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ON EXISTENCE OF PROPER SOLUTIONS OF QUASILINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract

In the paper, the nonlinear differential equation $(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = e(t)$ is studied. Sufficient conditions for the nonexistence of singular solutions of the first and second kind are given. Hence, sufficient conditions for all nontrivial solutions to be proper are derived. Sufficient conditions for the nonexistence of weakly oscillatory solutions are given.

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1 Introduction

In this paper, we study the existence of proper solutions of a forced second order nonlinear differential equation of the form

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = e(t) \tag{1}$$

where $p > 0$, $a \in C^0(R_+)$, $b \in C^0(R_+)$, $r \in C^0(R_+)$, $e \in C^0(R_+)$, $f \in C^0(R)$, $g \in C^0(R)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$ and $a > 0$ on R_+.

A special case of Equation (1) is the unforced equation

$$(a(t)|y'|^{p-1}y')' + b(t)g(y') + r(t)f(y) = 0. \tag{2}$$

We will often use of the following assumptions

$$f(x)x \geq 0 \quad \text{ on } R \tag{3}$$

and

$$g(x)x \geq 0 \quad \text{ on } R_+. \tag{4}$$

Definition 1. A solution y of (1) is called proper if it is defined on R_+ and $sup_{t\in[\tau,\infty)}|y(t)| > 0$ for every $\tau \in (0, \infty)$. It is called singular of the 1-st kind if it is defined on R_+, there exists $\tau \in (0, \infty)$ such that $y \equiv 0$ on $[\tau, \infty)$ and $sup_{T\leq t\leq \tau}|y(t)| > 0$ for every $T \in [0, \tau)$. It is called singular of the 2-nd kind if it is defined on $[0, \tau), \tau < \infty$ and $sup_{0\leq t<\tau}|y'(t)| = \infty$.

Note, that a singular solution y of the 2-nd kind is sometimes called noncontinuable.

Definition 2. A proper solution y of (1) is called oscillatory if there exists a sequence of its zeros tending to ∞. Otherwise, it is called nonoscillatory. A nonoscillatory solution y of (1) is called weakly oscillatory if there exists a sequence of zeros of y' tending to ∞.

It is easy to see that (1) can be transformed into the system

$$\begin{aligned} y_1' &= a(t)^{-\frac{1}{p}}|y_2|^{\frac{1}{p}} \operatorname{sgn} y_2, \\ y_2' &= -b(t)g(a(t)^{-\frac{1}{p}}|y_2|^{\frac{1}{p}} \operatorname{sgn} y_2) - r(t)f(y_1) + e(t); \end{aligned} \tag{5}$$

the relation between a solution y of (1) and a solution of (5) is $y_1(t) = y(t)$, $y_2(t) = a(t)|y'(t)|^{p-1}y'(t)$.

An important problem is the existence of solutions defined on R_+ or of proper solutions (for Equation (2)). Their asymptotic behaviour is studied by many authors (see e.g. monographs [7], [9] and [10], and the references therein). So, it is very important to know conditions under the validity of which all solutions of (1) are defined on R_+ or are proper. For a special type of the equation of (2), for the equation

$$(a(t)|y'|^{p-1}y')' + r(t)f(y) = 0, \tag{6}$$

sufficient conditions for all nontrivial solutions to be proper are given e.g. in [1], [8], [9] and [10]. It is known that for half-linear equations, i.e., if $f(x) = |x|^p \operatorname{sgn} x$, all nontrivial solutions of (4) are proper, see e.g. [6]. For the forced equation (1) with (3) holding, $a \in C^1(R_+)$, $a^{\frac{1}{p}} r \in AC^1_{loc}(R_+)$ and $b \equiv 0$, it is proved in [2] that all solutions are defined on R_+, i.e., the set of all singular solutions of the second kind is empty. On the other hand, in [4] and [5] examples are given for which Equation (6) has singular solutions of the first and second kinds (see [1], as well). Moreover, Lemma 4 in [3] gives sufficient conditions for the equation

$$(a(t)y')' + r(t)f(y) = 0$$

to have no proper solutions.

In the present paper, these problems are solved for (1). Sufficient conditions for the nonexistence of singular solutions of the first and second kinds are given, and so, sufficient conditions for all nontrivial solutions of (2) to be proper are given. In the last section, simple asymptotic properties of solutions of (2) are given.

Note that it is known that Equation (6) has no weakly oscillatory solutions (see e.g. [10]), but as we will see in Section 4, Equation (1) may have them.

It will be convenient to define the following constants:

$$\gamma = \frac{p+1}{p(\lambda+1)}, \delta = \frac{p+1}{p}.$$

We define the function $R : R_+ \to R$ by

$$R(t) = a^{\frac{1}{p}}(t)r(t).$$

For any solution y of (1), we let

$$y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t)$$

and if (3) and $r > 0$ on R_+ hold, let us define

$$\begin{aligned} V(t) &= \frac{a(t)}{r(t)}|y'(t)|^{p+1} + \gamma \int_0^{y(t)} f(s)\ \mathrm{d}s \\ &= \frac{|y^{[1]}(t)|^{\delta}}{R(t)} + \gamma \int_0^{y(t)} f(s)\ \mathrm{d}s \geq 0. \end{aligned} \tag{7}$$

For any continuous function $h : R_+ \to R$, we let $h_+(t) = \max\{h(t), 0\}$ and $h_-(t) = \max\{-h(t), 0\}$ so that $h(t) = h_+(t) - h_-(t)$.

2 Singular solutions of the second kind

In this section, the nonexistence of singular solutions of the second kind will be studied. The following theorem is a generalization of the well-known Wintner's Theorem to (1).

Theorem 1. *Let $M > 0$ and*

$$|g(x)| \le |x|^p \qquad \text{and} \qquad |f(x)| \le |x|^p \text{ for } |x| \ge M.$$

Then there exist no singular solution y of the second kind of (1) *and all solutions of* (1) *are defined on R_+.*

Proof. Let, to the contrary, y be a singular solution of the second kind defined on $[0, \tau)$, $\tau < \infty$. Then,

$$\sup_{0 \le t < \tau} |y'(t)| = \infty \qquad \text{and} \qquad \sup_{0 \le t < \tau} |y^{[1]}(t)| = \infty. \tag{8}$$

The assumptions of the theorem yield

$$|f(x)| \le M_1 + |x|^p \qquad \text{and} \qquad |g(x)| \le M_2 + |x|^p \tag{9}$$

with $M_1 = \max_{|s| \le M} |f(s)|$ and $M_2 = \max_{|s| \le M} |g(s)|$. Let $t_0 \in [0, \tau)$ be such that

$$\tau - t_0 \le 1, \quad \int_{t_0}^{\tau} a^{-1}(s)|b(s)| \, \mathrm{d}s \le \frac{1}{2}, \tag{10}$$

and

$$2^p \max_{0 \le s \le \tau} |r(s)| \Big(\int_{t_0}^{\tau} a^{-\frac{1}{p}}(s) \, \mathrm{d}s \Big)^p \le \frac{1}{3}. \tag{11}$$

Using system (5), by an integration we obtain

$$|y_1(t)| \le |y_1(t_0)| + \int_{t_0}^{t} a^{-\frac{1}{p}}(s) |y_2(s)|^{\frac{1}{p}} \, \mathrm{d}s \tag{12}$$

and

$$\begin{aligned} &|y_2(t)| \le |y_2(t_0)| \\ &+ \int_{t_0}^{t} [|b(s) g(a(s)^{-\frac{1}{p}} |y_2(s)|^{\frac{1}{p}} \operatorname{sgn} y_2(s))| + |r(s)||f(y_1(s))| + |e(s)|] \, \mathrm{d}s. \end{aligned} \tag{13}$$

Hence, using (9), (10) and (12), we have for $t \in [t_0, \tau)$,

$$
\begin{aligned}
|y_2(t)| &\le |y_2(t_0)| + \int_{t_0}^{t} |b(s)|[M_2 + a^{-1}(s)|y_2(s)|] \, \mathrm{d}s \\
&\quad + \int_{t_0}^{t} |r(s)|[M_1 + |y_1(s)|^p] \, \mathrm{d}s + \int_{t_0}^{t} |e(s)| \, \mathrm{d}s \\
&\le M_3 + \frac{1}{2} \max_{t_0 \le s \le t} |y_2(s)| \, \mathrm{d}s \\
&\quad + \int_{t_0}^{t} |r(s)|[|y_1(t_0)| + \int_{t_0}^{s} a^{-\frac{1}{p}}(\sigma)|y_2(\sigma)|^{\frac{1}{p}} \mathrm{d}\sigma]^p \, \mathrm{d}s
\end{aligned}
\tag{14}
$$

with $M_3 = |y_2(t_0)| + M_2 \int_{t_0}^{\tau} |b(s)| \, \mathrm{d}s + M_1 \int_{t_0}^{\tau} |r(s)| \, \mathrm{d}s + \int_{t_0}^{\tau} |e(s)| \, \mathrm{d}s$.

Denote $v(t_0) = |y_2(t_0)|$ and $v(t) = \max_{t_0 \le s \le t} |y_2(s)|$, $t \in (t_0, \tau)$. Then, (10), (12) and (14) yield

$$
\begin{aligned}
v(t) &\le M_3 + \frac{1}{2} v(t) + \int_{t_0}^{t} |r(s)|[|y_1(t_0)| + M_4 v(s)^{\frac{1}{p}}]^p \, \mathrm{d}s \\
&\le M_3 + \frac{1}{2} v(t) + 2^p M_5 \int_{t_0}^{t} [y_1^p(t_0) + M_4^p v(s)] \, \mathrm{d}s \\
&\le M_3 + \frac{1}{2} v(t) + 2^p M_5 y_1^p(t_0) + 2^p M_4^p M_5 v(t)
\end{aligned}
$$

with $M_4 = \int_{t_0}^{\tau} a^{-\frac{1}{p}}(\sigma) \mathrm{d}\sigma$, $M_5 = \max_{0 \le s \le \tau} |r(s)|$.
From this and from (11), we have

$$
\frac{1}{6} v(t) \le M_3 + 2^p M_5 y_1^p(t_0), t \in [t_0, \tau).
$$

But this inequality contradicts (8) and the definition of v. ☐

Remark 1. The results of Theorem 1 for Equation (1) with $p \le 1$ and without the damping ($b \equiv 0$) is a generalization of the well-known Wintner's Theorem, see e.g. Theorem 11.5. in [9] or Theorem 6.1. in [7].

The following result shows that singular solutions of the second kind of (1) do not exist if $r > 0$ and R is smooth enough under weakened assumptions on f.

Theorem 2. *Let* (3), $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and let either*

(i) $M \in (0, \infty)$ *exist such that* $|g(x)| \le |x|^p$ *for* $|x| \ge M$
or

(ii) (4) *holds and* $b(t) \ge 0$ *on* R_+.

Then Equation (1) *has no singular solution of the second kind and all solutions of* (1) *are defined on* R_+.

Proof. Suppose y is a singular solution of the second kind defined on $I = [0, \tau)$. Then $sup_{t\in[0,\tau)}|y'(t)| = \infty$ and (7) yields

$$\begin{aligned}
V'(t) &= \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^{\delta} + \frac{\delta}{r(t)} y'(t)(y^{[1]}(t))' + \delta f(y(t))y'(t) \\
&= \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^{\delta} + \frac{\delta}{r(t)} y'(t)[e(t) - b(t)g(y'(t)) \\
&- r(t)f(y(t))] + \delta f(t)y'(t)
\end{aligned}$$

or

$$V'(t) = \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^{\delta} + \frac{\delta}{r(t)} y'(t)e(t) - \frac{\delta b(t)g(y'(t))y'(t)}{r(t)} \tag{15}$$

for $t \in I$. We will estimate the summands in (15). We have on I,

$$\left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^{\delta} = \frac{-R'(t)}{R(t)} \frac{|y^{[1]}(t)|^{\delta}}{R(t)} \le \frac{R'_-(t)}{R(t)} V(t). \tag{16}$$

From $|x| \le |x|^s + 1$ for $s \ge 1$ and $x \in R$, we get

$$\begin{aligned}
\left|\frac{\delta e(t)}{r(t)} y'(t)\right| &= \left|\frac{\delta e(t)a^{\frac{1}{p}}(t)y'(t)}{R(t)}\right| \qquad (17) \\
&\le \delta|e(t)|a^{\frac{1}{p}}(t)\frac{|y'(t)|^{p+1} + 1}{R(t)} \\
&= \frac{\delta|e(t)||y^{[1]}(t)|^{\delta}}{a(t)R(t)} + \frac{\delta|e(t)|}{r(t)} \le \frac{\delta|e(t)|V(t)}{a(t)} + \frac{\delta|e(t)|}{r(t)}
\end{aligned}$$

on I. Furthermore, in case (ii), we have

$$\begin{aligned}
-\frac{\delta b(t)g(y'(t))y'(t)}{r(t)} &\le v(t) + \frac{\delta|b(t)||y'(t)|^{p+1}}{r(t)} \\
&= v(t) + \frac{\delta|b(t)||y^{[1]}(t)|^{\delta}}{a(t)R(t)} \le v(t) + \frac{\delta|b(t)|V(t)}{a(t)}
\end{aligned} \tag{18}$$

with

$$v(t) = \frac{\delta|b(t)|}{r(t)} \max_{|s|\le M} |sg(s)|.$$

Due to the fact that $b \geq 0$, inequality (18) holds in case (i) with $v(t) \equiv 0$. From this and (15), (16) and (17), we obtain

$$V'(t) \leq \left[\frac{R'_-(t)}{R(t)} + \frac{\delta}{a(t)}[|e(t)| + |b(t)|]\right] V(t) + \frac{\delta|e(t)|}{r(t)} + v(t). \qquad (19)$$

The integration of (19) on $[0, t] \in I$ yields

$$\begin{aligned} V(t) - V(0) \leq & \int_0^t \left[\frac{R'_-(s)}{R(s)} + \frac{\delta}{a(s)}[|e(s)| + |b(s)|]\right] V(s) \, \mathrm{d}s \\ & + \int_0^\tau \left[\frac{\delta|e(t)|}{r(t)} + v(t)\right] \mathrm{d}t. \end{aligned}$$

Hence, Gronwall's inequality yields

$$\begin{aligned} 0 \leq V(t) \leq & \left[V(0) + \int_0^\tau \left[\frac{\delta|e(t)|}{r(t)} + v(t)\right] \mathrm{d}t\right] \\ & \times \exp \int_0^\tau \left[\frac{R'_-(t)}{R(t)} + \frac{\delta}{a(t)}[|e(t)| + |b(t)|]\right] \mathrm{d}t. \qquad (20) \end{aligned}$$

Now $V(t)$ is bounded from above on I since I is a bounded interval, so (7) yields that $|y^{[1]}(t)|^\delta$ and $|y'(t)|$ are bounded above on I. But this inequality contradicts (8). □

Remark 2. It is clear from the proof of Theorem 2 (ii) that if $b \equiv 0$, then assumption (4) is not needed in case (ii).

Remark 3. Note that the condition $|g(x)| \leq |x|^p$ in (i) can not be improved upon even for Equation (2).

Example 1. Let $\varepsilon \in (0, 1)$. Then the function $y = \left(\frac{1}{1-t}\right)^{\frac{1-\varepsilon}{\varepsilon}}$ is a singular solution of the second kind of the equation

$$y'' - |y'|^\varepsilon y' + C|y|^{\frac{1+\varepsilon}{1-\varepsilon}} \operatorname{sgn} y = 0$$

on $[0, 1)$ with $C = \left(\frac{1-\varepsilon}{\varepsilon^2}\right)^{\varepsilon+1} - \frac{1-\varepsilon}{\varepsilon^2}$.

Remark 4.

(i) The result of Theorem 2 is obtained in [2] in case $b \equiv 0$ using a the similar method.

(ii) Note that Theorem 2 is not valid if $R \notin C^1(R_+)$; see [1] or [4] for the case $g \equiv 0$.

Remark 5. Theorem 2 is not valid if $r < 0$ on an interval of positive measure, see e.g. Theorem 11.3 in [9] (for (6) and $p = 1$). The existence of singular solutions of the second kind for (1) is an open problem.

3 Singular solutions of the first kind

In this section, the nonexistence of singular solutions of the first kind mainly for (2) will be studied. The following lemma shows that $e(t)$ has to be trivial in a neighbourhood of ∞ if Equation (1) has a singular solution of the first kind.

Lemma 1. *Let y be a singular solution of the first kind of* (1)*. Then $e(t) \equiv 0$ in a neighbourhood ∞.*

Proof. Let y be a singular solution of (1) and τ the number from its domain of definition. Then $y \equiv 0$ on $[\tau, \infty)$ and Equation (1) yields $e(t) \equiv 0$ on $[\tau, \infty)$. $\square$

In what follows, we will only consider Equation (2).

Theorem 3. *Let $M > 0$ and*

$$|g(x)| \leq |x|^p \quad \textit{and} \quad |f(x)| \leq |x|^p \quad \textit{for } |x| \leq M. \tag{21}$$

Then there exist no singular solution of the first kind of Equation (2)*.*

Proof. Assume that y is a singular solution of the first kind and τ is the number from Definition 1. Using system (5), we have $y_1 \equiv y_2 \equiv 0$ on $[\tau, \infty)$. Let $0 \leq T < \tau$ be such that

$$|y_1(t)| \leq M, \quad |y_2(t)| \leq M \quad \text{on } [T, \tau], \tag{22}$$

and

$$\int_T^\tau a(s)|b(s)| \, \mathrm{d}s + \left(\int_T^\tau a^{-\frac{1}{p}}(s) \, \mathrm{d}s \right)^p \int_T^\tau |r(s)| \, \mathrm{d}s \leq \frac{1}{2}. \tag{23}$$

Define $I = [T, \tau]$ and

$$v_1(t) = \max_{t \leq s \leq \tau} |y_1(s)|, \; t \in I, \tag{24}$$

$$v_2(t) = \max_{t \leq s \leq \tau} |y_2(s)|, \; t \in I. \tag{25}$$

From the definition of τ, (22), (24) and (25), we have

$$0 < v_1(t) \leq M, 0 < v_2(t) \leq M \text{ on } [T, \tau). \tag{26}$$

An integration of the first equality in (5) and (25) yield

$$\begin{aligned} |y_1(t)| &\leq \int_t^\tau a^{-\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}} \, \mathrm{d}s \leq \int_t^\tau a^{-\frac{1}{p}}(s)|v_2(s)|^{\frac{1}{p}} \, \mathrm{d}s \\ &\leq |v_2(t)|^{\frac{1}{p}} \int_t^\tau a^{-\frac{1}{p}}(s) \, \mathrm{d}s \end{aligned} \tag{27}$$

on I. If $M_1 = \int_T^\tau a^{-\frac{1}{p}}(s)\ \mathrm{d}s$, then

$$|y_1(t)| \le M_1 |v_2(t)|^{\frac{1}{p}} \tag{28}$$

and from (24) we obtain

$$v_1(t) \le M_1 |v_2(t)|^{\frac{1}{p}},\ t \in I. \tag{29}$$

Similarly, an integration of the second equality in (5) and (21) yield

$$\begin{aligned} |y_2(t)| &\le \int_t^\tau \left| b(s) g\big(a^{\frac{1}{p}}(s)|y_2(s)|^{\frac{1}{p}} \text{ sgn } y_2(s)\big)\right| \ \mathrm{d}s \\ &\quad + \int_t^\tau |r(s) f(y_1(s))|\ \mathrm{d}s \\ &\le \int_t^\tau |b(s)| (a^{\frac{1}{p}}(s)|v_2(s)|^{\frac{1}{p}})^p\ \mathrm{d}s + \int_t^\tau |r(s)| y_1(s)|^p\ \mathrm{d}s. \end{aligned} \tag{30}$$

Hence, from this, (21), (23) and (28)

$$\begin{aligned} |y_2(t)| &\le v_2(t)\left[\int_T^\tau a(s)|b(s)|\ \mathrm{d}s + v_1^p(t)\int_T^\tau |r(s)|\ \mathrm{d}s\right] \\ &\le v_2(t)\left[\int_T^\tau a(s)|b(s)|\ \mathrm{d}s + M_1^p \int_T^\tau |r(s)|\ \mathrm{d}s\right] \le \frac{v_2(t)}{2}. \end{aligned} \tag{31}$$

Hence $v_2(t) \le \frac{v_2(t)}{2}$ and so $v_2(t) \equiv 0$ on I. The contradiction with (26) proves the conclusion. □

Theorem 4. *Consider* (3), $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and let either*

(i) $M \in (0, \infty)$ *exist such that* $|g(x)| \le |x|^p$ *for* $|x| \le M$
or

(ii) (4) *and* $b(t) \le 0$ *on* R_+.

Then Equation (2) *has no singular solution of the first kind.*

Proof. Let $y(t)$ be singular solution of the first kind of (2). Then there exists $\tau \in (0, \infty)$ such that $y(t) \equiv 0$ on $[\tau, \infty)$ and $\sup_{T \le s < \tau} |y(s)| > 0$ for $T \in [0, \tau)$. Then, similar to the proof of Theorem 2, (15) and the equality in (16) hold with $e \equiv 0$. From this we have

$$\begin{aligned} V'(t) &= \left(\frac{1}{R(t)}\right)' |y^{[1]}(t)|^\delta - \frac{\delta b(t) g(y'(t)) y'(t)}{r(t)} \\ &\ge -\frac{R'_+(t)}{R^2(t)} a^\delta(t) |y'(t)|^{p+1} - \frac{\delta a^{\frac{1}{p}}(t) b(t) g(y'(t)) y'(t)}{R(t)}. \end{aligned} \tag{32}$$

Let (i) be valid. Let $T \in [0,\tau)$ be such that $|y'(t)| \le M$ on $[T,\tau]$, and let $\varepsilon > 0$ be arbitrary. Then,

$$
\begin{aligned}
\frac{V'(t)}{V(t)+\varepsilon} &\ge -\frac{|y'(t)|^{p+1}}{R(t)[V(t)+\varepsilon]}\left(a^{\delta}(t)\frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t)|b(t)|\right) \\
&\ge -\frac{V(t)}{V(t)+\varepsilon}\left(a^{\delta}(t)\frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t)|b(t)|\right) \\
&\ge -\left(a^{\delta}(t)\frac{R'_+(t)}{R(t)} + \delta a^{\frac{1}{p}}(t)|b(t)|\right). \qquad (33)
\end{aligned}
$$

An integration on the interval $[t,\tau] \subset [T,\tau]$ yields

$$
\frac{\varepsilon}{V(t)+\varepsilon} = \frac{V(\tau)+\varepsilon}{V(t)+\varepsilon} \ge \exp\left\{-\int_t^{\tau}\left[a^{\delta}(s)\frac{R'_+(s)}{R(s)} + \delta a^{\frac{1}{p}}(s)|b(s)|\right]\,\mathrm{d}s\right\}.
$$

As $\varepsilon > 0$ is arbitrary, we have

$$
0 \ge V(t)\exp\left\{-\int_t^{\tau}\left[a^{\delta}(s)\frac{R'_+(s)}{R(s)} + \delta a^{\frac{1}{p}}(s)b(s)\right]\,\mathrm{d}s\right\},\ t \in [T,\tau].
$$

Hence, $V(t) \equiv 0$ on $[T,\tau]$ and (7) yields $y(t) = 0$ on $[T,\tau]$. The contradiction to $\sup_{t\in[T,\tau]}|y(t)| > 0$ proves that the conclusion holds in this case.
Let (ii) hold; then from (7) and (32) we have

$$
\begin{aligned}
\frac{V'(t)}{V(t)+\varepsilon} &\ge \left\{-a^{\delta}(t)\frac{R'_+(t)}{R^2(t)}|y'(t)|^{p+1} - \frac{\delta a^{\frac{1}{p}}(t)b(t)g(y'(t))y'(t)}{R(t)}\right\} \\
&\quad\times (V(t)+\varepsilon)^{-1} \\
&\ge -\frac{V(t)}{V(t)+\varepsilon}a^{\delta}(t)\frac{R'_+(t)}{R(t)} \ge -a^{\delta}(t)\frac{R'_+(t)}{R(t)} \qquad (34)
\end{aligned}
$$

for $t \in [0,\tau]$. Hence, we have a similar situation to that in (33) and the proof is similar to case (i). □

Remark 6. Theorem 3 generalized results of Theorem 1.2 in [10], obtained in case $b \equiv 0$. Results of Theorem 9.4 in [7] with ($b \equiv 0, f(x) = |x|^p \operatorname{sgn} x$) and of Theorem 1 in [1] ($b \equiv 0$) are special cases of Theorem 1 here.

Remark 7. Theorem 4 is not valid if $r < 0$ on an interval of positive measure; see e.g.Theorem 11.1 in [9] (for (6) and $p = 1$). The existence of singular solutions of the first kind of (2) is an open problem.

Remark 8. If $R \notin C^1(R_+)$, then the statement of Theorem 4 does not hold (see [1] for $g \equiv 0$ or [5]).

Note that condition (i) in Theorem 4 can not be improved.

Example 2. Let $\varepsilon \in (0,1)$. Then function $y = (1-t)^{(1+\frac{1}{\varepsilon})}$ for $t \in [0,1]$ and $y \equiv 0$ on $(1,\infty)$ is a singular solution of the first kind of the equation

$$y'' + \left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right)\left(1 + \frac{1}{\varepsilon}\right)^{\varepsilon - 1} |y'|^{1-\varepsilon} \operatorname{sgn} y' + |y|^{\frac{1-\varepsilon}{1+\varepsilon}} = 0.$$

Note that $p = 1$ in this case.

Theorems 1, 2, 3 and 4 gives us sufficient conditions for all nontrivial solutions of (2) to be proper.

Corollary 1. *Let $|g(x)| \le |x|^p$ and $|f(x)| \le |x|^p$ for $x \in R$. Then every nontrivial solution y of* (2) *is proper.*

Corollary 2. *Let* (3)*, $R \in C^1(R_+)$, $r > 0$ on R_+ and $|g(x)| \le |x|^p$ on R hold. Then every nontrivial solution y of* (2) *is proper.*

Remark 9. The results of Corollary 1 and Corollary 2 are obtained in [1] for $b \equiv 0$.

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4 Further properties of solutions of (2)

In this section, simple asymptotic properties of solutions of (2) are studied. Mainly, sufficient conditions are given under which zeros of a nontrivial solutions are simple and zeros of a solution and its derivative separate from each other.

Corollary 3. *Let the assumptions either of Theorem 3 or of Theorem 4 hold. Then any nontrivial solution of* (2) *has no double zeros on R_+.*

Proof. Let y be a nontrivial solution of (2) defined on R_+ with a double zero at $\tau \in R_+$, i.e., $y(\tau) = y'(\tau) = 0$. Then it is clear that the function

$$\bar{y}(t) = y(t) \text{ on } [0,\tau], \ \ \bar{y}(t) = 0 \text{ for } t > \tau$$

is also solution of (2). As $\bar{y}$ is a singular solution of the first kind, we obtain contradiction with either Theorem 3 or with Theorem 4. ☐

Lemma 2. *Let $g(0) = 0$, $r \ne 0$ on R_+, and $f(x)x > 0$ for $x \ne 0$. Let y be a nontrivial solution of* (2) *such that $y'(t_1) = y'(t_2) = 0$ with $0 \le t_1 < t_2 < \infty$. Then there exists $t_3 \in [t_1, t_2]$ such that $y(t_3) = 0$.*

Proof. We may suppose without loss of generality that t_1 and t_2 are consecutive zeros of y'; if t_1 or t_2 is an accumulation point of zeros of y', the result holds. If we define $z(t) = y^{[1]}(t), t \in R_+$, then

$$z(t_1) = z(t_2) = 0 \text{ and } z(t) \neq 0 \text{ on } (t_1, t_2). \tag{35}$$

Suppose, contrarily, that $y(t) \neq 0$ on (t_1, t_2). Then either

$$y(t_1)y(t_2) > 0 \quad \text{on} \quad [t_1, t_2] \tag{36}$$

or

$$y(t_1)y(t_2) = 0 \tag{37}$$

holds. If (36) is valid, then (2) and the assumptions of the lemma yields

$$\text{sgn } z'(t_1) = \text{ sgn } z'(t_2) \neq 0$$

and the contradiction with (35) proves the statement in this case.

If (37) holds the conclusion is valid. □

Corollary 4. *Let $f(x)x > 0$ for $x \neq 0$ and one of the following possibilities hold:*

(i) $r \neq 0$ *on* R_+ *and*

$$|g(x)| \leq |x|^p \quad \textit{and} \quad |f(x)| \leq |x|^p \quad \textit{for} \quad x \in R;$$

(ii) $R \in C^1(R_+)$, $r > 0$ *on* R_+ *and*

$$|g(x)| \leq |x|^p \quad \textit{for} \quad |x| \in R;$$

(iii) $R \in C^1(R_+)$, $b \leq 0$ *on* R_+, $r > 0$ *on* R_+, $g(x)x \geq 0$ *on* R_+ *and* $M > 0$ *exists such that*

$$|g(x)| \geq |x|^p \quad \textit{for} \quad |x| \geq M;$$

(iv) $R \in C^1(R_+)$, $r > 0$ *on* R_+, $b \geq 0$ *on* R_+, $g(x)x \geq 0$ *on* R *and* M *exists such that*

$$|g(x)| \leq |x|^p \quad for \quad |x| \leq M.$$

Then the zeros of y and y' (if any) separate from each other, i.e. between two consecutive zeros of $y(y')$ there is the only zero of $y(y')$.

Proof. Accounting to our assumptions, Corollary 3 holds and hence all zeros of any nontrivial solution y of (2) are simple, there exists no accumulation point of zeros of y on R_+, and there exists no interval $[\alpha, \beta] \in R_+, \alpha < \beta$ of zeros of y. Then, the statement follows from Lemma 2 and Rolle's Theorem. □

Theorem 5. *Let $g(0) = 0$, $r \neq 0$ on R_+ and $f(x)x > 0$ for $x \neq 0$. Then* (2) *has no weakly oscillatory solution and every nonoscillatory solution y of* (2) *has a limit as $t \to \infty$.*

Proof. Let y be a weakly oscillatory solution of (2). Then there exist t_0, t_1 and t_2 such that $0 \leq t_0 < t_1 < t_2$, $y(t) \neq 0$ on $[t_0, \infty)$ and $y'(t_1) = y'(t_2) = 0$. But this fact contradicts Lemma 2. □

The following examples show that some of the assumptions of Theorem 5 cannot be omitted.

Example 3. The function $y = 2 + \sin t$, $t \in R_+$ is a weakly oscillatory solution of the equation

$$y'' - y' + \frac{\sin t + \cos t}{2 + \sin t} y = 0.$$

In this case $r \neq 0$ is not valid.

Example 4. The function $y = 2 + \sin t$, $t \in R_+$ is a weakly oscillatory solution of the equation

$$y'' - g(y') + 2y = 0 \quad \textit{with} \quad g(x) = \begin{cases} 4 + \sqrt{1 - x^2} & \text{for } |x| \leq 1; \\ 4 & \text{for } |x| > 1. \end{cases}$$

In this case $g(0) \neq 0$.

Remark 11. If $g \equiv 0$, the result of Theorem 5 is known, see e.g. Lemma 5.1 in [10] or a direct application of (5).

References

[1] Bartušek, M., *Singular Solution for the Differential Equation with p-Laplacian,* Arch. Math. (Brno) **41** (2005) 123-128.

[2] Bartušek, M., Graef, J. R., *Asymptotic Properties of Solutions of Forced Second Order Differential Equation with p-Laplacian.* Pan Amer. Math. J. **16** (2006) 41-59.

[3] Bartušek, M., Graef, J. R., *On the Limit-Point/Limit-Circle Problem for Second Order Nonlinear Equations,* Nonlin. Studies **9** (2002) 361-369.

[4] Coffman, C. V., Ulrich, D. F., *On the Continuation of Solutions of a certain Non-linear Differential Equation,* Monath. Math. B. **71** (1967) 385-392.

[5] Coffman, C. V., Wong, J. S. W., *Oscillation and Nonoscillation Theorems for Second Order Differential Equations,* Funkcial. Ekvac. **15** (1972) 119-130.

[6] Došlý, O., Řehák, P., *Half-Linear Differential Equations,* Math. Studies 202, Elsevier, Amsterodam-Boston-New York etc., 2005.

[7] Hartman, P., *Ordinary Differential Equations.*, John-Wiley & Sons, New York-London-Sydney, 1964.

[8] Heidel, J. W., *Uniqueness, Continuation and Nonoscillation for a Second Order Differential Equation,* Pacif. J. Math. **32** (1970) 715-721.

[9] Kiguradze, I. T., Chanturia, T., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations,* Kluwer, Dordrecht, 1993.

[10] Mirzov, D., *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations,* Maikop (1993)(in Russian); Folia Fac. Sci. Natur. Univ. Masaryk. Brunen. Math. (Brno) 14, 2004.

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