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Location of Resonances Generated by Degenerate Potential Barrier

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Abstract

We study resonances of the semi-classical Schrödinger operator $H = -h^2\Delta + V$ on $L^2(I\!\!R^N)$. We consider the case where the potential V have an absolute degenerate maximum. Then we prove that H has resonances with energies $E = V_0 + e^{-i\frac{\pi}{\sigma+1}} h^{\frac{2\sigma}{\sigma+1}} k_j + \mathcal{O}(h^{\frac{2\sigma+1}{\sigma+1}})$, where k_j is in the spectrum of some quartic oscillator.

1 Introduction and main results.

We are interested in this paper to study the asymptotic behavior for Schrödinger operator:

$$H = -h^2\Delta + V \tag{1}$$

on $L^2(I\!\!R^N)$, where V is a bounded real function having an absolute maximum V_0 realized at a unique point that we suppose to be $x = 0$. Under some assumptions on V, one can define on $L^2(I\!\!R^N)$ the operator $H_\theta(x, hD)$ with domain the Sobolev space $W^2(I\!\!R^N)$ obtained by analytic dilation :

$$H_\theta(x, hD) = U_\theta H(x, hD) U_{-\theta} = e^{-2i\theta} h^2 D^2 + V(xe^{i\theta}).$$

Here U_θ denotes the group of unitary operators on $L^2(I\!\!R^N)$, given by $U_\theta u(x) = e^{-\frac{iN\theta}{2}} u(e^{i\theta}x)$, for θ in $I\!\!R$ such that $0 \le \theta < \theta_0 < \frac{\pi}{2}$. D denotes the differential operator $(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, ..., -i\frac{\partial}{\partial x_N})$. Isolated eigenvalues of H_θ with finite

multiplicities which are not discrete eigenvalues of H are called resonances of H.
In [2] and [3] Ph.Briet, J.M.Combes and P.Duclos consider the same problem with V having a non degenerate maximum at $x = 0$. They prove existence of resonance values of H near 0. They prove too that these values are located outside a ball of radius $\mathcal{O}(h)$. In the second paper they give the first factor in the asymptotic expansion of resonance values of H. We are interested here to generalize results of [2] and [3] to cases of degenerate potential barrier of order σ i.e $V(x) \sim -(\lambda_1 x_1^{2\sigma} + ... + \lambda_N x_N^{2\sigma}) + \mathcal{O}(x^{2\sigma+1})$ as $x \to 0$ in $I\!R^N$. Here $\sigma > 1$ is an integer and $\lambda_1, \dots \lambda_N$ some strictly positive real constants. We prove that resonance values of H are located outside a ball centered at 0 and of radius $\mathcal{O}(h^{\frac{2\sigma}{\sigma+1}})$. This power of h exists already in the results of Martinez-Rouleux [5]. The authors study there asymptotic of eigenvalues of the operator H in the case of degenerate minimum of order σ for V. They prove that H has no eigenvalue inside a ball centered at 0 and of radius $\mathcal{O}(h^{\frac{2\sigma}{\sigma+1}})$. In fact this result is closely related to the Taylor expansion of V near 0. Our second aim is to get more precise result under more precise assumption on the Taylor expansion of V. We notice that in a recent paper [1] we study the resonances of H in the one dimensional case where the potential V have a degenerate maximum of quartic type. We give in that paper the full asymptotic expansion of resonance values near the barrier maximum. The method used there is the BKW techniques. This method is specific to the one dimensional case. Here we use local analysis on the resolvent of the operator. To state the results assume the following hypothesis on V:
(A_1) V is a bounded real function having an absolute degenerate maximum of order σ at $x = 0$.
(A_2) $V_\theta(x) = V(e^\theta x), \theta \in I\!R$ has an analytic continuation to complex θ in $S_\alpha = \{\theta \in I\!\!\!C, \mid \mathrm{Im}\theta \mid < \alpha\}$ as a family of bounded operators.
(A_3) $\exists \theta_0 = i\beta_0, 0 < \beta_0 < \alpha, \forall \delta > 0, \exists C_\delta > 0$, such that

$$\forall x, \mid x \mid > \delta, \mathrm{Im}(e^{2\theta_0} V_{\theta_0}) < -C_\delta \beta_0.$$

(A_4) Near $x = 0$, V_θ has the Taylor expansion :

$$V_\theta(x) = -e^{2\sigma\theta}(\lambda_1 x_1^{2\sigma} + ... + \lambda_N x_N^{2\sigma}) + \mathcal{O}(x^{2\sigma+1})$$

for all $\theta \in S_\alpha$, with $\lambda_1, ...\lambda_N$ non vanishing positive real constants.
Consider the operator given by :

$$K = -\Delta + (\lambda_1 x_1^{2\sigma} + ... + \lambda_N x_N^{2\sigma}).$$

It is well known that the spectrum of K is discrete. We use $(k_j)_{j\geq 1}$ for eigenvalues of K. We recall that the operator H_θ has a domain $D(H_\theta) = W^2(\mathbb{R}^N)$ independent of θ. This operator has an analytic continuation to an analytic family of type A in S_α (see [4] and [6], section XIII). We get :

Théorème 1 *Consider the operator $H_{\theta_0} = -h^2 e^{-2\theta_0}\Delta + V_{\theta_0}$ and assume the hypothesis $(A_{1,2,3,4})$ on V. For all $k_j \in \sigma(K)$ there exits a ball $B(h)$ centered at $e^{-i\frac{\pi}{\sigma+1}} h^{\frac{2\sigma}{\sigma+1}} k_j$ with radius $\mathcal{O}(h^{\frac{2\sigma+1}{\sigma+1}})$ such that for h small enough , H_{θ_0} has purely discrete spectrum inside $B(h)$ with total algebraic multiplicity equal to the multiplicity of k_j.*

In the direction to get the full asymptotic expansion of resonance values of H, we try more subtle approach and we get the following result: Assume (A_5) Near $x = 0$, V_θ has the Taylor expansion :

$$V_\theta(x) = -e^{2\sigma\theta}(\lambda_1 x_1^{2\sigma} + ... + \lambda_N x_N^{2\sigma}) + \mathcal{O}(x^{2\sigma+p})$$

with $p > 0$ an integer.

Théorème 2 *Assume $(A_{1,2,3,5})$. For all $k_j \in \sigma(K)$ there exits a ball $B(h)$ centered at $e^{-i\frac{\pi}{\sigma+1}} h^{\frac{2\sigma}{\sigma+1}} k_j$ with radius $\mathcal{O}(h^{\frac{2\sigma+p}{\sigma+1}})$ such that for h small enough, H_{θ_0} has purely discrete spectrum inside $B(h)$ with the same algebraic multiplicity as k_j.*

Finally to interpret the eigenvalues of H_{θ_0} mentioned in the Theorem 1 as resonance values of H we state the following theorem :

Théorème 3 *Let ϕ be in the domain D of analytic dilation dense in $L^2(\mathbb{R}^N)$. For all $C > 0$ and h small enough $((H - z)^{-1}\phi, \phi)$ has meromorphic continuation from $\mathbb{C}^+ = \{z \in \mathbb{C}, \mathrm{Im} z > 0\}$ into a complex disk centered at 0 with radius $Ch^{\frac{2\sigma}{\sigma+1}}$. The poles of this continuation belong to the set of the eigenvalues of H_{θ_0} given in theorem 1.*

This theorem shows in particular that there is no other part of the spectrum of H_{θ_0} in the disk other than the eigenvalues mentioned in the Theorem 1. The authors of [2] and [3] established the same result in the case $\sigma = 1$.

2 Localisation Formula

To prove theorems we decompose the operator H into a direct sum of an interior operator and an exterior one. To do this we use j_i and $j_e \in C^\infty(I\!R^N)$, with $j_i = 0$ outside the ball $B(0, h^{\frac{1}{\sigma+1}}\delta_i)$ et $j_e = 0$ inside the ball $B(0, h^{\frac{1}{\sigma+1}}\delta_e)$, satisfying to $j_i^2 + j_e^2 = 1$. Here $0 < \delta_e < \delta_i$ are two positive constants to be chosen later. The support of ∇j_i and ∇j_e is located in $\Omega_0 = \{x \in I\!R^N, 0 < h^{\frac{1}{\sigma+1}}\delta_e <| x |< h^{\frac{1}{\sigma+1}}\delta_i\}$. Let $\mathcal{H} = L^2(I\!R^N)$ and $\mathcal{H}_d = L^2(I\!R^N) \oplus L^2($ support of $j_e)$, we define the application J by

$$J : \mathcal{H}_d \to \mathcal{H}$$
$$J(u \oplus v) = j_i u + j_e v.$$

We have $JJ^* = id_{\mathcal{H}}$.
For $\mathrm{Im}\theta \neq 0$, we define $H_\theta^d = H_\theta^i \oplus H_\theta^e$ where

$$H_\theta^i = -h^2 e^{-2\theta}\Delta - e^{2\sigma\theta}(\lambda_1 x_1^{2\sigma} + ... + \lambda_N x_N^{2\sigma}) \qquad (2)$$

on $L^2(I\!R^N)$ and

$$H_\theta^e = -h^2 e^{-2\theta}\Delta + V_\theta$$

on $L^2($ support of $j_e)$ with Dirichlet conditions on the boundary of the support of j_e. Hence we have $J(D(H_\theta^d)) \subset D(H_\theta)$ and we can write the resolvent equation:

$$(H_\theta - z)^{-1} = J(H_\theta^d - z)^{-1}J^* - (H_\theta - z)^{-1}\Pi(H_\theta^d - z)^{-1}J^* \qquad (3)$$

where Π is the operator given by $\Pi = H_\theta J - JH_\theta^d$ acting on $\mathcal{H}_d$ as follows:

$$\Pi(u \oplus v) = wu - h^2 e^{-2\theta}([\Delta, j_i]u + [\Delta, j_e]v)$$

where

$$w = (V_\theta + e^{2\sigma\theta}(\lambda_1 x_1^{2\sigma} + \ldots + \lambda_N x_N^{2\sigma}))j_i.$$

For the study of the interior operator H_θ^i one prove the following lemma:

Lemme 1 *Let H_θ^i defined by (2) with domain $D(H_\theta^i) = W^2(I\!R^N) \cap D(x^{2\sigma})$ we have :*
i) $\{H_\theta^i, 0 < \mathrm{Im}\theta < \frac{\pi}{\sigma+1}\}$ is an analytic family of type A of sector-operators with sector

$$\Sigma = \{z \in \mathbb{C}, -\pi + 2\mathrm{Im}\theta \leq arg z \leq -2\mathrm{Im}\theta\}.$$

ii) The spectrum of H^i_θ is θ independent, purely discrete and given by :

$$\sigma(H^i_\theta) = \sigma(H^i_{i\frac{\pi}{2\sigma+2}}) = e^{-i\frac{\pi}{\sigma+1}} h^{\frac{2\sigma}{\sigma+1}} \sigma(K).$$

Proof : The proof of *i)* can be reduced to prove the same properties for the one dimensional operator :

$$h_\theta = -h^2 e^{-2\theta} \frac{\partial^2}{\partial x^2} - e^{2\sigma\theta} x^{2\sigma}.$$

the domain of h_θ is $D(h_\theta) = W^2(I\!R) \cap D(x^{2\sigma})$. To prove that the family (h_θ) is analytic of type A, one proves that h_θ is closed for all θ, $0 < \mathrm{Im}(\theta) < \frac{\pi}{\sigma+1}$. For this we have to establish for all $\theta = i\beta$ such that $0 <\mid \beta - \frac{\pi}{2\sigma+2} \mid< \frac{\pi}{2\sigma+2}$, the following inequality :

$$\| h_\theta u \|^2 + \| u \|^2 \geq c(h^4 \| u'' \|^2 + \| x^{2\sigma} u \|^2) \tag{4}$$

on $D(h_\theta)$, where c is a positive constant. We have:

$$\|h_\theta u\|^2 = h^4 \|u''\|^2 + \|x^{2\sigma} u\|^2 + 2h^2 \mathrm{Re}(e^{-i(2+2\sigma)\beta}(u'', x^{2\sigma} u))$$

We write

$$h^2(u'', x^{2\sigma} u) = -h^2 \int x^{2\sigma} u' \overline{u}' dx - h^2 \int 2\sigma x^{2\sigma-1} u' \overline{u} dx. \tag{5}$$

One can estimate the second integral in this equation as follows :

$$h^2 |\int 2\sigma x^{2\sigma-1} u' \overline{u} dx| \leq \|h^2 u'\|^2 + \|2\sigma x^{2\sigma-1} u\|^2.$$

By integration by parts we get:

$$\|h^2 u'\|^2 \leq h^8 \|u''\|^2 + \|u\|^2. \tag{6}$$

Let $B(R)$ denotes the ball centered at 0 with radius $R > 0$ and $B(R)^c$ the complementary set of this ball. We write :

$$\|2\sigma x^{2\sigma-1} u\|^2 = \int_{B(R)\cup B(R)^c} (2\sigma x^{2\sigma-1})^2 u \overline{u} dx$$

$$\leq 4\sigma^2 R^{4\sigma-2} \|u\|^2 + \frac{4\sigma^2}{R^2} \|x^{2\sigma} u\|^2.$$

This yields in combination with (6):

$$h^2|\int 2\sigma x^{2\sigma-1}u'\overline{u}dx| \leq h^8\|u''\|^2 + \|u\|^2 + 4\sigma^2 R^{4\sigma-2}\|u\|^2 + \frac{4\sigma^2}{R^2}\|x^{2\sigma}u\|^2. \quad (7)$$

For the first integral in the equation 5 :

$$2h^2|\int x^{2\sigma}u'\overline{u}'dx| \leq h^4\|u''\|^2 + \|x^{2\sigma}u\|^2 + 2h^2|\int 2\sigma x^{2\sigma-1}u'\overline{u}dx|. \quad (8)$$

We now write :

$$\|h_\theta u\|^2 = h^4\|u''\|^2 + \|x^{2\sigma}u\|^2 + 2\mathrm{Re}[e^{-i(2+2\sigma)\beta}(-h^2\int x^{2\sigma}u'\overline{u}'dx - h^2\int 2\sigma x^{2\sigma-1}u'\overline{u}dx)]$$

$$\geq h^4\|u''\|^2 + \|x^{2\sigma}u\|^2 - 2|\cos((2+2\sigma)\beta)|h^2\int x^{2\sigma}u'\overline{u}'dx - 2h^2|\int 2\sigma x^{2\sigma-1}u'\overline{u}dx|$$

Therefore by (7) and (8)we get :

$$\begin{aligned}\|h_\theta u\|^2 \geq\ & [1 - |\cos((2+2\sigma)\beta)| - 2h^4(|\cos((2+2\sigma)\beta)| + 1)]\|h^2u''\|^2 + \\ & (1 - |\cos((2+2\sigma)\beta)| - 8(|\cos((2+2\sigma)\beta)| + 1)\tfrac{\sigma^2}{R^2})\|x^{2\sigma}u\|^2 - \\ & 2(|\cos((2+2\sigma)\beta)| + 1)(1 + 4\sigma^2R^{4\sigma-2})\|u\|^2.\end{aligned}$$

Finally choosing R big enough we get (4).

To prove *ii)*, notice that H_θ^i has a compact resolvent for $0 < \mathrm{Im}\theta < \frac{\pi}{\sigma+1}$. By analyticity of $(H_\theta^i)_\theta$ the spectrum of H_θ^i is independent of θ, hence $\sigma(H_\theta^i) = \sigma(H^i_{i\frac{\pi}{2\sigma+2}})$. Using the scaling of order $h^{\frac{1}{\sigma+1}}$ we get:

$$H^i_{i\frac{\pi}{2\sigma+2}} = e^{-i\frac{\pi}{\sigma+1}}h^{\frac{2\sigma}{\sigma+1}}K.$$

Therefore:

$$\sigma(H_\theta^i) = e^{-i\frac{\pi}{\sigma+1}}h^{\frac{2\sigma}{\sigma+1}}\sigma(K).$$

■

In the exterior domain our operator H here has the same shape as this one studied in [3]. Here we localize more closely to the boss of the barrier. So we get more precise results than lemma II-5 in [3].

Lemme 2 *Under assumptions* $(A_{1,2,3,4})$ *and for* $\theta_0 = i\beta_0$ *we have :*
i) The resolvent set of $H_{\theta_0}^e$ *contains a complex neighborhood* Υ *of* 0 *in the form:*

$$\Upsilon = \{z \in \mathbb{C}, \mathrm{Im}(e^{2\theta_0}(-z)) < C\beta_0 h^{\frac{2\sigma}{\sigma+1}}\}.$$

Here $C > 0$ is h-independent constant.
ii) For all $z \in \Upsilon$ we have,

$$\| (H^e_{\theta_0} - z)^{-1} \| \leq \frac{1}{dist(z, \partial\Upsilon)}.$$

Proof: We first notice that by assumption A_4 there exists $\epsilon > 0$ independent of h such that for $\theta_0 = i\beta_0$ with $\beta_0 > 0$ small enough we get for all $| x | < \epsilon$:

$$\begin{aligned} \mathrm{Im}(e^{2\theta_0} V_{\theta_0}(x)) &= -sin((2\sigma+1)\beta_0)(\lambda_1 x_1^{2\sigma} + \ldots + \lambda_N x_N^{2\sigma}) + \mathcal{O}(x^{2\sigma+1}) \\ &\leq -\tfrac{1}{2} sin((2\sigma+1)\beta_0)(\lambda_1 x_1^{2\sigma} + \ldots + \lambda_N x_N^{2\sigma}). \end{aligned}$$

Therefore we get $\forall x$ such that $h^{\frac{1}{\sigma+1}} \delta_e < | x | < \epsilon$:

$$\mathrm{Im}(e^{2\theta_0} V_{\theta_0}) < -C\beta_0 h^{\frac{2\sigma}{\sigma+1}},$$

for a convenient $C > 0$ independent of h. Hence by this and assumption A_3 we get:

$$\forall x, \ \ | x | > h^{\frac{1}{\sigma+1}} \delta_e, \ \ \mathrm{Im}(e^{2\theta_0} V_{\theta_0}) < -C\beta_0 h^{\frac{2\sigma}{\sigma+1}}. \tag{9}$$

Let $u \in D(H^e_{\theta_0})$ be such that $\| u \| = 1$, and let $z \in \Upsilon$. Since $(\Delta u, u)$ has a real value, we get by (9):

$$\begin{aligned} \mathrm{Im}(e^{2i\beta_0}(H^e_{\theta_0} - z)u, u) &= \mathrm{Im}(e^{2i\beta_0}(V_{\theta_0} - z)u, u) \\ &\leq \mathrm{Im}(-e^{2i\beta_0} z - C\beta_0 h^{\frac{2\sigma}{\sigma+1}}) \ , \\ &\leq -dist(z, \partial\Upsilon) \end{aligned}$$

where $(.,.)$ denotes the inner product in $L^2(\mathbb{R}^N)$. Hence we get:

$$\begin{aligned} \| (H^e_{\theta_0} - z)u \| &\geq | \mathrm{Im}(e^{2i\beta_0}(H^e_{\theta_0} - z)u, u) | \\ &\geq dist(z, \partial\Upsilon). \end{aligned} \tag{10}$$

This proves that $Ker(H^e_{\theta_0} - z) = \{0\}$ and $(H^e_{\theta_0} - z)$ has a closed image. On the other hand $(H^e_{\theta_0} - z)^* = H^e_{\bar{\theta}_0} - \bar{z}$, and we get by the same way:

$$\| (H^e_{\theta_0} - z)^* u \| \geq dist(\bar{z}, \partial\bar{\Upsilon}).$$

Therefore $Ker(H^e_{\theta_0} - z)^* = \{0\}$, and hence the image of $(H^e_{\theta_0} - z)$ is the whole $L^2(support j_e)$. This leads to *i)*. For *ii)* it is immediately given by (10). ∎

This lemma means that the operator valued function $z \mapsto (H^e_{\theta_0} - z)^{-1}$ is holomorphic inside Υ. By the same argument one gets the following:

Lemme 3 *Let $g \in C_0^\infty(\mathbb{R}^N)$ with support$(g) \subset \{x \in \mathbb{R}^N, 0 < h^{\frac{1}{\sigma+1}}\delta <| x |\}$. The operator valued functions $z \mapsto g(H^d_{\theta_0} - z)^{-1}J^*$ and $z \mapsto (H_{\theta_0} - z)^{-1}g$ have no poles inside Υ for a convenient C.*

Let now $k_j \in \sigma(K)$ and γ denotes the closed curve given by:

$$\gamma = \{z \in \mathbb{C}, | z - E^d |= \rho h^{\frac{2\sigma}{\sigma+1}}\}. \tag{11}$$

Here $E^d = e^{-i\frac{\pi}{\sigma+1}} h^{\frac{2\sigma}{\sigma+1}} k_j$ and $\rho > 0$ is a constant independent of h to be chosen later small enough. We have:

Lemme 4 *Let γ be the closed curve defined by* (11)*, then for ρ small enough we have by assumptions $(A_{1,2,3,4})$:*
i) γ is in the resolvent set of H_{θ_0}.
ii) $\| (H_{\theta_0} - z)^{-1} \|= \mathcal{O}(h^{\frac{-2\sigma}{\sigma+1}})$ uniformly for $z \in \gamma$ as $h \longrightarrow 0$.

By our localization formula close to the boss of the barrier, the proof of this lemma seems to be much more complicated than if we do this using standard localization formula as in [3]. This is largely due to the contribution of the commutator part in Π in the equation (3). The estimation of this contribution becomes here not precise. By the same way as in [3] we get the same results. To avoid needless repetition we shall not rewrite proof.

3 Proof of Theorems

The following lemma shows that the spectrum of H_{θ_0} is discrete inside γ. The total algebraic multiplicity is equal to the multiplicity of E^d as an eigenvalue of $H^i_{\theta_0}$.

Lemme 5 *Let γ be the closed curve given by* (11)*, with ρ small enough but independent of h. Then under assumptions $(A_{1,2,3,4})$ for h small enough, P_{θ_0} and $P^i_{\theta_0}$ have the same rank. where P_{θ_0} and $P^i_{\theta_0}$ denote the projectors defined respectively by:*

$$P_{\theta_0} = -\frac{1}{2i\pi}\oint_\gamma (H_{\theta_0} - z)^{-1} dz$$

and

$$P^i_{\theta_0} = -\frac{1}{2i\pi}\oint_\gamma (H^i_{\theta_0} - z)^{-1} dz.$$

Proof : The lemma 4 shows that the operator P_{θ_0} is well defined for h small enough. By (3) we get

$$P_{\theta_0} - j_i P^i_{\theta_0} j_i = -\frac{1}{2i\pi}\oint_\gamma (H_{\theta_0} - z)^{-1}\Pi(H^d_{\theta_0} - z)^{-1}J^* dz.$$

We have for $s = i$ and $s = e$, $[\Delta, j_s] = (\Delta j_s + 2\nabla j_s \nabla)$ is supported in Ω_0. Let $g \in C_0^\infty(I\!R^N)$ with $support(g) \subset \{x \in I\!R^N, 0 < h^{\frac{1}{\sigma+1}}\delta <\mid x \mid\}$, with $\delta < \delta_e$ and satisfying to $g(x) = 1$ inside Ω_0. We have $[\Delta, j_s] = g[\Delta, j_s]g$. Then by lemma 3 the commutator part in the expression of Π has no contribution to this integral. We get

$$P_{\theta_0} - j_i P^i_{\theta_0} j_i = -\frac{1}{2i\pi}\oint_\gamma (H_{\theta_0} - z)^{-1} w (H^i_{\theta_0} - z)^{-1} j_i dz.$$

Let us now prove that :

$$\| (H_{\theta_0} - z)^{-1} w (H^i_{\theta_0} - z)^{-1} j_i \| = \mathcal{O}(h^{\frac{1-2\sigma}{\sigma+1}}) \tag{12}$$

uniformly for $z \in \gamma$. We have:

$$\| (H_{\theta_0} - z)^{-1} w (H^i_{\theta_0} - z)^{-1} j_i \| \leq \| (H_{\theta_0} - z)^{-1} \| . \| w (H^i_{\theta_0} - z)^{-1} j_i \| .$$

By lemma 1 *ii)* one can choose ρ such that $dist(\gamma, \sigma(H^i_{\theta_0})) > ch^{\frac{2\sigma}{\sigma+1}}$. Hence one gets

$$\| (H^i_{\theta_0} - z)^{-1} \| = \mathcal{O}(h^{\frac{-2\sigma}{\sigma+1}}) \tag{13}$$

uniformly for $z \in \gamma$. Now by (A_4) we have $w = \mathcal{O}(h^{\frac{1+2\sigma}{\sigma+1}})$ on the support of j_i. Then we get:

$$\| w (H^i_{\theta_0} - z)^{-1} \| = \mathcal{O}(h^{\frac{1}{\sigma+1}}). \tag{14}$$

This equation in combination with lemma 4 *ii)* leads to (12). Since $\mid \gamma \mid = \mathcal{O}(h^{\frac{2\sigma}{\sigma+1}})$, we get :

$$P_{\theta_0} - j_i P^i_{\theta_0} j_i = \mathcal{O}(h^{\frac{1}{\sigma+1}}).$$

This leads to :

$$P_{\theta_0} - P^i_{\theta_0} = \mathcal{O}(h^{\frac{1}{\sigma+1}}),$$

and then smaller than 1 for h small enough, and the lemma is proved. ∎

To finish the proof of theorem 1, let E be an eigenvalue of H_{θ_0} inside γ, and let ϕ be the corresponding normalized eigenvector. We have :

$$(E-E^d)\phi = (H_{\theta_0}-E^d)\phi = -\frac{1}{2i\pi}\int_\gamma (z-E^d)(H_{\theta_0}-z)^{-1}\phi dz.$$

By equation (3) and the fact that the commutator part has no contribution to the integral we get:

$$(E-E^d)\phi = -\frac{1}{2i\pi}\int_\gamma (z-E^d)(H_{\theta_0}-z)^{-1}w(H^i_{\theta_0}-z)^{-1}j_i\phi dz. \tag{15}$$

Then

$$\mid E-E^d \mid \leq \rho^2 h^{\frac{4\sigma}{\sigma+1}} \sup_{z\in\gamma} \|(H_{\theta_0}-z)^{-1}w(H^i_{\theta_0}-z)^{-1}j_i\|.$$

Therefore equation (12) leads to

$$\begin{aligned} \mid E-E^d \mid \ & \leq \rho^2 h^{\frac{4\sigma}{\sigma+1}}\mathcal{O}(h^{\frac{1-2\sigma}{\sigma+1}}) \\ & = \mathcal{O}(h^{\frac{1+2\sigma}{\sigma+1}}). \end{aligned}$$

This proves the theorem 1.
To prove theorem 2, Assuming moreover (A_5), the equation (14) becomes

$$\| \ w(H^i_{\theta_0}-z)^{-1} \ \| = \mathcal{O}(h^{\frac{p}{\sigma+1}}).$$

Then equation (12) becomes

$$\| \ (H_{\theta_0}-z)^{-1}w(H^i_{\theta_0}-z)^{-1}j_i \ \| = \mathcal{O}(h^{\frac{p-2\sigma}{\sigma+1}}).$$

Hence by equation (15) the following holds true:

$$\mid E-E^d \mid = \mathcal{O}(h^{\frac{p+2\sigma}{\sigma+1}}).$$

Therefore all eigenvalues of H_{θ_0} inside γ are inside the ball centered at E^d and of radius $\mathcal{O}(h^{\frac{p+2\sigma}{\sigma+1}})$. This in combination with the theorem 1 proves the theorem 2.

The proof of theorem 3 is similar to the proof of theorem (2.4) in [3], we have only to replace h by $h^{\frac{2\sigma}{\sigma+1}}$.

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