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# Stability of Volterra difference delay equations

**ERNEST YANKSON**

**Abstract**

We study the asymptotic stability of the zero solution of the Volterra difference delay equation

$$x(n+1) = a(n)x(n) + c(n)\Delta x(n-g(n)) + \sum_{s=n-g(n)}^{n-1} k(n,s)h(x(s)).$$

A Krasnoselskii fixed point theorem is used in the analysis.

## 1 Introduction

Fixed point theorems have been used extensively in recent times to study some of the qualitative properties of solutions of difference and differential equations.
In the current paper we use Krasnoselskii's fixed point theorem to study the asymptotic stability of the zero solution of the difference equation,

$$x(n+1) = a(n)x(n) + c(n)\Delta x(n-g(n)) + \sum_{s=n-g(n)}^{n-1} k(n,s)h(x(s)) \tag{1.1}$$

where $a(n), c(n) : \mathbb{Z} \to \mathbb{R}$, $k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, $h : \mathbb{Z} \to \mathbb{R}$, and $g(n) : \mathbb{Z} \to \mathbb{Z}^+$. The operator $\Delta$ is defined as

$$\Delta x(n) = x(n+1) - x(n).$$

We assume that $a(n)$ and $c(n)$ are bounded discrete functions
whereas $0 \le g(n) \le g_0$ for some integer $g_0$. We also assume that $h(0) = 0$ and

$$|h(x) - h(z)| \le L|x - z| \tag{1.2}$$

for some positive constant $L$.

We refer to [1],[12] and [17] for some of the reasons why fixed point theorems are used to study stability. Equation (1.1) is the discrete version of one of the differential equations studied in [17].
Continuous versions of (1.1) are generally known as neutral Volterra differential equations.
For more on stability of ordinary and functional differential equations we refer to [1],[2],[7],[9],[11],[18], and for difference equations we refer to [4],[5],[6],[7],[10],[13], and [16].
For any integer $n_0 \geq 0$, we define $\mathbb{Z}_0$ to be the set of integers in $[-g_0, n_0]$.
Let $\psi(n) : \mathbb{Z}_0 \to \mathbb{R}$ be an initial discrete bounded function.

**Definition 1.1.** We say $x(n) := x(n, n_0, \psi)$ is a solution of (1.1) if $x(n) = \psi(n)$ on $\mathbb{Z}_0$ and satisfies (1.1) for $n \geq n_0$.

**Definition 1.2.** The zero solution of (1.1) is Liapunov stable if for any $\epsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(n)| \leq \delta$ on $\mathbb{Z}_0$ imply $|x(n, n_0, \psi)| \leq \epsilon$ for $n \geq n_0$.

**Definition 1.3.** The zero solution of (1.1) is asymptotically stable if it is Liapunov stable and if for any integer $n_0 \geq 0$ there exists $r(n_0) > 0$ such that $|\psi(n)| \leq r(n_0)$ on $\mathbb{Z}_0$ implies $|x(n, n_0, \psi)| \to 0$ as $n \to \infty$.

# 2 Asymptotic Stability

**Lemma 2.1.** Suppose that $a(n) \neq 0$ for all $n \in \mathbb{Z}$ . Then $x(n)$ is a solution of equation (1.1) if and only if

$$\begin{aligned} x(n) = &\Big[x(n_0) - c(n_0 - 1)x(n_0 - g(n_0))\Big] \prod_{s=n_0}^{n-1} a(s) + c(n-1)x(n - g(n)) \\ &+ \sum_{r=n_0}^{n-1} \Big[ - x(r - g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(x(u))\Big] \prod_{s=r+1}^{n-1} a(s), \ n \geq n_0 \end{aligned}$$

where $\Phi(r) = c(r) - c(r-1)a(r)$.

**Proof.** Rewrite (1.1) as

$$[\Delta x(n)\prod_{s=n_0}^{n-1}a^{-1}(s)]=\Big[c(n)\Delta x(n-g(n))+\sum_{u=n-g(n)}^{n-1}k(n,u)h(x(u))\Big]\prod_{s=n_0}^{n}a^{-1}(s) \tag{2.1}$$

Summing equation (2.1) from $n_0$ to n-1 gives

$$\sum_{r=n_0}^{n-1}[\Delta x(r)\prod_{s=n_0}^{r-1}a^{-1}(s)]=\sum_{r=n_0}^{n-1}[c(r)\Delta x(r-g(r))+\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))]\prod_{s=n_0}^{r}a^{-1}(s)$$

which gives,

$$|x(r)\prod_{s=n_0}^{r-1}a^{-1}(s)|_{n_0}^{n}=\sum_{r=n_0}^{n-1}[\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))+c(r)\Delta x(r-g(r))]\prod_{s=n_0}^{r}a^{-1}(s)$$

Thus,

$$x(n)=x(n_0)\prod_{s=n_0}^{n-1}a(s)+\sum_{r=n_0}^{n-1}[\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))+c(r)\Delta x(r-g(r))]\prod_{s=n_0}^{r}a^{-1}(s)\prod_{s=n_0}^{n-1}a(s)$$

Since

$$\prod_{s=n_0}^{r}a^{-1}(s)\prod_{s=n_0}^{n-1}a(s)=\prod_{s=r+1}^{n-1}a(s),$$

we obtain,

$$x(n)=x(n_0)\prod_{s=n_0}^{n-1}a(s)+\sum_{r=n_0}^{n-1}[\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))]\prod_{s=r+1}^{n-1}a(s)+\sum_{r=n_0}^{n-1}\Big[c(r)\Delta x(r-g(r))\prod_{s=r+1}^{n-1}a(s)\Big] \tag{2.2}$$

But,

$$
\begin{aligned}
\sum_{r=n_0}^{n-1}\Big[c(r)\Delta x(r-g(r))\prod_{s=r+1}^{n-1}a(s)\Big] &= |c(r-1)x(r-g(r))\prod_{s=r}^{n-1}a(s)\Big|_{n_0}^{n} \\
&\quad -\sum_{r=n_0}^{n-1}x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{n-1}a(s)] \\
&= c(n-1)x(n-g(n))\prod_{s=n}^{n-1}a(s)-c(n_0-1)x(n_0-g(n_0))\prod_{s=n_0}^{n-1}a(s) \\
&\quad -\sum_{r=n_0}^{n-1}x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{n-1}a(s)]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{r=n_0}^{n-1}\Big[c(r)\Delta x(r-g(r))\prod_{s=r+1}^{n-1}a(s)\Big] &= c(n-1)x(n-g(n))-c(n_0-1)x(n_0-g(n_0))\prod_{s=n_0}^{n-1}a(s) \\
&\quad -\sum_{r=n_0}^{n-1}x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{n-1}a(s)]
\end{aligned}
$$

Thus equation (2.2) becomes

$$
\begin{aligned}
x(n) &= x(n_0)\prod_{s=n_0}^{n-1}a(s)+\sum_{r=n_0}^{n-1}[\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))]\prod_{s=r+1}^{n-1}a(s)+c(n-1)x(n-g(n)) \\
&\quad -c(n_0-1)x(n_0-g(n_0))\prod_{s=n_0}^{n-1}a(s)-\sum_{r=n_0}^{n-1}x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{n-1}a(s)] \\
&= [x(n_0)-c(n_0-1)x(n_0-g(n_0))]\prod_{s=n_0}^{n-1}a(s)+c(n-1)x(n-g(n)) \\
&\quad +\sum_{r=n_0}^{n-1}\Big(-x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{n-1}a(s)]+\sum_{u=r-g(r)}^{r-1}k(r,u)h(x(u))\prod_{s=r+1}^{n-1}a(s)\Big)
\end{aligned}
$$

Therefore we have,

$$
\begin{aligned}
x(n) &= [x(n_0) - c(n_0-1)x(n_0 - g(n_0))] \prod_{s=n_0}^{n-1} a(s) + c(n-1)x(n-g(n)) \\
&\quad + \sum_{r=n_0}^{n-1} [-x(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(x(u))] \prod_{s=r+1}^{n-1} a(s) \\
&= [x(n_0) - c(n_0-1)x(n_0 - g(n_0))] \prod_{s=n_0}^{n-1} a(s) + c(n-1)x(n-g(n)) \\
&\quad + \sum_{r=n_0}^{n-1} \Big[ - x(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(x(u))\Big] \prod_{s=r+1}^{n-1} a(s),\ n \ge n_0
\end{aligned}
$$

This completes the proof of lemma 2.1.

Define

$$
S = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} \mid ||\varphi|| \to 0 \text{ as } n \to \infty\Big\}, \tag{2.3}
$$

where

$$
||\varphi|| = \max_{n \in \mathbb{Z}} |\varphi(n)|.
$$

Then $\Big(S, ||\cdot||\Big)$ is a Banach space.
Define mapping $H : S \to S$ by

$$
(H\varphi)(n) = \psi(n) \text{ on } \mathbb{Z}_0,
$$

and

$$
\begin{aligned}
(H\varphi)(n) &= \Big[\psi(n_0) - c(n_0-1)\psi(n_0 - g(n_0))\Big] \prod_{s=n_0}^{n-1} a(s) + c(n-1)\varphi(n-g(n)) \\
&\quad + \sum_{r=n_0}^{n-1} \Big[ - \varphi(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(x(u))\Big] \prod_{s=r+1}^{n-1} a(s),\ n \ge n_0
\end{aligned} \tag{2.4}
$$

**Lemma 2.2.** Suppose (1.2) hold. Also, suppose that

$$\prod_{s=n_0}^{n-1} a(s) \to 0 \text{ as } n \to \infty, \tag{2.5}$$

$$n - g(n) \to \infty \text{ as } n \to \infty, \tag{2.6}$$

and there exist $\alpha \in (0,1)$ such that,

$$\sum_{r=n_0}^{n-1} \Big[|\Phi(r)| + L \sum_{u=r-g(r)}^{r-1} k(r,u)\Big] \Big| \prod_{s=r+1}^{n-1} a(s)\Big| \leq \alpha,\ n \geq n_0. \tag{2.7}$$

Then the mapping $H$ defined by (2.4) $\to 0$ as $n \to \infty$.

**Proof.** The first term on the right of (2.4) goes to zero because of condition (2.5). The second term on the right goes to zero because of condition (2.6) and the fact that $\varphi \in S$.
Finally we show that the last term

$$\sum_{r=n_0}^{n-1} \Big[ - \Phi(r)\varphi(r-g(r)) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(\varphi(u))\Big] \prod_{s=r+1}^{n-1} a(s)$$

on the right of (2.4) goes to zero as $n \to \infty$.
Let $m > 0$ such that for $\varphi \in S$, $|\varphi(n-g(n))| < \sigma$ for $\sigma > 0$. Also, since $\varphi(n - g(n)) \to 0$ as $n - g(n) \to \infty$, there exists a $n_2 > m$ such that for $n > n_2$, $|\varphi(n-g(n))| < \epsilon_2$ for $\epsilon_2 > 0$. Due to condition (2.5) there exists a $n_3 > n_2$ such that for $n > n_3$ implies that

$$\Big| \prod_{s=n_2}^{n-1} a(s)\Big| < \frac{\epsilon_2}{\alpha\sigma}.$$

Thus for $n > n_3$, we have

$$\begin{aligned}
&\Big|\sum_{r=n_0}^{n-1}\big[-\varphi(r-g(r))\Phi(r)+\sum_{u=r-g(r)}^{r-1}k(r,u)h(\varphi(u))\big]\prod_{s=r+1}^{n-1}a(s)\Big| \\
\leq\ & \sum_{r=n_0}^{n-1}\Big|\big[-\varphi(r-g(r))\Phi(r)+\sum_{u=r-g(r)}^{r-1}k(r,u)h(\varphi(u))\big]\prod_{s=r+1}^{n-1}a(s)\Big| \\
\leq\ & \sum_{r=n_0}^{n_2-1}\Big|\big[\varphi(r-g(r))\Phi(r)+L\sum_{u=r-g(r)}^{r-1}k(r,u)\varphi(u)\big]\prod_{s=r+1}^{n-1}a(s)\Big| \\
& +\sum_{r=n_2}^{n-1}\Big|\big[\varphi(r-g(r))\Phi(r)+L\sum_{u=r-g(r)}^{r-1}k(r,u)\varphi(u)\big]\prod_{s=r+1}^{n-1}a(s)\Big| \\
\leq\ & \sigma\sum_{r=n_0}^{n_2-1}\Big[|\Phi(r)|+L\sum_{u=r-g(r)}^{r-1}k(r,u)\Big]\Big|\prod_{s=r+1}^{n-1}a(s)\Big|+\epsilon_2\alpha \\
\leq\ & \sigma\sum_{r=n_0}^{n_2-1}\Big[|\Phi(r)|+L\sum_{u=r-g(r)}^{r-1}k(r,u)\Big]\Big|\prod_{s=r+1}^{n_2-1}a(s)\prod_{s=n_2}^{n-1}a(s)\Big|+\epsilon_2\alpha \\
\leq\ & \sigma\alpha\Big|\prod_{s=n_2}^{n-1}a(s)\Big|+\epsilon_2\alpha \\
\leq\ & \epsilon_2+\epsilon_2\alpha.
\end{aligned}$$

This completes the proof of lemma 2.2.

We state below Krasnoselskii's fixed point Theorem which is the main mathematical tool in this paper and we refer to [19] for the proof.

**Theorem 2.3** ( Krasnoselskii's ) Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that $A$ and $Q$ map $\mathbb{M}$ into $\mathbb{B}$ such that
(i) $x, y \in \mathbb{M}$, implies $Ax + Qy \in \mathbb{M}$,
(ii) $A$ is continuous and $A\mathbb{M}$ is contained in a compact set,
(iii) $Q$ is a contraction mapping.
Then there exists $z \in \mathbb{M}$ with $z = Az + Qz$.

The application of the above Theorem requires the construction of two mappings which we obtain by expressing (2.4) as

$$(H\varphi)(n) = (Q\varphi)(n) + (A\varphi)(n),$$

where $A, Q : S \to S$ are given by

$$(Q\varphi)(n) = \Big[\psi(n_0) - c(n_0-1)\psi(n_0 - g(n_0))\Big] \prod_{s=n_0}^{n-1} a(s) + c(n-1)\varphi(n-g(n)) \tag{2.8}$$

$$(A\varphi)(n) = \sum_{r=n_0}^{n-1} \Big[ -\varphi(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(\varphi(u))\Big] \prod_{s=r+1}^{n-1} a(s) \tag{2.9}$$

**Lemma 2.4** Suppose that (1.2) and (2.7) hold. Also suppose that there exist a constant $\gamma$ such that,

$$\max_{n\in\mathbb{Z}} |a(n)| = \gamma. \tag{2.10}$$

Then the mapping $A$ defined by (2.9) is continuous and compact.

**Proof.** Let $\varphi, \zeta \in S$. Given any $\epsilon_1 > 0$, choose $\delta = \frac{\epsilon_1}{\alpha}$ such that $||\varphi - \zeta|| < \delta$.

$$\begin{aligned}
||(A\varphi) - (A\zeta)|| &\leq \sum_{r=n_0}^{n-1} |\Phi(r)| \Big|\varphi(r-g(r)) - \zeta(r-g(r))\Big| \Big| \prod_{s=r+1}^{n-1} a(s)\Big| \\
&+ \sum_{r=n_0}^{n-1} \Big| \sum_{u=r-g(r)}^{r-1} k(r,u)h(\varphi(u)) - \sum_{u=r-g(r)}^{r-1} k(r,u)h(\zeta(u))\Big| \Big| \prod_{s=r+1}^{n-1} a(s)\Big| \\
&\leq \sum_{r=n_0}^{n-1} \Big\{ \Big[ |\Phi(r)| + L \sum_{u=r-g(r)}^{r-1} k(r,u)\Big] \Big| \prod_{s=r+1}^{n-1} a(s)\Big| \Big\} ||\varphi - \zeta|| \\
&\leq \alpha ||\varphi - \zeta|| \\
&\leq \epsilon_1.
\end{aligned}$$

We next show that $A$ is compact. Let $\{\varphi_n\} \subset S$ denote a sequence of uniformly bounded functions with $||\varphi_n|| \leq \lambda$, where $n$ is a positive integer and $\lambda > 0$. Thus,

$$\begin{aligned}
||(A\varphi_n)|| &\leq \sum_{r=n_0}^{n-1} |\Big[\varphi(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(\varphi(u))\Big] \prod_{s=r+1}^{n-1} a(s)| \\
&\leq \sum_{r=n_0}^{n-1} \Big|\Big[\Phi(r) + L \sum_{u=r-g(r)}^{r-1} k(r,u)\Big] \prod_{s=r+1}^{n-1} a(s)\Big| ||\varphi|| \\
&\leq \alpha||\varphi|| \\
&\leq \alpha\lambda
\end{aligned}$$

Thus showing that $||(A\varphi_n)|| \leq \iota$ for some positive constant $\iota$.

Also,

$$\begin{aligned}
||\Delta(A\varphi)|| &\leq |a(n)|| - \varphi(n-g(n))\Phi(n) + \sum_{u=n-g(n)}^{n-1} k(r,u)h(x(u))| \\
&\leq \gamma|\Phi(n) + L \sum_{u=n-g(n)}^{n-1} k(r,u)|||\varphi|| \\
&\leq \gamma\alpha\lambda \\
&\leq \beta.
\end{aligned}$$

For some positive constant $\beta$. Thus the sequence $(A\varphi_n)$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that $(A\varphi_{n_k})$ converges uniformly to a function $\varphi^*$. Thus $A$ is compact.

**Lemma 2.5** Let $Q$ be defined by (2.8) and

$$|c(n-1)| \leq \eta < 1. \tag{2.11}$$

Then $Q$ is a contraction.

**Proof.** Let $\varphi, \zeta \in S$,

$$\begin{aligned}
||(Q\varphi) - (Q\zeta)|| &\leq |c(n-1)|||\varphi - \zeta|| \\
&\leq \eta||\varphi - \zeta||.
\end{aligned}$$

**Theoem 2.6.** Suppose the hypotheses of lemma 2.2, lemma 2.4, and lemma 2.5 hold. Also, suppose that there is a positive constant $\rho$ such that

$$\Big|\prod_{s=n_0}^{n-1} a(s)\Big| \leq \rho, \tag{2.12}$$

and there exist an $\alpha \in (0,1)$

$$\Big|c(n-1)\Big| + \sum_{r=n_0}^{n-1}\Big[|\Phi(r)| + L\sum_{u=r-g(r)}^{r-1} k(r,u)\Big]\Big|\prod_{s=r+1}^{n-1} a(s)\Big| \leq \alpha,\ n \geq n_0. \tag{2.13}$$

Then the zero solution of (1.1) is asymptotically stable.

**Proof.** Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$\Big|1 - c(n_0-1)\Big|\delta\rho + \alpha\epsilon < \epsilon.$$

Let $\psi(n)$ be any given initial function such that $|\psi(n)| < \delta$.

Define $\mathbb{M} = \{\varphi \in S : ||\varphi|| < \epsilon\}$. Let $\varphi, \zeta \in \mathbb{M}$, then,

$$\begin{aligned}
||(Q\zeta) + (A\varphi)|| \;\leq\; & \Big|\psi(n_0) - c(n_0-1)\psi(n_0 - g(n_0))\Big]\prod_{s=n_0}^{n-1} a(s)\Big| + \Big|c(n-1)\zeta(n-g(n))\Big| \\
& + \sum_{r=n_0}^{n-1}\Big|\varphi(r-g(r))\Phi(r) + \sum_{u=r-g(r)}^{r-1} k(r,u)h(\varphi(u))\Big]\prod_{s=r+1}^{n-1} a(s)\Big| \\
\leq\; & \Big|1 - c(n_0-1)\Big|\delta\rho + |c(n-1)|\epsilon + \sum_{r=n_0}^{n-1}\Big|\Phi(r) + L\sum_{u=r-g(r)}^{r-1} k(r,u)\Big|\Big|\prod_{s=r+1}^{n-1} a(s)\Big|\epsilon \\
\leq\; & \Big|1 - c(n_0-1)\Big|\delta\rho + \Big\{|c(n-1)| + \sum_{r=n_0}^{n-1}\Big|\Phi(r) + L\sum_{u=r-g(r)}^{r-1} k(r,u)\Big|\Big|\prod_{s=r+1}^{n-1} a(s)\Big|\Big\}\epsilon \\
\leq\; & \Big|1 - c(n_0-1)\Big|\delta\rho + \alpha\epsilon \\
<\; & \epsilon
\end{aligned}$$

It therefore follows from the above work that all the conditions of the Krasnoselskii's Theorem are satisfied on $\mathbb{M}$. Thus there exists a fixed point $z$ in $\mathbb{M}$ such that $z = Az + Qz$. By lemma 2.1 this fixed point $x(n)$ is a solution of (1.1) and in particular, $||x|| < \epsilon$.

Therefore showing that the zero solution of (1.1) is stable. In addition $||x|| \to 0$ as $n \to \infty$ by the fact that $x(n) \in S$. Therefore the zero solution of (1.1) is asymptotically stable.

**Example.** Consider the difference equation

$$x(n+1) = \frac{1}{1+n}x(n) + \frac{2^{n+1}}{16(n+1)!}\Delta x(n-2) + \sum_{s=n-2}^{n-1} \frac{2^n}{8(1-n)!(s+2)}x(s), \; n \geq 0. \tag{2.14}$$

In this example we take $n_0 = 0$.
We observe that

$$\prod_{s=0}^{n-1} \frac{1}{1+s} = \frac{1}{n!} \to 0 \text{ as } n \to \infty,$$

thus condition (2.5) is satisfied. Condition (2.6) also satisfied since

$$n - 2 \to \infty \text{ as } n \to \infty.$$

Using (2.7) we obtain,

$$\begin{aligned}
&\sum_{r=0}^{n-1} \left[\frac{2^{r+1}}{16(r+1)!} - \frac{2^r}{16r!(r+1)}\right] \prod_{s=r+1}^{n-1} \frac{1}{1+s} \\
&+ \sum_{r=0}^{n-1} \sum_{u=r-2}^{r-1} \frac{2^r}{8(1-r)!(u+2)} \prod_{s=r+1}^{n-1} \frac{1}{1+s} \\
&= \frac{1}{8n!} \sum_{r=0}^{n-1} 2^r - \frac{1}{16n!} \sum_{r=0}^{n-1} 2^r + \sum_{r=0}^{n-1} \frac{2^r}{8n!} \\
&\leq \frac{1}{8n!}(2^n - 1) + \frac{1}{8n!}(2^n - 1) \\
&\leq \frac{1}{8n!}2^n + \frac{1}{8n!}2^n \\
&\leq \frac{1}{2} < 1.
\end{aligned}$$

Thus condition (2.7) is satisfied.

Using (2.13) we obtain,

$$\Big|\frac{2^n}{16n!}\Big| + \sum_{r=0}^{n-1}\Big[\frac{2^{r+1}}{16(r+1)!} - \frac{2^r}{16r!(r+1)}\Big]\prod_{s=r+1}^{n-1}\frac{1}{1+s}$$
$$+\sum_{r=0}^{n-1}\sum_{u=r-2}^{r-1}\frac{2^r}{8(1-r)!(u+2)}\prod_{s=r+1}^{n-1}\frac{1}{1+s} \le \frac{5}{8} < 1.$$

Thus condition (2.13) is satisfied.
We also have,

$$\Big|\frac{1}{1+n}\Big| \le 1,$$

$$\Big|c(n-1)\Big| = \Big|\frac{2^n}{16n!}\Big| \le \frac{1}{8} < 1,$$

and,

$$\Big|\prod_{s=0}^{n-1}\frac{1}{1+s}\Big| \le 1.$$

Thus conditions (2.10),(2.11) and (2.12) are satisfied respectively.

Therefore the zero solution of (2.14) is asymptotically stable.

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Ernest Yankson
Department of Mathematical Sciences, Kent State University, Kent, OH
**email:** eyankson@math.kent.edu