}{N^{\alpha}} ||y_j - \widetilde{y}_j||_1,
\end{aligned}$$

then

$$\begin{aligned}
||y_i - \widetilde{y}_i||_1 &\leq \frac{||a_{ij}^*||}{N^{\alpha}} ||x_j^0 - \widetilde{x}_j^0||_2 + \frac{||a_{ij}^*||}{N^{\alpha}} ||y_j - \widetilde{y}_j||_1, \\
\left(1 - \frac{||a_{ij}^*||}{N^{\alpha}}\right) ||y_j - \widetilde{y}_j||_1 &\leq \frac{||a_{ij}^*||}{N^{\alpha}} ||x_j^0 - \widetilde{x}_j^0||_2 \\
\Rightarrow ||y_i - \widetilde{y}_i||_1 &\leq \frac{||a_{ij}^*||}{N^{\alpha}} \left(1 - \frac{||a_{ij}^*||}{N^{\alpha}}\right)^{-1} ||x_j^0 - \widetilde{x}_j^0||_2 \\
&\leq \left(\frac{||a_{ij}^*||}{N^{\alpha} - ||a_{ij}^*||}\right) ||x_j^0 - \widetilde{x}_j^0||_2.
\end{aligned}$$

Now, since

$$x(t) = x^0 + I\, y(t),$$

then

$$||x_i - \widetilde{x}_i||_2 \leq ||x_i^0 - \widetilde{x}_i^0||_2 + ||y_i - \widetilde{y}_i||_1$$

$$\leq \; ||x_i^0 - \widetilde{x}_i^0||_2 + \left(\frac{||a_{ij}^*||}{N^\alpha - ||a_{ij}^*||}\right) ||x_j^0 - \widetilde{x}_j^0||_2$$

$$\leq \; \left(\frac{N^\alpha}{N^\alpha - ||a_{ij}^*||}\right) ||x_j^0 - \widetilde{x}_j^0||_2.$$

Therefore

$$\sum_{i=1}^{n} ||x_i - \widetilde{x}_i||_2 \leq \sum_{i=1}^{n} \left(\frac{N^\alpha}{N^\alpha - ||a_{ij}^*||}\right) ||x_j^0 - \widetilde{x}_j^0||_2,$$

$$||x - \widetilde{x}||_2 \leq \left(\frac{N^\alpha}{N^\alpha - ||A^*||}\right) ||x^0 - \widetilde{x}^0||_2$$

Therefore, if $||x^0 - \widetilde{x}^0||_2 < \delta(\varepsilon)$, then $||x - \widetilde{x}||_2 < \varepsilon$, which complete the proof of the theorem.

4 Autonomous systems

Now we study the problems:

$$D_{t_0}^\alpha \, x(t) = A \, x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0 \tag{8}$$

and

$$x'(t) = A \, \frac{d}{dt} \, I_{t_0}^\alpha \, x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0 \tag{9}$$

which are the special cases of the initial-value problems (2) and (5) when $A(t) = A$, where A is a real constant matrix and $f(t)$ is the zero vector.

Theorem 4.1

The solution of the initial-value problem (8) or (9) is given by the formula

$$x(t) = \sum_{k=0}^{\infty} (A \, I_{t_0}^\alpha)^k \, x^0 = \sum_{k=0}^{\infty} \frac{A^k \, (t - t_0)^{k\alpha}}{\Gamma(1 + k\alpha)} \, x^0$$

Proof. Firstly we prove that the two problems are equivalent to each other, indeed: Let $x(t)$ be a solution of (8), then

$$I_{t_0}^{1 - \alpha} \, \frac{dx}{dt} = A \, x(t),$$

operating by $I_{t_0}^\alpha$ on both sides of the last relation, we get

$$x(t) = x^0 + A \, I_{t_0}^\alpha \, x(t),$$

differentiating both sides, we obtain (9) and when $t = t_0$, we obtain $x(t_0) = x^0$.
Conversely let $x(t)$ be a solution of (9), operating by $I_{t_0}^{1\,-\,\alpha}$ on both sides of it, we get

$$\begin{aligned} I_{t_0}^{1\,-\,\alpha}\,\frac{dx}{dt} &= A\ I_{t_0}^{1\,-\,\alpha}\left(x^0\ \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\ +\ I_{t_0}^{\alpha}\ x'(t)\right) \\ \Rightarrow\ D_{t_0}^{\alpha}\ x(t) &= A\ (x^0\ +\ x(t)\ -\ x^0)\ =\ A\ x(t). \end{aligned}$$

Now since

$$\begin{aligned} e^{-Nt}\ |a_{ij}\ I_{t_0}^{\alpha}\ x_j(t)| &\leq |a_{ij}|\ e^{-Nt}\int_{t_0}^{t}\frac{(t\ -\ s)^{\alpha\,-\,1}}{\Gamma(\alpha)}\ |x_j(s)|\ ds \\ &\leq |a_{ij}|\int_{t_0}^{t} e^{-N(t\,-\,s)}\ \frac{(t\ -\ s)^{\alpha\,-\,1}}{\Gamma(\alpha)}\ e^{-Ns}\ |x_j(s)|\ ds, \\ ||a_{ij}\ I_{t_0}^{\alpha}\ x_j||_2 &\leq |a_{ij}|\ ||x_j||_2\int_0^{N(t-t_0)} e^{-u}\ \frac{u^{\alpha-1}}{N^{\alpha-1}\ \Gamma(\alpha)}\ \frac{du}{N} \\ &\leq \frac{|a_{ij}|}{N^{\alpha}}\ ||x_j||_2, \end{aligned}$$

then

$$\begin{aligned} \sum_{i=1}^{n}\ ||a_{ij}\ I_{t_0}^{\alpha}\ x_j||_2 &\leq \frac{1}{N^{\alpha}}\sum_{i=1}^{n}\ |a_{ij}|\ ||x_j||_2, \qquad i\ =\ 1,2,...,n, \\ ||A\ I_{t_0}^{\alpha}\ x||_2 &\leq \frac{||A||}{N^{\alpha}}\ ||x||_2\ <\ ||x||_2, \end{aligned}$$

where $||A|| < N^{\alpha}$, it follows that $||A\ I_{t_0}^{\alpha}||_2\ <\ 1$, then from Neumann expansion (see [3]) we complete the proof.

Theorem 4.2

If A is a real constant matrix with the characteristic roots all having negative real parts, then the solution of the initial-value problem (8) (or (9)) is uniformly asymptotically stable.
Proof.

$$\begin{aligned} |x_i(t)\ -\ \widetilde{x}_i(t)| &= |x_i(t)\ -\ e^{a_{ij}(t\,-\,t_0)}\ x_j^0\ -\ \widetilde{x}_i(t)\ +\ e^{a_{ij}(t\,-\,t_0)}\ \widetilde{x}_j^0 \\ &+\ e^{a_{ij}(t\,-\,t_0)}\ x_j^0\ -\ e^{a_{ij}(t\,-\,t_0)}\ \widetilde{x}_j^0| \\ &= |(x_i(t)\ -\ e^{a_{ij}(t\,-\,t_0)}\ x_j^0)\ -\ (\widetilde{x}_i(t)\ -\ e^{a_{ij}(t\,-\,t_0)}\ \widetilde{x}_j^0) \\ &+\ e^{a_{ij}(t\,-\,t_0)}\ (x_j^0\ -\ \widetilde{x}_j^0)| \\ &= |(\sum_{k=0}^{\infty}\frac{a_{ij}^k\ (t\ -\ t_0)^{k\alpha}}{\Gamma(1\ +\ k\alpha)}\ x_j^0\ -\ e^{a_{ij}(t\,-\,t_0)}\ x_j^0) \\ &-\ (\sum_{k=0}^{\infty}\frac{a_{ij}^k\ (t-t_0)^{k\alpha}}{\Gamma(1+k\alpha)}\ \widetilde{x}_j^0\ -\ e^{a_{ij}(t-t_0)}\ \widetilde{x}_j^0)\ +\ e^{a_{ij}(t-t_0)}\ (x_j^0-\widetilde{x}_j^0)| \\ &\leq |\sum_{k=0}^{\infty}\frac{a_{ij}^k\ (t\ -\ t_0)^{k\alpha}}{\Gamma(1\ +\ k\alpha)}\ -\ \sum_{k=0}^{\infty}\frac{a_{ij}^k\ (t\ -\ t_0)^{k}}{\Gamma(1\ +\ k)}|\ |x_j^0\ -\ \widetilde{x}_j^0| \end{aligned}$$

$$
\begin{aligned}
& + \quad e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0| \\
& = \quad |-\sum_{k=0}^{\infty} \frac{a_{ij}^k}{\Gamma(1+k)} \left(1 \,-\, \frac{\Gamma(1+k)}{\Gamma(1+k\alpha)}\,(t-t_0)^{k\alpha-k}\right)\,(t-t_0)^k|\;|x_j^0 - \widetilde{x}_j^0| \\
& + \quad e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0|.
\end{aligned}
$$

Let $k\ \alpha \ = \ k \ - \ \beta, \ \alpha \ \in \ (0,1], \ \beta \ > \ 0$, then

$$
\begin{aligned}
|x_i(t) \,-\, \widetilde{x}_i(t)| \quad & \leq \quad |-\sum_{k=0}^{\infty} \frac{a_{ij}^k}{\Gamma(1+k)} \left(1 \,-\, \frac{\Gamma(1+k)}{\Gamma(1+k\,-\,\beta)}\,(t-t_0)^{-\beta}\right)\,(t-t_0)^k|\;|x_j^0 - \widetilde{x}_j^0| \\
& + \quad e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0| \\
& \leq \quad |\sum_{k=0}^{\infty} \frac{a_{ij}^k\,(t\,-\,t_0)^k}{\Gamma(1\,+\,k)}|\;|x_j^0 \,-\, \widetilde{x}_j^0\,| \,+\, e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0| \\
& = \quad e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0| \,+\, e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0| \\
& = \quad 2\,e^{a_{ij}(t\,-\,t_0)}\,|x_j^0 \,-\, \widetilde{x}_j^0|.
\end{aligned}
$$

Since A is a real constant matrix with the characteristic roots all having negative real parts, then there exists positive constants K and σ such that

$$
e^{At} \ \leq \ K\ e^{-\sigma t},
$$

therefore

$$
\begin{aligned}
e^{-Nt}\,|x_i(t) \,-\, \widetilde{x}_i(t)| \quad & \leq \quad 2\,K\,e^{-\sigma\,(t\,-\,t_0)}\,e^{-Nt}\,|x_i^0 \,-\, \widetilde{x}_i^0| \\
& \leq \quad 2\,K\,e^{-\sigma\,(t\,-\,t_0)}\,e^{-Nt_0}\,|x_i^0 \,-\, \widetilde{x}_i^0|, \\
||x_i \,-\, \widetilde{x}_i||_2 \quad & \leq \quad 2\,K\,e^{-\sigma\,(t\,-\,t_0)}\,||x_i^0 \,-\, \widetilde{x}_i^0||_2, \\
\sum_{i=1}^{n} ||x_i \,-\, \widetilde{x}_i||_2 \quad & \leq \quad 2\,K\,e^{-\sigma\,(t\,-\,t_0)}\,\sum_{i=1}^{n} ||x_i^0 \,-\, \widetilde{x}_i^0||_2, \\
||x \,-\, \widetilde{x}||_2 \quad & \leq \quad 2\,K\,e^{-\sigma\,(t\,-\,t_0)}\,||x^0 \,-\, \widetilde{x}^0||_2.
\end{aligned}
$$

Therefore, we deduce that the solution is uniformly asymptotically stable.

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