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PRINCIPAL MATRIX SOLUTIONS AND VARIATION OF PARAMETERS FOR A VOLTERRA INTEGRO-DIFFERENTIAL EQUATION AND ITS ADJOINT

LEIGH C. BECKER

Department of Mathematics, Christian Brothers University
650 E. Parkway South, Memphis, TN 38104, USA
E-mail: lbecker@cbu.edu

Abstract. We define the principal matrix solution $Z(t,s)$ of the linear Volterra vector integro-differential equation

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du$$

in the same way that it is defined for $x' = A(t)x$ and prove that it is the unique matrix solution of

$$\frac{\partial}{\partial t}Z(t,s) = A(t)Z(t,s) + \int_s^t B(t,u)Z(u,s)\,du, \quad Z(s,s) = I.$$

Furthermore, we prove that the solution of

$$x'(t) = A(t)x(t) + \int_\tau^t B(t,u)x(u)\,du + f(t), \quad x(\tau) = x_0$$

is unique and given by the variation of parameters formula

$$x(t) = Z(t,\tau)x_0 + \int_\tau^t Z(t,s)f(s)\,ds.$$

We also define the principal matrix solution $R(t,s)$ of the adjoint equation

$$r'(s) = -r(s)A(s) - \int_s^t r(u)B(u,s)\,du$$

and prove that it is identical to Grossman and Miller's resolvent, which is the unique matrix solution of

$$\frac{\partial}{\partial s}R(t,s) = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)\,du, \quad R(t,t) = I.$$

Finally, we prove that despite the difference in their definitions $R(t,s)$ and $Z(t,s)$ are in fact identical.

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1. Introduction: Resolvent vs Principal Matrix Solution

The variation of parameters formula

$$x(t) = R(t,0)x_0 + \int_0^t R(t,s)f(s)\,ds \tag{1.1}$$

gives the unique solution of the linear nonhomogeneous Volterra vector integro-differential equation

$$x'(t) = A(t)x(t) + \int_0^t B(t,u)x(u)\,du + f(t) \tag{1.2}$$

satisfying the initial condition $x(0) = x_0$. It has been around for at least 36 years: Grossman and Miller defined the matrix function $R(t,s)$, called the *resolvent*, and used it to derive (1.1) in 1970 in their classic paper [12]. They formally defined $R(t,s)$ by

$$R(t,s) = I + \int_s^t R(t,u)\Psi(u,s)\,du \quad (0 \le s \le t < \infty) \tag{1.3}$$

where I is the identity matrix and

$$\Psi(t,s) = A(t) + \int_s^t B(t,v)\,dv. \tag{1.4}$$

They proved that $R(t,s)$ exists and is continuous for $0 \le s \le t$ and that it satisfies

$$\frac{\partial}{\partial s}R(t,s) = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)\,du, \quad R(t,t) = I \tag{1.5}$$

on the interval $[0,t]$, for each $t > 0$. With this they were able to derive the variations of parameters formula (1.1) (cf. [12, Lemma 1, p. 459]).

Despite the prominence of the resolvent $R(t,s)$ in the literature and its indispensability, its definition (1.3) is not as conceptually simple as one would like. However, there is a more fundamental way to look at $R(t,s)$ and that is from the standpoint of linear systems of ODEs. In 1979 in my dissertation [1, Ch. II], results for (1.2) were obtained with this point of view. There the *principal matrix solution* $Z(t,s)$ of the homogeneous Volterra equation

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du \tag{1.6}$$

was first introduced. Its definition looks exactly like the definition of the principal matrix solution of the homogeneous vector differential equation

$$x'(t) = A(t)x(t)$$

that is given by Hale in [14, p. 80]: $Z(t,s)$ is a matrix solution of (1.6) with columns that are linearly independent such that $Z(s,s) = I$. Using $Z(t,s)$ instead of $R(t,s)$, the variation of parameters formula

$$x(t) = Z(t,0)x_0 + \int_0^t Z(t,s)f(s)\,ds \tag{1.7}$$

for (1.2) is a natural extension of the variation of parameters formula for the nonhomogeneous vector differential equation

$$x'(t) = A(t)x(t) + f(t).$$

The principal matrix version of the resolvent equation (1.5), namely,

$$\frac{\partial}{\partial t}Z(t,s) = A(t)Z(t,s) + \int_s^t B(t,u)Z(u,s)\,du, \quad Z(s,s) = I \tag{1.8}$$

has been instrumental in a number of papers for obtaining results that might not have otherwise been obtained with (1.5) alone.

The principal matrix solution $Z(t,s)$, the variation of parameters formula (1.7), and the principal matrix equation (1.8) are used and cited in papers by Becker et al. [3], Burton [6, 7], Eloe et al. [11], Islam and Raffoul [17], Raffoul [19], Hino and Murakami [20, 21], Zhang [22], and in the monographs [4, Ch. 7] and [8, Ch. 5] by T. A. Burton. Burton synopsizes some of the results from [1] in Section 7.1 of [4] and perceptively contrasts the difference in the definitions of the principal matrix solution $Z(t,s)$ and Grossman and Miller's resolvent $R(t,s)$. However, complete proofs of these results and concomitant definitions and applications have never been published—for that reason we do so now in Sections 2–5 of this paper.

Not found in [1] is an alternative to Grossman and Miller's definition of $R(t,s)$. It is this: $R(t,s)$ is the transpose of the principal matrix solution of the adjoint equation

$$y'(s) = -A^T(s)y(s) - \int_s^t B^T(u,s)y(u)\,du \tag{1.9}$$

for $0 \le s \le t$. Details are given in Section 6. The paper culminates with the proof in Section 7 that, notwithstanding the difference in their definitions, $Z(t,s)$ and $R(t,s)$ are identical.

2. Existence and Uniqueness

Let us begin by establishing the existence and uniqueness of solutions of the nonhomogeneous equation

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du + f(t), \tag{2.1}$$

where s is a fixed nonnegative number, A is an $n \times n$ matrix function that is continuous on $[0,\infty)$, B is an $n \times n$ matrix function that is continuous for $0 \le u \le t < \infty$, and f is a continuous n-vector function on $[0,\infty)$.

In [1, pp. 6–13], we established that solutions of (2.1) exist on $[s,\infty)$ and are unique by referring to the existence and uniqueness theorems in Driver [10, pp. 406–408] for the general Volterra functional differential equation

$$x'(t) = F(t, x(\cdot)). \tag{2.2}$$

However, here we present a proof that avoids such references; instead we change the initial value problem consisting of (2.1) and an initial condition $x(s) = x_0$ to an equivalent integral equation from which we construct a contraction mapping with a unique fixed point—which will be the unique solution. Burton [4, p. 221-222] does just that for the associated homogeneous equation (i.e., (2.1) with $f(t) \equiv 0$) by constructing a contraction mapping on a variant of the complete metric space $(C[a,b], \rho_r)$ that is described in the next paragraph. The proof for the nonhomogeneous equation (2.1) is essentially the same aside from an additional term: an integral of the forcing term $f(t)$. Nonetheless, we present the proof for the sake of clarity and unity with the rest of this paper. But first let us describe the metric space $(C[a,b], \rho_r)$.

Let $|\cdot|$ be any vector norm for $\mathbf{R}^n$. Let $|\cdot|$ also denote the matrix norm induced by the vector norm; that is, for an $n \times n$ matrix A

$$|A| = \sup\{\, |Ax| : |x| \le 1 \,\}.$$

Let $C[a,b]$ be the vector space of continuous functions $\phi\colon [a,b] \to \mathbf{R}^n$. For a fixed real number r, let $|\cdot|_r$ be the norm on $C[a,b]$ that is defined as follows: for $\phi \in C[a,b]$,

$$|\phi|_r := \sup\{\, |\phi(t)|e^{-r(t-a)} : a \le t \le b \,\}.$$

Let ρ_r denote the induced norm metric; that is, for $\phi, \eta \in C[a,b]$,

$$\rho_r(\phi,\eta) := |\phi - \eta|_r = \sup\{\, |\phi(t) - \eta(t)|e^{-r(t-a)} : a \le t \le b \,\}. \tag{2.3}$$

The space $C[a,b]$ with the metric ρ_r is complete, which we denote by $(C[a,b], \rho_r)$.

Definition 2.1. Let $x_0 \in \mathbf{R}^n$. A *solution* of (2.1) on the interval $[s,T)$, where $s < T \le \infty$, with the initial value x_0 at $t = s$ is a differentiable function $x\colon [s,T) \to \mathbf{R}^n$ that satisfies (2.1) on (s,T) and the initial condition $x(s) = x_0$.

Theorem 2.2. *For a given $x_0 \in \mathbf{R}^n$, there is a unique solution of*

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du + f(t)$$

on the interval $[s,\infty)$ satisfying the initial condition $x(s) = x_0$.

Proof. We begin by inverting (2.1) to obtain an equivalent integral equation from which we will be able to define a contraction mapping. Integrating (2.1) from s to t and replacing $x(s)$ with x_0, we get

$$x(t) = x_0 + \int_s^t A(v)x(v)\,dv + \int_s^t \int_s^v B(v,u)x(u)\,du\,dv + \int_s^t f(v)\,dv.$$

Interchanging the order of integration in the iterated integral, we have

$$x(t) = x_0 + \int_s^t \left[A(u) + \int_u^t B(v,u)\,dv\right] x(u)\,du + \int_s^t f(u)\,du. \tag{2.4}$$

This shows that a differentiable function $x(t)$ that satisfies (2.1) and the initial condition $x(s) = x_0$ also satisfies the integral equation (2.4). For such a function the integrand $B(v,u)x(u)$ is continuous, which justifies the interchange in the order of integration.

Conversely, if $x(t)$ is a continuous function that satisfies (2.4), then the integrands in (2.4) are continuous; as a result, $x(t)$ is also differentiable. Differentiating (2.4) with the aid of Leibniz's rule, we find that $x(t)$ also satisfies (2.1). Setting $t = s$ in (2.4), we have $x(s) = x_0$.

The point is that Theorem 2.2 is equivalent to the statement that there is a unique continuous function x that satisfies (2.4) on $[s,\infty)$ for a given $x_0 \in \mathbf{R}^n$. So proving the latter would prove Theorem 2.2. In other words, we need to prove that a unique continuous function x exists such that $x(t) = (Px)(t)$, where $(Px)(t)$ is the right-hand side of (2.4). To this end, choose any $T > s$ and let

$$C_{x_0}[s,T] := \{\, \phi \in C[s,T] : \phi(s) = x_0 \,\}.$$

That is, $C_{x_0}[s,T]$ is the set of continuous functions $\phi\colon [s,T] \to \mathbf{R}^n$ with $\phi(s) = x_0$. Now define the mapping P by

$$(2.5)\quad (P\phi)(t) := x_0 + \int_s^t \left[A(u) + \int_u^t B(v,u)\,dv \right] \phi(u)\,du + \int_s^t f(u)\,du$$

for all $\phi \in C_{x_0}[s,T]$. For each such ϕ, $P\phi$ is continuous on $[s,T]$. This and $(P\phi)(s) = x_0$ establishes that P maps $C_{x_0}[s,T]$ into itself.

For a fixed real number r, whose value will be addressed shortly, let $(C[s,T], \rho_r)$ be the complete metric space described earlier. Then $C_{x_0}[s,T]$ with the metric ρ_r is also complete since it is a closed subset of $C[s,T]$.

Now we can show that P is a contraction mapping on $C_{x_0}[s,T]$. For any $\phi, \eta \in C_{x_0}[s,T]$,

$$\begin{aligned}|(P\phi)(t) - (P\eta)(t)| &= \left| \int_s^t \left[A(u) + \int_u^t B(v,u)\,dv \right] (\phi(u) - \eta(u))\,du \right| \\ &\leq \int_s^t \left[|A(u)| + \int_u^t |B(v,u)|\,dv \right] |\phi(u) - \eta(u)|\,du.\end{aligned}$$

Since $A(t)$ and $B(t,u)$ are continuous for $s \leq u \leq t \leq T$, there is an $r > 1$ such that

$$|A(u)| + \int_u^t |B(v,u)|\,dv \leq r - 1.$$

For such an r,

$$|(P\phi)(t) - (P\eta)(t)| \leq \int_s^t (r-1)\,|\phi(u) - \eta(u)|\,du.$$

Thus,

$$\begin{aligned}&|(P\phi)(t) - (P\eta)(t)|e^{-r(t-s)} \\ &\qquad \leq \int_s^t (r-1)e^{-r(t-s)+r(u-s)}\,|\phi(u) - \eta(u)|e^{-r(u-s)}\,du \\ &\qquad\qquad \leq |\phi - \eta|_r \int_s^t (r-1)e^{-r(t-u)}\,du \leq \frac{r-1}{r}|\phi - \eta|_r.\end{aligned}$$

From this it follows that

$$\rho_r(P\phi, P\eta) \leq \frac{r-1}{r}\,\rho_r(\phi, \eta).$$

By Banach's contraction mapping principle, P has a unique fixed point in $C_{x_0}[s,T]$. It follows that there is a unique continuous solution x of (2.4) on $[s,\infty)$ since T was arbitrarily chosen from (s,∞). $\square$

3. Joint Continuity

For a given $x_0 \in \mathbf{R}^n$, the homogeneous equation

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du \tag{3.1}$$

has a unique solution x_s satisfying the initial condition $x_s(s) = x_0$ by Theorem 2.2 (with $f(t) \equiv 0$). Equivalently, by (2.4), x_s is the unique continuous solution of

$$x(t) = x_0 + \int_s^t \Phi(t,u)x(u)\,du \tag{3.2}$$

where

$$\Phi(t,u) := A(u) + \int_u^t B(v,u)\,dv. \tag{3.3}$$

Up to now the value of s has been fixed. But with that restriction removed, the totality of values $x_s(t)$ defines a function, x say, on the set

$$\Omega := \{\,(t,s) : 0 \le s \le t < \infty\,\}$$

whose value at $(t_1,s_1) \in \Omega$ is the value of the solution x_{s_1} at $t = t_1$.

Definition 3.1. For a given $x_0 \in \mathbf{R}^n$, let x denote the function with domain Ω whose value at (t,s) is

$$x(t,s) := x_s(t) \tag{3.4}$$

where x_s is the unique solution of (3.1) on $[s,\infty)$ satisfying the initial condition $x_s(s) = x_0$.

Since $x(t,s)$ is continuous in t for a fixed s, it is natural to ask if it is also continuous in s for a fixed t—and if so, is it jointly continuous in t and s? The next theorem answers both of these in the affirmative. This will play an essential role in the proof of the variation of parameters formula for (2.1) that is given in Section 5.

Theorem 3.2. *The function $x(t,s)$ defined by* (3.4) *is continuous for $0 \le s \le t < \infty$.*

Proof. First extend the domain Ω of the function x to the entire plane by defining $x(t,s) = x_0$ for $s > t$. For any $T > 0$, consider $x(t,s)$ on $[0,T] \times [0,T]$. We will prove $x(t,s)$ is continuous in s uniformly for $t \in [0,T]$, which means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|s_1 - s_2| < \delta$ implies that

$$|x(t,s_1) - x(t,s_2)| < \epsilon \tag{3.5}$$

for all $s_1, s_2 \in [0,T]$ and all $t \in [0,T]$. This and the continuity of $x(t,s)$ in t for each fixed s would establish that $x(t,s)$ is jointly continuous in both variables on the set $[0,T] \times [0,T]$ by the Moore-Osgood theorem (cf. [13, Thm. 5, p. 102], [16, p. 13], or [18, Ex. 31, p. 310]).

Proving (3.5) will require bounds for $x(t,s)$. For a fixed $s \in [0,T]$ and for $t \in [s,T]$, we see from (3.2) that

$$\begin{aligned} |x(t,s)| &\leq |x_0| + \int_s^t |\Phi(t,u)|\,|x(u,s)|\,du \\ &\leq |x_0| + \int_s^t \left[|A(u)| + \int_u^t |B(v,u)|\,dv\right] |x(u,s)|\,du \\ &\leq |x_0| + \int_s^t k\,|x(u,s)|\,du, \end{aligned} \tag{3.6}$$

where k is a constant chosen so that

$$|\Phi(t,u)| \leq |A(u)| + \int_u^t |B(v,u)|\,dv \leq k \tag{3.7}$$

for $0 \leq u \leq t \leq T$. By Gronwall's inequality,

$$|x(t,s)| \leq |x_0| e^{\int_s^t k\,du} = |x_0| e^{k(t-s)}$$

for $0 \leq s \leq t \leq T$. Since $|x(t,s)| = |x_0|$ for $s > t$, we have

$$|x(t,s)| \leq |x_0| e^{kT} \tag{3.8}$$

for all $(t,s) \in [0,T] \times [0,T]$.

With the aid of (3.8) we now prove (3.5). For definiteness, suppose $s_2 > s_1$. For $t \in [0,s_1]$,

$$|x(t,s_1) - x(t,s_2)| = 0 \tag{3.9}$$

as $x(t,s) = x_0$ for $t \leq s$.

For $t \in (s_1, s_2]$, it follows from (3.2) and (3.7) that

$$|x(t,s_1) - x(t,s_2)| = |x(t,s_1) - x_0| \le \int_{s_1}^{t} |\,\Phi(t,u)\,|\,|x(u,s_1)|\,du$$
$$\le \int_{s_1}^{s_2} |\,\Phi(t,u)\,|\,|x(u,s_1)|\,du \le \int_{s_1}^{s_2} k|x(u,s_1)|\,du.$$

Then by (3.8),

$$|x(t,s_1) - x(t,s_2)| \le \int_{s_1}^{s_2} k|x_0|e^{kT}du = k|x_0|e^{kT}(s_2 - s_1). \tag{3.10}$$

For $t \in (s_2, T]$, we have

$$|x(t,s_1) - x(t,s_2)| = \left| \int_{s_1}^{t} \Phi(t,u)x(u,s_1)\,du - \int_{s_2}^{t} \Phi(t,u)x(u,s_2)\,du \right|$$
$$= \left| \int_{s_1}^{t} \Phi(t,u)x(u,s_1)\,du - \int_{s_2}^{t} \Phi(t,u)x(u,s_1)\,du \right.$$
$$\left. + \int_{s_2}^{t} \Phi(t,u)x(u,s_1)\,du - \int_{s_2}^{t} \Phi(t,u)x(u,s_2)\,du \right|$$
$$\le \int_{s_1}^{s_2} |\,\Phi(t,u)\,|\,|x(u,s_1)|\,du + \int_{s_2}^{t} |\,\Phi(t,u)\,|\,|x(u,s_1) - x(u,s_2)|\,du$$
$$\le \int_{s_1}^{s_2} k\,|x(u,s_1)|\,du + \int_{s_2}^{t} k\,|x(u,s_1) - x(u,s_2)|\,du.$$

Applying (3.8) again,

$$|x(t,s_1) - x(t,s_2)| \le k|x_0|e^{kT}(s_2 - s_1) + \int_{s_2}^{t} k\,|x(u,s_1) - x(u,s_2)|\,du.$$

By (3.10), this holds at $t = s_2$ as well. Therefore, for $t \in [s_2, T]$,

$$|x(t,s_1) - x(t,s_2)| \le k|x_0|e^{kT}(s_2 - s_1)e^{k(t-s_2)} \tag{3.11}$$

by Gronwall's inequality.

It follows from (3.9)–(3.11) that

$$|x(t,s_1) - x(t,s_2)| \le k|x_0|e^{2kT}(s_2 - s_1) \tag{3.12}$$

for all $t \in [0, T]$ and $s_2 > s_1$. Of course, it is also true for $s_2 = s_1$.

We conclude

$$|x(t,s_1) - x(t,s_2)| \le k|x_0|e^{2kT}|s_1 - s_2| \tag{3.13}$$

for all $s_1, s_2 \in [0,T]$ and $t \in [0,T]$, which implies (3.5). Therefore, $x(t,s)$ is continuous on $[0,T] \times [0,T]$. Since T is arbitrary, $x(t,s)$ is continuous on $[0,\infty) \times [0,\infty)$, a fortiori, for $0 \le s \le t < \infty$. □

4. Principal Matrix Solution

For a fixed $s \ge 0$, let S denote the set of all solutions of (3.1) on the interval $[s,\infty)$ that correspond to initial vectors. Let $x(t,s)$ and $\tilde{x}(t,s)$ be two such solutions satisfying the initial conditions $x(s,s) = x_0$ and $\tilde{x}(s,s) = x_1$, respectively. Linearity of (3.1) implies the *principle of superposition*, namely, that the linear combination $c_1 x(t,s) + c_2 \tilde{x}(t,s)$ is a solution of (3.1) on $[s,\infty)$ for any $c_1, c_2 \in \mathbf{R}$. Consequently, the set S is a vector space. Note that S comprises all solutions that have their initial values specified at $t = s$, but not those for which an initial function is specified on an initial interval $[s, t_0]$ for some $t_0 > s$.

Theorem 4.1. *For a fixed $s \in [0,\infty)$, let S be the set of all solutions of* (3.1) *on the interval $[s,\infty)$ corresponding to initial vectors. Then S is an n-dimensional vector space.*

Proof. We have already established that S is a vector space. To complete the proof, we must find n linearly independent solutions spanning S. To this end, let $e^1, \ldots, e^n$ be the standard basis for $\mathbf{R}^n$, where e^i is the vector whose ith component is 1 and whose other components are 0. By Theorem 2.2, there are n unique solutions $x^i(t,s)$ of (3.1) on $[s,\infty)$ with $x^i(s,s) = e^i$ $(i = 1, \ldots, n)$. By the usual argument, these solutions are linearly independent.

To show they span S, choose any $x(t,s) \in S$. Suppose its value at $t = s$ is the vector x_0. Let $\xi_1, \ldots, \xi_n$ be the unique scalars such that $x_0 = \xi_1 e^1 + \cdots + \xi_n e^n$. By the principle of superposition, the linear combination

$$\xi_1 x^1(t,s) + \cdots + \xi_n x^n(t,s) = \sum_{i=1}^{n} \xi_i x^i(t,s) \tag{4.1}$$

is a solution of (3.1). Since its value at $t = s$ is x_0, the uniqueness part of Theorem 2.2 implies

$$x(t,s) = \sum_{i=1}^{n} \xi_i x^i(t,s). \tag{4.2}$$

Hence, the n solutions $x^1(t,s), \ldots, x^n(t,s)$ span S. This and their linear independence make them a basis for S. □

If we define an $n \times n$ matrix function $Z(t,s)$ by

$$Z(t,s) := \begin{bmatrix} x^1(t,s) & x^2(t,s) & \cdots & x^n(t,s) \end{bmatrix}, \tag{4.3}$$

where the columns $x^1(t,s), \ldots, x^n(t,s)$ are the basis for S defined in the proof of Theorem 4.1, then (4.2) can be written as

$$x(t,s) = Z(t,s)x_0. \tag{4.4}$$

Since $x^i(s,s) = e^i$,

$$Z(s,s) = I, \tag{4.5}$$

the $n \times n$ identity matrix.

If B is the zero matrix, then the columns of $Z(t,s)$ become linearly independent solutions of the ordinary vector differential equation

$$x'(t) = A(t)x(t). \tag{4.6}$$

This makes $Z(t,s)$ a *fundamental matrix solution* of (4.6) (cf. [9, p. 64], [14, p. 80]). In fact, because $Z(s,s) = I$, it is the so-called *principal matrix solution* (cf. [5, p. 29], [14, p. 80]). These terms are also used for the integro-differential equation (3.1) (cf. [1, p. 10], [5, p. 78], [20]).

Definition 4.2. The *principal matrix solution* of (3.1) is the $n \times n$ matrix function $Z(t,s)$ defined by (4.3). In other words, $Z(t,s)$ is a matrix with n columns that are linearly independent solutions of (3.1) and whose value at $t = s$ is the identity matrix I.

Remark 4.3. An alternative term from integral equations is *resolvent* (cf. [4, Sec 7.1], [6], [8, Sec. 5.2]), an apt term in view of (4.2), which states that every solution of (3.1) can be resolved into the n columns constituting $Z(t,s)$.

Theorem 3.2 implies that each of the columns $x^i(t,s)$ of $Z(t,s)$ are continuous for $0 \le s \le t < \infty$. Consequently, we have the following.

Theorem 4.4. *$Z(t,s)$, the principal matrix solution of equation* (3.1)*, is continuous for* $0 \le s \le t < \infty$.

Since the ith column of $Z(t,s)$ is the unique solution of (3.1) whose value at $t = s$ is e^i, $Z(t,s)$ is the unique matrix solution of the initial value problem

$$\frac{\partial}{\partial t} Z(t,s) = A(t)Z(t,s) + \int_s^t B(t,u)Z(u,s)\,du, \quad Z(s,s) = I \tag{4.7}$$

for $0 \le s \le t < \infty$. Equivalently, it is the unique matrix solution of

$$Z(t,s) = I + \int_s^t \left[A(u) + \int_u^t B(v,u)\,dv\right] Z(u,s)\,du \tag{4.8}$$

by (3.2) and (3.3). Note that this is the principal matrix counterpart of Grossman and Miller's resolvent equation (1.3).

5. Variation of Parameters Formula

Let $X(t)$ be any fundamental matrix solution of the homogeneous differential equation

$$x'(t) = A(t)x(t). \tag{5.1}$$

By definition, the columns of a fundamental matrix solution $X(t)$ are linearly independent solutions of (5.1). So for $c \in \mathbf{R}^n$, $x(t) = X(t)c$ is a solution of (5.1) by the principle of superposition. If $x(\tau) = x_0$, then $X(\tau)c = x_0$. Since $X(\tau)$ is nonsingular (cf. [9, p. 62]), the unique solution $x(t)$ of (5.1) satisfying $x(\tau) = x_0$ is

$$x(t) = X(t)X^{-1}(\tau)x_0. \tag{5.2}$$

Now compare (5.2) to the unique solution of the nonhomogeneous equation

$$x'(t) = A(t)x(t) + f(t) \tag{5.3}$$

satisfying $x(\tau) = x_0$. The method of variation of parameters applied to (5.3) (cf. [9, p. 65]) yields the following well-known formula for the solution

$$x(t) = X(t)X^{-1}(\tau)x_0 + \int_\tau^t X(t)X^{-1}(s)f(s)\,ds. \tag{5.4}$$

Of course, (5.4) reduces to (5.2) if $f \equiv 0$.

As for the integro-differential equation (3.1), the counterpart of (5.2) is (4.4), which is stated next as a lemma.

Lemma 5.1. *The solution of*

$$x'(t) = A(t)x(t) + \int_\tau^t B(t,u)x(u)\,du \quad (\tau \ge 0) \tag{5.5}$$

on $[\tau, \infty)$ *satisfying the initial condition* $x(\tau) = x_0$ *is*

$$x(t) = Z(t,\tau)x_0, \tag{5.6}$$

where $Z(t,\tau)$ *is the principal matrix solution of* (5.5).

Suppose $B \equiv 0$ (zero matrix). Then (5.2), (5.6), and uniqueness of solutions imply that

$$Z(t,\tau) = X(t)X^{-1}(\tau).$$

In that case, the variation of parameters formula (5.4) simplifies to

$$x(t) = Z(t,\tau)x_0 + \int_\tau^t Z(t,s)f(s)\,ds. \tag{5.7}$$

Lemma 5.1 extends a classical result for the homogeneous differential equation (5.1) to the homogeneous integro-differential equation (3.1). This suggests that a variation of parameters formula similar to (5.7) may also hold for the nonhomogeneous integro-differential equation (2.1).

The essential element in the derivation of the variation of parameters formula (5.4) is the nonsingularity of $X(t)$ for each t. If the same were true of the principal matrix solution $Z(t,s)$ of (3.1), then a variation of parameters formula could be derived for (2.1) as well. In fact, as Theorem 5.2 shows, there are examples of (3.1) other than (5.1) for which $\det Z(t,s)$ is never zero.

Theorem 5.2. (Becker [2, Cor. 3.4]) *Assume $a, b\colon [0,\infty) \to \mathbf{R}$ are continuous functions and $b(t) \geq 0$ on $[0,\infty)$. Let $x(t)$ be the unique solution of the scalar equation*

$$x'(t) = -a(t)x(t) + \int_s^t b(t-u)x(u)\,du \quad (s \geq 0) \tag{5.8}$$

on $[s,\infty)$ satisfying the initial condition $x(s) = x_0$. If $x_0 \geq 0$, then

$$x_0 e^{-\int_s^t a(u)\,du} \leq x(t) \leq x_0 e^{-\int_s^t p(u)\,du} \tag{5.9}$$

for all $t \geq s$, where

$$p(u) := a(u) - \int_0^{u-s} e^{\int_{u-v}^u a(r)\,dr} b(v)\,dv. \tag{5.10}$$

It follows that the *principal solution* $x(t,s)$ of (5.8) (i.e., the solution whose value at $t = s$ is 1) is always positive. In our notation, $Z(t,s)$ is the 1×1 matrix $[x(t,s)]$ and so

$$\det Z(t,s) = x(t,s) > 0$$

for all $t \geq s \geq 0$.

However, unlike the differential equation (5.1), the principal matrix solution of the integro-differential equation (3.1) may be singular at points as the next theorem of Burton shows.

Theorem 5.3. (Burton [5, p. 86]) *Assume $a \geq 0$ and $b\colon [0,\infty) \to \mathbf{R}$ is continuous, where $b(t) \leq 0$ on $[0,\infty)$. Let $x(t)$ be the unique solution of*

$$x'(t) = -ax(t) + \int_0^t b(t-u)x(u)\,du \tag{5.11}$$

satisfying the initial condition $x(0) = 1$. If there exists a $t_1 > 0$ such that

$$\int_{t_1}^t \int_0^{t_1} b(v-u)\,du\,dv \to -\infty \tag{5.12}$$

as $t \to \infty$, then there exists a $t_2 > 0$ such that $x(t_2) = 0$.

Theorem 5.3 establishes that the determinant of the principal matrix solution $Z(t,s)$ of (3.1) may vanish. Consequently, unlike the nonhomogeneous differential equation (5.3), we cannot derive a formula like (5.7) in general for the nonhomogeneous integro-differential equation (2.1) by directly applying the method of variation of parameters to it. Nevertheless, in the next proof we use uniqueness of solutions to verify that the function given by the variation of parameters formula (5.14) always satisfies (5.13), regardless of the values of $\det Z(t,s)$.

Theorem 5.4. (Variation of Parameters) *The solution of*

$$x'(t) = A(t)x(t) + \int_\tau^t B(t,u)x(u)\,du + f(t) \quad (\tau \geq 0) \tag{5.13}$$

on $[\tau,\infty)$ satisfying the initial condition $x(\tau) = x_0$ is

$$x(t) = Z(t,\tau)x_0 + \int_\tau^t Z(t,s)f(s)\,ds, \tag{5.14}$$

where $Z(t,s)$ is the principal matrix solution of

$$x'(t) = A(t)x(t) + \int_s^t B(t,u)x(u)\,du.$$

Proof. By Theorem 2.2, there is a unique solution $x(t)$ of (5.13) on $[\tau,\infty)$ such that $x(\tau) = x_0$. Let us show that

$$\varphi(t) := Z(t,\tau)x_0 + \int_\tau^t Z(t,s)f(s)\,ds \tag{5.15}$$

is also a solution of (5.13) by differentiating it. To this end, define $Z(t,s) = I$ for $s > t$. Then $Z(t,s)$ is continuous on $[0,\infty) \times [0,\infty)$ by Theorem 4.4. This and (4.7) imply the same is true of its partial derivative $Z_t(t,s)$. Consequently, the integral term in (5.15) is differentiable by Leibniz's rule. Differentiating $\varphi(t)$, we obtain

$$\varphi'(t) = \left[A(t)Z(t,\tau) + \int_\tau^t B(t,u)Z(u,\tau)\,du \right] x_0 \\ + Z(t,t)f(t) + \int_\tau^t \frac{\partial}{\partial t} Z(t,s)f(s)\,ds$$

by (4.7) and Leibniz's rule. Applying (4.7) again, we have

$$\begin{aligned} \varphi'(t) &= A(t)Z(t,\tau)x_0 + \int_\tau^t B(t,u)Z(u,\tau)x_0\,du + If(t) \\ &\quad + \int_\tau^t \left[A(t)Z(t,s) + \int_s^t B(t,u)Z(u,s)\,du \right] f(s)\,ds \\ &= f(t) + A(t)\left[Z(t,\tau)x_0 + \int_\tau^t Z(t,s)f(s)\,ds \right] \\ &\quad + \int_\tau^t B(t,u)Z(u,\tau)x_0\,du + \int_\tau^t \int_s^t B(t,u)Z(u,s)f(s)\,du\,ds. \end{aligned}$$

An interchange in the order of integration yields

$$\varphi'(t) = f(t) + A(t)\varphi(t) + \int_\tau^t B(t,u)Z(u,\tau)x_0\,du \\ + \int_\tau^t \int_\tau^u B(t,u)Z(u,s)f(s)\,ds\,du,$$

which simplifies to

$$\begin{aligned} \varphi'(t) &= f(t) + A(t)\varphi(t) \\ &\quad + \int_\tau^t B(t,u)\left[Z(u,\tau)x_0 + \int_\tau^u Z(u,s)f(s)\,ds \right] du \\ &= f(t) + A(t)\varphi(t) + \int_\tau^t B(t,u)\varphi(u)\,du. \end{aligned}$$

Thus, $\varphi(t)$ is a solution on $[\tau,\infty)$. By (5.15), $\varphi(\tau) = x_0$. Therefore, $x(t) \equiv \varphi(t)$ on $[\tau,\infty)$ by uniqueness of solutions. $\square$

Note that (5.14) reduces to (5.6) when $f \equiv 0$.

Corollary 5.5. *Let $\varphi \in C[0,\tau]$ for any $\tau > 0$. The solution of*

$$x'(t) = A(t)x(t) + \int_0^t B(t,u)x(u)\,du + f(t) \tag{5.16}$$

on $[\tau,\infty)$ satisfying the condition $x(t) = \varphi(t)$ for $0 \le t \le \tau$ is

$$\begin{aligned} x(t) = Z(t,\tau)\varphi(\tau) &+ \int_\tau^t Z(t,s)f(s)\,ds \\ &+ \int_\tau^t Z(t,s)\left\{\int_0^\tau B(s,u)\varphi(u)\,du\right\}ds. \end{aligned} \tag{5.17}$$

Proof. Since $x(t) \equiv \varphi(t)$ on $[0,\tau]$, we can rewrite (5.16) as follows:

$$x'(t) = A(t)x(t) + \int_\tau^t B(t,u)x(u)\,du + g(t) \tag{5.18}$$

where

$$g(t) := f(t) + \int_0^\tau B(t,u)\varphi(u)\,du. \tag{5.19}$$

By Theorem 2.2, equation (5.18) has a unique solution on $[\tau,\infty)$ such that $x(\tau) = \varphi(\tau)$. By the variation of parameters formula (5.14), the solution is

$$x(t) = Z(t,\tau)\varphi(\tau) + \int_\tau^t Z(t,s)g(s)\,ds,$$

which is (5.17). □

6. The Adjoint Equation

The differential equation

$$y'(t) = -A^T(t)y(t), \tag{6.1}$$

where A^T is the transpose of A, is the so-called adjoint to (5.1). The associated nonhomogeneous adjoint equation (cf. [15, p. 62]) is

$$y'(t) = -A^T(t)y(t) - g(t). \tag{6.2}$$

Let us extend this definition to the integro-differential equation (2.1).

Definition 6.1. The *adjoint* to (2.1) is

$$y'(s) = -A^T(s)y(s) - \int_s^t B^T(u,s)y(u)\,du - g(s) \tag{6.3}$$

where $s \in [0,t]$.

The next theorem establishes that solutions of (6.3) do exist and are unique.

Theorem 6.2. *For a fixed $t > 0$ and a given $y_0 \in \mathbf{R}^n$, there is a unique solution $y(s)$ of*

$$y'(s) = -A^T(s)y(s) - \int_s^t B^T(u,s)y(u)\,du - g(s)$$

on the interval $[0,t]$ satisfying the condition $y(t) = y_0$.

Proof. The objective, as it was in proving Theorem 2.2, is to find a suitable contraction mapping. To this end, integrate (6.3) from s to t:

$$y(t) - y(s) = -\int_s^t A^T(v)y(v)\,dv \\ - \int_s^t \int_v^t B^T(u,v)y(u)\,du\,dv - \int_s^t g(v)\,dv.$$

Replacing $y(t)$ with y_0 and interchanging the order of integration, this becomes

$$y(s) = y_0 + \int_s^t A^T(v)y(v)\,dv + \int_s^t \int_s^u B^T(u,v)y(u)\,dv\,du + \int_s^t g(v)\,dv$$

or

$$y(s) = y_0 + \int_s^t \left[A^T(u) + \int_s^u B^T(u,v)\,dv\right] y(u)\,du + \int_s^t g(u)\,du. \tag{6.4}$$

Clearly, the appropriate set of functions on which to define a mapping is

$$C_{y_0}[0,t] := \{\, \phi \in C[0,t] : \phi(t) = y_0 \,\}.$$

Now define the mapping $\tilde{P}$ by

$$(\tilde{P}\phi)(s) := y_0 + \int_s^t \left[A^T(u) + \int_s^u B^T(u,v)\,dv\right] \phi(u)\,du + \int_s^t g(u)\,du$$

for all $\phi \in C_{y_0}[0,t]$.

For a given $\phi \in C_{y_0}[0,t]$, it is apparent that $\tilde{P}\phi$ is continuous on $[0,t]$ and that $(\tilde{P}\phi)(t) = y_0$. Thus, $\tilde{P} \colon C_{y_0}[0,t] \to C_{y_0}[0,t]$.

For an arbitrary pair of functions $\phi, \eta \in C_{y_0}[0,t]$,

$$\begin{aligned} |(\tilde{P}\phi)(s) - (\tilde{P}\eta)(s)| &= \left| \int_s^t \left[A^T(u) + \int_s^u B^T(u,v)\,dv \right] (\phi(u) - \eta(u))\,du \right| \\ &\le \int_s^t \left[|A^T(u)| + \int_s^u |B^T(u,v)|\,dv \right] |\phi(u) - \eta(u)|\,du. \end{aligned}$$

So if $r > 1$ is chosen so that

$$|A^T(u)| + \int_s^u |B^T(u,v)|\,dv \le r - 1$$

for $0 \le s \le u \le t$, then

$$|(\tilde{P}\phi)(s) - (\tilde{P}\eta)(s)| \le \int_s^t (r-1)\,|\phi(u) - \eta(u)|\,du. \tag{6.5}$$

The proof of Theorem 2.2 takes place in the complete metric space $(C_{x_0}[s,T], \rho_r)$, where s is fixed and t varies. But now that s varies and t is fixed, let us alter the metric ρ_r (cf. (2.3)) slightly in order to show that $\tilde{P}$ is a contraction mapping: replacing $-r$ with r yields the metric

$$d_r(\phi, \eta) := \sup\{\, |\phi(s) - \eta(s)| e^{rs} : 0 \le s \le t \,\}.$$

The metric space $(C_{y_0}[0,t], d_r)$ is complete (for the same reason that $(C_{x_0}[s,T], \rho_r)$ is).

What remains then is to show that $\tilde{P}$ is a contraction on $C_{y_0}[0,t]$. Returning to (6.5), we have

$$\begin{aligned} |(\tilde{P}\phi)(s) - (\tilde{P}\eta)(s)| e^{rs} &\le \int_s^t (r-1) e^{rs - ru}\, |\phi(u) - \eta(u)| e^{ru}\,du \\ &\le d_r(\phi,\eta) \int_s^t (r-1) e^{r(s-u)}\,du \\ &\le \frac{r-1}{r}\, d_r(\phi,\eta). \end{aligned}$$

Hence,

$$d_r(P\phi, P\eta) \le \frac{r-1}{r}\, d_r(\phi,\eta).$$

Therefore, $\tilde{P}$ has a unique fixed point in $C_{y_0}[0,t]$, which translates to the existence of a unique solution of (6.4) on the interval $[0,t]$. □

Definition 6.3. The *principal matrix solution* of

$$y'(s) = -A^T(s)y(s) - \int_s^t B^T(u,s) y(u)\,du \tag{6.6}$$

is the $n \times n$ matrix function

$$Q(t,s) := \begin{bmatrix} y^1(t,s) & y^2(t,s) & \cdots & y^n(t,s) \end{bmatrix}, \tag{6.7}$$

where $y^i(t,s)$ (t fixed) is the unique solution of (6.6) on $[0,t]$ that satisfies the condition $y^i(t,t) = e^i$.

By virtue of this definition, $Q(t,s)$ is the unique matrix solution of

$$\frac{\partial}{\partial s} Q(t,s) = -A^T(s)Q(t,s) - \int_s^t B^T(u,s)Q(t,u)\,du,\ Q(t,t) = I \tag{6.8}$$

on the interval $[0,t]$. Reasoning as in the proof of Theorem 4.1, we conclude that for a given $y_0 \in \mathbf{R}^n$, the unique solution of (6.6) satisfying the condition $y(t) = y_0$ is

$$y(s) = Q(t,s)y_0 \tag{6.9}$$

for $0 \le s \le t$.

Taking the transpose of (6.6) and letting $r(s)$ be the row vector $y^T(s)$, we obtain

$$r'(s) = -r(s)A(s) - \int_s^t r(u)B(u,s)\,du.$$

The solution satisfying the condition $r(t) = y_0^T =: r_0$ is the transpose of (6.9), namely,

$$y^T(s) = y_0^T Q^T(t,s) \tag{6.10}$$

or

$$r(s) = r_0 R(t,s)$$

where

$$R(t,s) := Q^T(t,s). \tag{6.11}$$

Consequently, $R(t,s)$ is the principal matrix solution of the transposed equation. As a result, Lemma 5.1 has the following adjoint counterpart.

Lemma 6.4. *The solution of*

$$r'(s) = -r(s)A(s) - \int_s^t r(u)B(u,s)\,du \tag{6.12}$$

on $[0,t]$ satisfying the condition $r(t) = r_0$ is

$$r(s) = r_0 R(t,s), \tag{6.13}$$

where $R(t,s)$ is the principal matrix solution of (6.12).

It follows from (6.8) that $R(t,s)$ is the unique matrix solution of

$$\frac{\partial}{\partial s}R(t,s) = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)\,du, \quad R(t,t) = I \tag{6.14}$$

on the interval $[0,t]$. Moreover, it is the unique matrix solution of

$$R(t,s) = I + \int_s^t R(t,u)\left[A(u) + \int_s^u B(u,v)\,dv\right]du \tag{6.15}$$

for $0 \leq s \leq t < \infty$, which is derived by integrating (6.14) from s to t and then interchanging the order of integration.

Now it becomes apparent from comparing (6.14) and (6.15) to (1.5) and (1.3), respectively, that the principal matrix solution of the adjoint equation (6.12) is identical to Grossman and Miller's resolvent.

7. Equivalence of $R(t,s)$ and $Z(t,s)$

The solutions of (3.1) and its adjoint

$$r'(s) = -r(s)A(s) - \int_s^t r(u)B(u,s)\,du$$

are related via the equation

$$\frac{\partial}{\partial u}[r(u)Z(u,s)] = r(u)\frac{\partial}{\partial u}Z(u,s) + r'(u)Z(u,s) \tag{7.1}$$

for $0 \leq s \leq u \leq t$. We exploit this to prove that the principal matrix solution and Grossman and Miller's resolvent are one and the same.

Theorem 7.1. *$R(t,s) \equiv Z(t,s)$.*

Proof. Select any $t > 0$. For a given row n-vector r_0, let $r(s)$ be the unique solution of (6.12) on $[0,t]$ such that $r(t) = r_0$. Now integrate both sides of (7.1) from s to t:

$$r(t)Z(t,s) - r(s)Z(s,s) = \int_s^t [r(u)Z_u(u,s) + r'(u)Z(u,s)]\,du.$$

By (6.12), we have

$$\begin{aligned} r_0 Z(t,s) - r(s) = \int_s^t \Bigg[& r(u)Z_u(u,s) - r(u)A(u)Z(u,s) \\ & - \left(\int_u^t r(v)B(v,u)\,dv\right) Z(u,s)\Bigg]\,du. \end{aligned} \tag{7.2}$$

With an interchange in the order of integration, the iterated integral becomes

$$(7.3)\quad \int_s^t \left(\int_u^t r(v)B(v,u)\,dv \right) Z(u,s)\,du$$

$$= \int_s^t r(v) \left(\int_s^v B(v,u)Z(u,s)\,du \right) dv$$

$$= \int_s^t r(u) \left(\int_s^u B(u,v)Z(v,s)\,dv \right) du.$$

Making this change in (7.2), we obtain

$$(7.4)\quad r_0 Z(t,s) - r(s) = \int_s^t r(u) \Big[Z_u(u,s) - A(u)Z(u,s) - \int_s^u B(u,v)Z(v,s)\,dv \Big]\,du.$$

By (4.7), the integrand is zero. Hence,

$$(7.5)\qquad r(s) = r_0 Z(t,s).$$

On the other hand,

$$r(s) = r_0 R(t,s)$$

by (6.13). Therefore, by uniqueness of the solution $r(s)$,

$$(7.6)\qquad r_0 R(t,s) = r_0 Z(t,s).$$

Now let r_0 be the transpose of the ith basis vector e^i. Then (7.6) implies that the ith rows of $R(t,s)$ and $Z(t,s)$ are equal for $0 \le s \le t$. The theorem follows as t is arbitrary. □

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