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EXISTENCE OF PSEUDO ALMOST PERIODIC SOLUTIONS TO SOME CLASSES OF PARTIAL HYPERBOLIC EVOLUTION EQUATIONS

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Abstract. The paper examines the existence of pseudo almost periodic solutions to some classes of partial hyperbolic evolution equations. Namely, some sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions to those classes of hyperbolic evolution equations are given. As an application, we consider the existence of pseudo almost periodic solutions to the heat equations with delay.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a sectorial linear operator (see Definition 2.1). For $\alpha \in (0, 1)$, the space $\mathbb{X}_\alpha$ denotes an abstract *intermediate* Banach space between $D(A)$ and $\mathbb{X}$. Examples of those $\mathbb{X}_\alpha$ include, among others, the fractional spaces $D((-A)^\alpha)$ for $\alpha \in (0, 1)$, the real interpolation spaces $D_A(\alpha, \infty)$ due to J. L. Lions and J. Peetre, and the Hölder spaces $D_A(\alpha)$, which coincide with the continuous interpolation spaces that both G. Da Prato and P. Grisvard introduced in the literature.

In [7, 11, 12, 22], some sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions to the abstract (semilinear) differential equations,

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \quad \text{and} \tag{1.1}$$

$$u'(t) + Au(t) = f(t, Bu(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $-A$ is a Hille-Yosida linear operator (respectively, the infinitesimal generator of an analytic semigroup, and the infinitesimal generator of a C_0-semigroup), B is a densely defined closed linear operator on $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is a jointly continuous function, were given. Similarly, in [13], some reasonable sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions to the class of partial evolution equations

$$\frac{d}{dt}\left[u(t) + f(t, Bu(t))\right] = Au(t) + g(t, Cu(t)), \quad t \in \mathbb{R} \tag{1.3}$$

where A is the infinitesimal generator of an exponentially stable semigroup acting on $\mathbb{X}$, B, C are arbitrary densely defined closed linear operators on $\mathbb{X}$, and f, g are some jointly continuous functions, were given.

The assumptions made in [13] require much more regularity for the operator A, that is, being the infinitesimal generator of an analytic semigroup. In this paper

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we address such an issue by studying pseudo almost periodic solutions to (1.3) in the case when A is a sectorial operator whose corresponding analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic, equivalently,

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

where $\sigma(A)$ denotes the spectrum of A.

Note that (1.3) in the case when A is sectorial corresponds to several interesting situations encountered in the literature. Applications include, among others, the existence and uniqueness of pseudo almost periodic solutions to the hyperbolic heat equation with delay.

As in [5, 14] in this paper we consider a general intermediate space $\mathbb{X}_\alpha$ between $D(A)$ and $\mathbb{X}$. In contrast with the fractional power spaces considered in some recent papers of the author et al. [11, 12], the interpolation and Hölder spaces, for instance, depend only on $D(A)$ and $\mathbb{X}$ and can be explicitly expressed in many concrete cases. The literature related to those intermediate spaces is very extensive, in particular, we refer the reader to the excellent book by A. Lunardi [23], which contains a comprehensive presentation on this topic and related issues.

The concept of pseudo almost periodicity, which is the central question in this paper was introduced in the literature in the early nineties by C. Zhang [29, 30, 31] as a natural generalization of the well-known Bohr almost periodicity. Thus this new concept is welcome to implement another existing generalization of almost periodicity, that is, the concept of asymptotically almost periodicity due to Fréchet [6, 16].

The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is one of the most attractive topics in qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory, physics and others.

Some contributions on almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions to abstract differential and partial differential equations have recently been made in [1, 2, 3, 7, 9, 11, 12, 13, 22]. However, the existence and uniqueness of pseudo almost periodic solutions to (1.3) in the case when A is sectorial is an important topic with some interesting applications, which is still an untreated question, is the main motivation of the present paper. Among other things, we will make extensive use of the method of analytic semigroups associated with sectorial operators and the Banach's fixed-point principle to derive sufficient conditions for the existence and uniqueness of a pseudo almost periodic (mild) solution to (1.3).

2. Preliminaries

This section is devoted to some preliminary facts needed in the sequel. Throughout the rest of this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space, A is a sectorial linear operator (see Definition 2.1), which is not necessarily densely defined, and B, C are (possibly unbounded) linear operators such that $A + B + C$ is not trivial, as each solution to (1.3) belongs to $D(A + B + C) = D(A) \cap D(B) \cap D(C)$. Now if A is a linear operator on $\mathbb{X}$, then $\rho(A)$, $\sigma(A)$, $D(A)$, $N(A)$, $R(A)$ stand for the resolvent, spectrum, domain, kernel, and range of A. The space $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural norm with $B(\mathbb{X},\mathbb{X}) = B(\mathbb{X})$.

2.1. **Sectorial Linear Operators and their Associated Semigroups.**

Definition 2.1. A linear operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

$$\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, \quad |\arg(\lambda - \omega)| < \theta\}, \quad \text{and} \tag{2.1}$$

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}. \tag{2.2}$$

The class of sectorial operators is very rich and contains most of classical operators encountered in the literature. Two examples of sectorial operators are given as follows:

Example 2.2. Let $p \geq 1$ and let $\mathbb{X} = L^p(\mathbb{R})$ be the Lebesgue space equipped with its norm $\|\cdot\|_p$ defined by

$$\|\varphi\|_p = \left(\int_{\mathbb{R}} |\varphi(x)|^p dx\right)^{1/p}.$$

Define the linear operator A on $L^p(\mathbb{R})$ by

$$D(A) = W^{2,p}(\mathbb{R}), \quad A(\varphi) = \varphi'', \quad \forall \varphi \in D(A).$$

It can be checked that the operator A is sectorial on $L^p(\mathbb{R})$.

Example 2.3. Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be open bounded subset with C^2 boundary $\partial\Omega$. Let $\mathbb{X} := L^p(\Omega)$ be the Lebesgue space equipped with the norm, $\|\cdot\|_p$ defined by,

$$\|\varphi\|_p = \left(\int_{\Omega} |\varphi(x)|^p dx\right)^{1/p}.$$

Define the operator A as follows:

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad A(\varphi) = \Delta\varphi, \quad \forall \varphi \in D(A),$$

where $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

It can be checked that the operator A is sectorial on $L^p(\Omega)$.

It is well-known that [23] if A is sectorial, then it generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathbb{X})$ and such that there exist $M_0, M_1 > 0$ with

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{2.3}$$

$$\|t(A - \omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{2.4}$$

Throughout the rest of the paper, we suppose that the semigroup $(T(t))_{t\geq 0}$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that $T(t)$

commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold

$$\|T(t)Px\| \le Me^{-\delta t}\|x\| \qquad \text{for } t \ge 0, \tag{2.5}$$

$$\|T(t)Qx\| \le Me^{\delta t}\|x\| \qquad \text{for } t \le 0, \tag{2.6}$$

where $Q := I - P$ and, for $t \le 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t\ge 0}$ associated with A is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

see, e.g., [15, Prop. 1.15, pp.305].

Definition 2.4. Let $\alpha \in (0, 1)$. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathbb{X}$, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \le c\|x\|^{1-\alpha}\|x\|_A^\alpha, \qquad x \in D(A), \tag{2.7}$$

where $\|\cdot\|_A$ is the graph norm of A.

Concrete examples of $\mathbb{X}_\alpha$ include $D((-A^\alpha))$ for $\alpha \in (0, 1)$, the domains of the fractional powers of A, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0, 1)$, defined as follows

$$\begin{cases} D_A(\alpha,\infty) := \{x \in \mathbb{X} : [x]_\alpha = \sup\limits_{0<t\le 1} \|t^{1-\alpha}AT(t)x\| < \infty\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{cases}$$

the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$ as well as the complex interpolation spaces $[\mathbb{X}, D(A)]_\alpha$, see A. Lunardi [23] for details.

For a hyperbolic analytic semigroup $(T(t))_{t\ge 0}$, one can easily check that similar estimations as both (2.5) and (2.6) still hold with norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from (2.6) that

$$\|AT(t)Qx\| \le C'e^{\delta t}\|x\| \qquad \text{for } t \le 0.$$

Hence, from (2.7) there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \le c(\alpha)e^{\delta t}\|x\| \qquad \text{for } t \le 0. \tag{2.8}$$

In addition to the above, the following holds

$$\|T(t)Px\|_\alpha \le \|T(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\| \qquad \text{for } t \ge 1,$$

and hence from (2.5), one obtains

$$\|T(t)Px\|_\alpha \le M'e^{-\delta t}\|x\|, \qquad t \ge 1,$$

where M' depends on α. For $t \in (0, 1]$, by (2.4) and (2.7)

$$\|T(t)Px\|_\alpha \le M''t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \le M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \qquad \text{for } t > 0. \tag{2.9}$$

2.2. **Pseudo Almost Periodic Functions.** Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be another Banach space. Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$). The space $BC(\mathbb{R},\mathbb{X})$ equipped with its natural norm, that is, the sup norm defined by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$$

is a Banach space. Furthermore, $C(\mathbb{R},\mathbb{Y})$ (respectively, $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$).

Definition 2.5. A function $f\in C(\mathbb{R},\mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau)-f(t)\|<\varepsilon \ \text{ for each } \ t\in\mathbb{R}.$$

The number τ above is called an *ε-translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 2.6. A function $F\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called (Bohr) almost periodic in $t\in\mathbb{R}$ uniformly in $y\in\mathbb{Y}$ if for each $\varepsilon>0$ and any compact $K\subset\mathbb{Y}$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t+\tau,y)-F(t,y)\|<\varepsilon \ \text{ for each } \ t\in\mathbb{R}, \quad y\in K.$$

The collection of those functions is denoted by $AP(\mathbb{R}\times\mathbb{Y})$.

Set

$$AP_0(\mathbb{X}) := \{f\in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|f(s)\|ds=0\},$$

and define $AP_0(\mathbb{R}\times\mathbb{X})$ as the collection of functions $F\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ such that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|F(t,u)\|dt=0$$

uniformly in $u\in\mathbb{Y}$.

Definition 2.7. A function $f\in BC(\mathbb{R},\mathbb{X})$ is called pseudo almost periodic if it can be expressed as $f=g+\phi$, where $g\in AP(\mathbb{X})$ and $\phi\in AP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAP(\mathbb{X})$.

Remark 2.8. The functions g and ϕ in Definition 2.7 are respectively called the *almost periodic* and the *ergodic perturbation* components of f. Moreover, the decomposition given in Definition 2.7 is unique.

Similarly,

Definition 2.9. A function $F\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to pseudo almost periodic in $t\in\mathbb{R}$ uniformly in $y\in\mathbb{Y}$ if it can be expressed as $F=G+\Phi$, where $G\in AP(\mathbb{R}\times\mathbb{Y})$ and $\phi\in AP_0(\mathbb{R}\times\mathbb{Y})$. The collection of such functions will be denoted by $PAP(\mathbb{R}\times\mathbb{Y})$.

3. Main results

To study the existence and uniqueness of pseudo almost periodic solutions to (1.3) we need to introduce the notion of mild solution to it.

Definition 3.1. Let $\alpha \in (0,1)$. A bounded continuous function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to (1.3) provided that the function $s \to AT(t-s)Pf(s, Bu(s))$ is integrable on $(-\infty, t)$, $s \to AT(t-s)Qf(s, Bu(s))$ is integrable on (t, ∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} u(t) &= -f(t, Bu(t)) - \int_{-\infty}^{t} AT(t-s)Pf(s, Bu(s))ds \\ &+ \int_{t}^{\infty} AT(t-s)Qf(s, Bu(s))ds + \int_{-\infty}^{t} T(t-s)Pg(s, Cu(s))ds \\ &- \int_{t}^{\infty} T(t-s)Qg(s, Cu(s))ds \end{aligned}$$

for each $\forall t \in \mathbb{R}$.

Throughout the rest of the paper we denote by $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, the nonlinear integral operators defined by

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} AT(t-s)Pf(s, Bu(s))ds, \quad (\Gamma_2 u)(t) := \int_{t}^{\infty} AT(t-s)Qf(s, Bu(s))ds,$$

$$(\Gamma_3 u)(t) := \int_{-\infty}^{t} T(t-s)Pg(s, Cu(s))ds, \quad \text{and}$$

$$(\Gamma_4 u)(t) := \int_{t}^{\infty} T(t-s)Qg(s, Cu(s))ds.$$

To study (1.3) we require the following assumptions:

- (**H1**) The operator A is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\geq 0}$.
- (**H2**) Let $0 < \alpha < 1$. Then $\mathbb{X}_\alpha = D((-A^\alpha))$, or $\mathbb{X}_\alpha = D_A(\alpha, p), 1 \leq p \leq +\infty$, or $\mathbb{X}_\alpha = D_A(\alpha)$, or $\mathbb{X}_\alpha = [\mathbb{X}, D(A)]_\alpha$. We also assume that $B, C : \mathbb{X}_\alpha \longrightarrow \mathbb{X}$ are bounded linear operators.
- (**H3**) Let $0 < \alpha < \beta < 1$, and $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}_\beta$ be a pseudo almost periodic function in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$.
- (**H4**) The functions f, g are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K > 0$ such that

  $$\|f(t,u) - f(t,v)\|_\beta \leq K\|u - v\|,$$

  and

  $$\|g(t,u) - g(t,v)\| \leq K\|u - v\|$$

  for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

In order to show that Γ_1 and Γ_2 are well defined, we need the following estimates.

Lemma 3.2. *Let $0 < \alpha, \beta < 1$. Then*

$$\|AT(t)Qx\|_\alpha \leq ce^{\delta t}\|x\|_\beta \qquad \textit{for } t \leq 0, \tag{3.1}$$

$$\|AT(t)Px\|_\alpha \leq ct^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta, \qquad \textit{for } t > 0. \tag{3.2}$$

Proof. As for (2.8), the fact that the part of A in $R(Q)$ is bounded yields

$$\|AT(t)Qx\| \leq ce^{\delta t}\|x\|_\beta, \quad \|A^2T(t)Qx\| \leq ce^{\delta t}\|x\|_\beta, \qquad \text{for } t \leq 0,$$

since $X_\beta \hookrightarrow X$. Hence, from (2.7) there is a constant $c(\alpha) > 0$ such that

$$\|AT(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\|_\beta \qquad \text{for } t \leq 0.$$

Furthermore,

$$\begin{aligned}\|AT(t)Px\|_\alpha &\leq \|AT(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\| \\ &\leq ce^{-\delta t}\|x\|_\beta, \qquad \text{for } t \geq 1.\end{aligned}$$

Now for $t \in (0, 1]$, by (2.4) and (2.7), one has

$$\|AT(t)Px\|_\alpha \leq ct^{-\alpha-1}\|x\|,$$

and

$$\|AT(t)Px\|_\alpha \leq ct^{-\alpha}\|Ax\|,$$

for each $x \in D(A)$. Thus, by reiteration Theorem (see [23]), it follows that

$$\|AT(t)Px\|_\alpha \leq ct^{\beta-\alpha-1}\|x\|_\beta$$

for every $x \in \mathbb{X}_\beta$ and $0 < \beta < 1$, and hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \qquad \text{for } t > 0.$$

□

Lemma 3.3. *Under assumptions* (**H1**)-(**H2**)-(**H3**)-(**H4**)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $PAP(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u \in PAP(\mathbb{X}_\alpha)$. Since $C \in B(\mathbb{X}_\alpha, \mathbb{X})$ it follows that $Cu \in PAP(\mathbb{X})$. Setting $h(t) = g(t, Cu(t))$ and using the theorem of composition of pseudo almost periodic functions [3, Theorem 5] it follows that $h \in PAP(\mathbb{X})$. Now, write $h = \phi + \zeta$ where $\phi \in AP(\mathbb{X})$ and $\zeta \in AP_0(\mathbb{X})$. Thus $\Gamma_3 u$ can be rewritten as

$$(\Gamma_3 u)(t) \;=\; \int_{-\infty}^t T(t-s)P\phi(s)ds + \int_{-\infty}^t T(t-s)P\zeta(s)ds.$$

Set

$$\Phi(t) = \int_{-\infty}^t T(t-s)P\phi(s)ds,$$

and

$$\Psi(t) = \int_{-\infty}^t T(t-s)P\zeta(s)ds$$

for each $t \in \mathbb{R}$. The next step consists of showing that $\Phi \in AP(\mathbb{X}_\alpha)$ and $\Psi \in AP_0(\mathbb{X}_\alpha)$.

Clearly, $\Phi \in AP(\mathbb{X}_\alpha)$. Indeed, since $\phi \in AP(\mathbb{X})$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $\tau \in [\xi, \xi + l(\varepsilon)]$ with

$$\|\Phi(t+\tau) - \Phi(t)\| < \mu \,.\, \varepsilon \;\text{ for each } t \in \mathbb{R},$$

where $\mu = \dfrac{\gamma^{1-\alpha}}{M(\alpha)\Gamma(1-\alpha)}$ with Γ being the classical gamma function.

Now using the expression

$$\Phi(t+\tau) - \Phi(t) = \int_{-\infty}^{t} T(t-s)P\left(\phi(s+\tau) - \phi(s)\right) ds$$

and (2.9) it easily follows that

$$\|\Phi(t+\tau) - \Phi(t)\|_\alpha < \varepsilon \;\text{ for each } t \in \mathbb{R},$$

and hence, $\Phi \in AP(\mathbb{X}_\alpha)$. To complete the proof for Γ_3, we have to show that $t \mapsto \Psi(t)$ is in $AP_0(\mathbb{X}_\alpha)$. First, note that $s \mapsto \Psi(s)$ is a bounded continuous function. It remains to show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Psi(t)\|_\alpha \, dt = 0.$$

Again using (2.9) one obtains that

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Psi(t)\|_\alpha \, dt &\leq \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} \int_{0}^{+\infty} s^{-\alpha} e^{-\gamma s} \, \|\zeta(t-s)\| ds \, dt \\
&\leq \lim_{r\to\infty} M(\alpha) \int_{0}^{+\infty} s^{-\alpha} e^{-\gamma s} \frac{1}{2r} \int_{-r}^{r} \|\zeta(t-s)\| dt \, ds = 0,
\end{aligned}$$

by using Lebesgue dominated Convergence theorem, and the fact that $AP_0(\mathbb{X})$ is invariant under translations. Thus Ψ belongs to $AP_0(\mathbb{X}_\alpha)$.

The proof for $\Gamma_4 u(\cdot)$ is similar to that of $\Gamma_3 u(\cdot)$. However one makes use of (2.8) rather than (2.9). $\square$

Lemma 3.4. *Under assumptions* (**H1**)-(**H2**)-(**H3**)-(**H4**)*, the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAP(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u \in PAP(\mathbb{X}_\alpha)$. Since $B \in B(\mathbb{X}_\alpha, \mathbb{X})$ it follows that the function $t \mapsto Bu(t)$ belongs to $PAP(\mathbb{X})$. Again, using the composition theorem of pseudo almost periodic functions [3, Theorem 5] it follows that $\psi(\cdot) = f(\cdot, Bu(\cdot))$ is in $PAP(\mathbb{X}_\beta)$ whenever $u \in PAP(\mathbb{X}_\alpha)$. In particular, $\|\psi\|_{\infty,\beta} = \sup_{t\in\mathbb{R}} \|f(t, Bu(t))\|_\beta < \infty$.

Now write $\psi = w + z$, where $w \in AP(\mathbb{X}_\beta)$ and $z \in AP_0(\mathbb{X}_\beta)$, that is, $\Gamma_1\phi = \Xi(w) + \Xi(z)$ where

$$\Xi w(t) := \int_{-\infty}^{t} AT(t-s)Pw(s)ds, \quad \text{and} \quad \Xi z(t) := \int_{-\infty}^{t} AT(t-s)Pz(s)ds.$$

Clearly, $\Xi(w) \in AP(\mathbb{X}_\alpha)$. Indeed, since $w \in AP(\mathbb{X}_\beta)$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $\tau \in [\xi, \xi + l(\varepsilon)]$ with the property:

$$\|w(t+\tau) - w(t)\|_\beta < \nu\varepsilon \;\text{ for each } t \in \mathbb{R},$$

where $\nu = \dfrac{1}{c\gamma^{\beta-\alpha}\Gamma(\beta-\alpha)}$ and c being the constant appearing in Lemma 3.2.

Now, the estimate (3.2) yields

$$\begin{aligned}\|\Xi(w)(t+\tau)-\Xi(w)(t)\|_\alpha &= \left\|\int_0^{+\infty} AT(s)P\left(w(t-s+\tau)-w(t-s)\right)\,ds\right\|_\alpha \\ &\le \int_0^{+\infty} s^{\beta-\alpha-1}e^{-\gamma s}\|w(t-s+\tau)-w(t-s)\|_\beta ds \\ &\le \varepsilon\end{aligned}$$

for each $t\in\mathbb{R}$, and hence $\Xi(w)\in AP(\mathbb{X}_\alpha)$.

Now, let $r>0$. Then, by (3.2), we have

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\|(\Xi z)(t)\|_{\mathbb{X}_\alpha}\,dt &\le \frac{1}{2r}\int_{-r}^{r}\int_0^{+\infty}\|AT(s)Pz(t-s)\|_\alpha\,ds\,dt \\ &\le \frac{1}{2r}\int_{-r}^{r}\int_0^{+\infty} s^{\beta-\alpha-1}e^{-\gamma s}\|z(t-s)\|_\beta\,ds\,dt \\ &\le \int_0^{+\infty} s^{\beta-\alpha-1}e^{-\gamma s}\frac{1}{2r}\int_{-r}^{r}\|z(t-s)\|_\beta\,dt\,ds.\end{aligned}$$

Obviously, $\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|(\Xi z)(t)\|_\alpha dt = 0$, since $t\mapsto z(t-s)\in AP_0(\mathbb{X}_\beta)$ for every $s\in\mathbb{R}$. Thus $\Xi z\in AP_0(\mathbb{X}_\alpha)$.

The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$. However, one uses (3.1) instead of (3.2). □

Throughout the rest of the paper, the constant $k(\alpha)$ denotes the bound of the embedding $\mathbb{X}_\beta \hookrightarrow \mathbb{X}_\alpha$, that is,

$$\|u\|_\alpha \le k(\alpha)\|u\|_\beta \quad \text{for each } u\in\mathbb{X}_\beta.$$

Theorem 3.5. *Under the assumptions* **(H1)-(H2)-(H3)-(H4)**, *the evolution equation* (1.3) *has a unique pseudo almost periodic mild solution whenever* $\Theta<1$, *where*

$$\Theta = K\varpi\left[k(\alpha)+\frac{c}{\delta}+c\frac{\Gamma(\beta-\alpha)}{\gamma^{\beta-\alpha}}+\frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}+\frac{c(\alpha)}{\delta}\right],$$

and $\varpi = \max(\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})},\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})})$.

Proof. Consider the nonlinear operator $\mathbb{M}$ on $PAP(\mathbb{X}_\alpha)$ given by

$$\begin{aligned}\mathbb{M}u(t) &= -f(t,Bu(t))-\int_{-\infty}^{t}AT(t-s)Pf(s,Bu(s))ds \\ &+ \int_t^{\infty}AT(t-s)Qf(s,Bu(s))ds+\int_{-\infty}^{t}T(t-s)Pg(s,Cu(s))ds \\ &- \int_t^{\infty}T(t-s)Qg(s,Cu(s))ds\end{aligned}$$

for each $t\in\mathbb{R}$.

As we have previously seen, for every $u \in PAP(\mathbb{X}_\alpha)$, $f(\cdot, Bu(\cdot)) \in PAP(\mathbb{X}_\beta) \subset PAP(\mathbb{X}_\alpha)$. In view of Lemma 3.3 and Lemma 3.4, it follows that $\mathbb{M}$ maps $PAP(\mathbb{X}_\alpha)$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed-point.

Let $v, w \in PAP(\mathbb{X}_\alpha)$

$$\begin{aligned}
\|\Gamma_1(v)(t) - \Gamma_1(w)(t)\|_\alpha &\leq \int_{-\infty}^{t} \|AT(t-s)P\left[f(s, Bv(s)) - f(s, Bw(s))\right]\|_\alpha \, ds \\
&\leq cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha} \int_{-\infty}^{t} (t-s)^{\beta-\alpha-1} e^{-\gamma(t-s)} ds \\
&= c\frac{\Gamma(\beta-\alpha)}{\gamma^{\beta-\alpha}} K\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\Gamma_2(v)(t) - \Gamma_2(w)(t)\|_\alpha &\leq \int_{t}^{\infty} \|AT(t-s)Q\left[f(s, Bv(s)) - f(s, Bw(s))\right]\|_\alpha ds \\
&\leq cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha} \int_{t}^{+\infty} e^{\delta(t-s)} ds \\
&= \frac{cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}}{\delta}\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Now for Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned}
\|\Gamma_3(v)(t) - \Gamma_3(w)(t)\|_\alpha &\leq \int_{-\infty}^{t} \|T(t-s)P\left[g(s, Cv(s)) - g(s, Cw(s))\right]\|_\alpha ds \\
&\leq \frac{K\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\|v-w\|_{\infty,\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\|\Gamma_4(v)(t) - \Gamma_4(w)(t)\|_\alpha &\leq \int_{t}^{+\infty} \|T(t-s)Q\left[g(s, Cv(s)) - g(s, Cw(s))\right]\|_\alpha ds \\
&\leq Kc(\alpha)\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha} \int_{t}^{+\infty} e^{\delta(t-s)} ds \\
&= \frac{K\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}c(\alpha)}{\delta}\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Consequently,

$$\|\mathbb{M}v - \mathbb{M}w\|_{\infty,\alpha} \leq \Theta \,.\, \|v-w\|_{\infty,\alpha}.$$

Clearly, if $\Theta < 1$, then (1.3) has a unique fixed-point by the Banach fixed point theorem, which obviously is the only pseudo almost periodic solution to (1.3). $\square$

Example 3.6. For $\sigma \in \mathbb{R}$, consider the (semilinear) heat equation with delay endowed with Dirichlet conditions:

$$\frac{\partial}{\partial t}[\varphi(t,x) + f(t,\varphi(t-p,x))] = \frac{\partial^2}{\partial x^2}\varphi(t,x) + \sigma\varphi(t,x) + g(t,\varphi(t-p,x)) \tag{3.3}$$

$$\varphi(t,0) = \varphi(t,1) = 0 \tag{3.4}$$

for $t \in \mathbb{R}$ and $x \in [0,1]$, where $p > 0$, and $f, g : \mathbb{R} \times C[0,1] \mapsto C[0,1]$ are some jointly continuous functions.

Take $\mathbb{X} := C[0,1]$, equipped with the sup norm. Define the operator A by

$$A(\varphi) := \varphi'' + \sigma\varphi, \quad \forall \varphi \in D(A),$$

where $D(A) := \{\varphi \in C^2[0,1], \varphi(0) = \varphi(1) = 0\} \subset C[0,1]$.

Clearly A is sectorial, and hence is the generator of an analytic semigroup. In addition to the above, the resolvent and spectrum of A are respectively given by

$$\rho(A) = \mathbb{C} - \{-n^2\pi^2 + \sigma : n \in \mathbb{N}\} \text{ and } \sigma(A) = \{-n^2\pi^2 + \sigma : n \in \mathbb{N}\}$$

so that $\sigma(A) \cap i\mathbb{R} = \{\emptyset\}$ whenever $\sigma \neq n^2\pi^2$.

Theorem 3.7. *Under assumptions* **(H3)-(H4)**, *if $\sigma \neq n^2\pi^2$ for each $n \in \mathbb{N}$, then the heat equation with delay* (3.3)-(3.4) *has a unique $\mathbb{X}_\alpha$-valued pseudo almost periodic mild solution whenever K is small enough.*

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