Electronic Journal of Qualitative Theory of Differential Equations
2006, No. **10**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

Integral Criteria for Second-Order Linear Oscillation

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Abstract

We present several new criteria for the oscillation of the second-order linear equation $y''(t) + q(t)y(t) = 0$, in which the coefficient q may or may not change signs. The criteria involve the integral $\int t^\gamma q(t)\, dt$ for some $\gamma > 0$. The special case $\gamma = 2$ is then studied in greater details.

AMS Subject Classification: 34C10

Keywords: oscillation second-order linear ordinary differential equation

1 Introduction

The second-order linear differential equation

$$y''(t) + q(t)y(t) = 0, \quad t > 0 \tag{1.1}$$

is said to be oscillatory if every nontrivial solution has an infinite number of zeros in $[0, \infty)$. Many oscillation criteria are known, covering a wide variety of interesting examples. Yet new identities and/or better approaches are still being discovered and they often find applications in other areas of the qualitative study of differential equations, especially in nonlinear oscillation.

Many of the classical oscillation criteria make use of the integral of the coefficient $q(t)$. In this paper we give some new oscillation criteria involving the integral of $t^\gamma q(t)$, $\gamma > 0$. The basic tools are Riccati integral equations and the theory of integral inequalities.

In this paper, oscillation criteria are stated using a combination of conditions denoted by (Cn) and nonoscillation criteria by conditions denoted by (Nn). Some examples are

$$q(t) \geq 0. \tag{C0}$$

$$\int_0^\infty t^\gamma q(t)\, dt = \infty, \text{ for some } \gamma \in [0, 1). \tag{C1}$$

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The oscillation criterion (C0) + (C1) has been attributed to Hille [3], Hartman [2], and Wintner [7]. (the special case $\gamma = 0$ is the classical Fite criterion), and extended to nonlinear equations by Wong [9]. In Section 5, we will show that just (C1) alone suffices to imply oscillation. This simple result appears to be new.

In [5], we established a new oscillation criterion: (C0) + (C2).

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T t^2 q(t)\,dt > 1. \tag{C2}$$

We also showed that (C1) implies (C2). The purpose of this paper is to derive more oscillation criteria of a similar type and also extend them to the more interesting situation in which (C0) no longer holds, namely, that $q(t)$ changes sign.

Section 2 is devoted to the derivation of some Riccati equations that are equivalent to (1.1). In Section 3, we collect some basic results in the theory of integral inequalities that we need in the proof of our theorems.

In Section 4, we take up the simpler situation in which (C0) holds, and extend the criterion (C2) to use the general multiplier t^γ, $\gamma > 1$. We complement (C1) with a criterion in which the integral in (C1) is assumed finite, but $\limsup T^{1-\gamma}\int_T^\infty t^\gamma q(t)\,dt$ is sufficiently large.

In Section 5, these results are extended to the case when $q(t)$ is allowed to change signs. Further results using lim inf instead of lim sup and a nonoscillation criterion are also given.

In Section 6, we focus on the special case $\gamma = 2$, and give further oscillation results that are useful when $Q_2(t)$ changes signs.

2 Equivalent Riccati Equations Involving $t^\gamma q(t)$

The use of an equivalent Riccati equation in the study of the oscillation of a linear second-order differential equation is well-known. In this section we derive from (1.1) Riccati equations that involve $t^\gamma q(t)$. The equations are probably not new.

Let $\gamma > 0$, $\gamma \neq 1$ be any positive number. Starting with the identity

$$t^\gamma q(t) = -\frac{t^\gamma y''(t)}{y(t)}, \tag{2.1}$$

applying an integration by parts, completing the square, and doing some algebraic manipulation, we arrive at the identity

$$\int_{T_1}^T t^\gamma q(t)\,dt = T^{\gamma-1}\left(\frac{\gamma}{2} - \frac{Ty'(T)}{y(T)}\right) + C - BT^{\gamma-1} - \int_{T_1}^T t^{\gamma-2}\left(\frac{\gamma}{2} - \frac{ty'(t)}{y(t)}\right)^2, \tag{2.2}$$

where

$$B = \frac{\gamma^2 - 2\gamma}{4(\gamma - 1)} \tag{2.3}$$

and C is a constant that can be easily determined by substituting $T = T_1$ in (2.2) :

$$C = T_1^{\gamma-1}\left(\frac{T_1 y'(T_1)}{y(T_1)} - \frac{\gamma}{2} + B\right) = T_1^{\gamma-1}\left(\frac{T_1 y'(T_1)}{y(T_1)} - \frac{\gamma^2}{4(\gamma-1)}\right). \tag{2.4}$$

With the new variable

$$R(t) = \frac{\gamma}{2} - \frac{t y'(t)}{y(t)} \tag{2.5}$$

and the notations

$$Q_\gamma(t) = \int_{T_1}^{T} t^\gamma q(t)\, dt, \tag{2.6}$$

$$A = C + Q_\gamma(T_1), \tag{2.7}$$

we can rewrite (2.2) as

$$T^{\gamma-1}R(T) = Q_\gamma(T) + BT^{\gamma-1} - A + \int_{T_1}^{T} t^{\gamma-2}R^2(t)\, dt. \tag{2.8}$$

Note that we cannot simply take T_1 to be 0 because $R(t)$ may not be defined for all t.

When $\gamma = 1$, the corresponding Riccati equation is slightly different because of an integration involving $1/t$.

$$R(T) = Q_1(T) - \frac{1}{4}\ln T - A_1 + \int_{T_1}^{T} \frac{R^2(t)}{t}\, dt, \tag{2.9}$$

where

$$A_1 = \frac{T_1 y'(T_1)}{y(T_1)} - \frac{1}{4}\ln T_1 - \frac{1}{2} + Q_1(T_1). \tag{2.10}$$

A Riccati integral equation, such as (2.8), although stated for all $T > T_1$, may or may not have a solution for all such T. When we talk about the solution of the equation, we always refer to a local solution that exists in $[T_1, T_1 + \delta)$ for some $\delta > 0$.

The use of a Riccati equation in the study of oscillation is based on the fact that if the solution $y(t)$ of (1.1) has a zero at a point $t = a$, then $R(a) = \infty$. Hence, the oscillation of (1.1) is equivalent to the blow up of all solutions of the Riccati equation at some finite point, and the nonoscillation of (1.1) is equivalent to the existence of a solution of the Riccati equation on $[a, \infty)$ for some a.

To conclude this section, we point out that the use of a more general weight function $f^2(t)$ instead of t^γ will lead to the Riccati equation

$$f(T)R(T) = Q_f(T) + f(T)f'(T) - \int_{T_1}^{T} f(t)f'^2(t)\, dt - A + \int_{T_1}^{T} R^2(t)\, dt, \tag{2.11}$$

of which (2.8) and (2.9) are special cases. Here A is a constant as before,

$$Q_f(T) = \int_0^T f(t)q(t)\,dt, \tag{2.12}$$

and

$$R(t) = f(t)\left(\frac{y'(t)}{y(t)} - f'(t)\right). \tag{2.13}$$

However, in the paper, we shall not make use of this general equation.

3 Integral Inequalities

The theory of integral inequalities has useful applications in the study of differential equations. In this section we collect two results in the theory that will be used to establish our theorems in this paper. For an in-depth introduction to the theory, consult, for instance, [8].

All the functions considered in this section are locally L^1 (i.e. integrable) functions. Equalities and inequalities are understood to hold almost everywhere. The first result is well-known and we omit the proof.

Lemma 1 *Let $u(t)$ and $v(t)$ satisfy respectively*

$$u(T) \ge (\le)\, f(T) + g(T)\int_a^T K(u,t)\,dt, \quad T > a \tag{3.1}$$

and

$$v(T) = f(T) + g(T)\int_a^T K(v,t)\,dt, \quad T > a. \tag{3.2}$$

In addition, the kernel function $K(v,t)$ satisfies

$$K(v,t) \text{ is nondecreasing in } v \text{ for each fixed } t. \tag{3.3}$$

Then,

$$u(T) \ge (\le)\, v(T), \quad T > a. \tag{3.4}$$

The second result concerns an inequality similar to (3.1), but having T in the lower limit of the integral.

$$u(T) \ge f(T) + g(T)\int_T^\infty K(u,t)\,dt, \quad T \in (a,\infty). \tag{3.5}$$

We assume in addition to (3.3), that $f(t)$ and $g(t)$ are differentiable and that

$$f(t), K(u,t) \geq 0, \;\; g(t) > 0. \tag{3.6}$$

We would like to find a criterion for the nonexistence of such a $u(t)$. One considers the comparison equation

$$w(T) = f(T) + g(T) \int_T^\infty K(w,t)\, dt, \quad T \in (a,\infty). \tag{3.7}$$

At this point, we do not have any knowledge whether (3.7) has a solution or not. If it does, the solution $w(t)$ must satisfy the differential equation

$$\left(\frac{w(t)}{g(t)}\right)' = \left(\frac{f(t)}{g(t)}\right)' - K(w,t). \tag{3.8}$$

We can solve (3.8) with any given initial value at $t = a$,

$$w(a) = \alpha. \tag{3.9}$$

Lemma 2 *Suppose that given any $\alpha > 0$, the solution to the initial value problem (3.8) and (3.9) always vanishes at some point $t_1 > a$, then the integral inequality (3.5) cannot have a solution.*

Proof. Suppose the contrary and $u(t)$ is a solution of (3.5). Take any $\alpha > u(a)$ and let $w_0(t)$ be the solution of the initial value problem (3.8) and (3.9). By hypotheses, $w_0(b) = 0$ for some $b > a$. Since

$$w_0(a) > u(a) > f(a), \tag{3.10}$$

and

$$w_0(b) = 0 \leq f(b), \tag{3.11}$$

by continuity, there is a point $c \in [a,b]$ such that

$$w_0(c) = f(c). \tag{3.12}$$

Then $w_0(t)$ satisfies the integral equation

$$w_0(T) = f(T) + g(T) \int_T^c K(w_0,t)\, dt, \quad T \in (a,c). \tag{3.13}$$

We can now use Lemma 1 (after a reflection) to compare $u(t)$ and $w_0(t)$, to conclude that

$$u(t) \geq w(t), \quad t \in [a,c]. \tag{3.14}$$

This contradicts the fact that $w(a) > u(a)$. ∎

4 Case $q(t) \geq 0$

In this section, we assume that (C0) holds. We deal with the case $\gamma > 1$ first.

Theorem 1 *Assume (C0). If for some $\gamma > 1$,*

$$\limsup_{T\to\infty} \frac{Q_\gamma(T)}{T^{\gamma-1}} > \frac{\gamma^2}{4(\gamma-1)}, \tag{C3}$$

then, (1.1) is oscillatory.

Proof. Given any a, assume that $y(t)$ is a solution of (1.1) with $y(a) = 0$ and $y'(a) > 0$. It suffices to show that y' vanishes at some point beyond a. From (2.8), with $T_1 = a$,

$$R(T) > \frac{Q_\gamma(T)}{T^{\gamma-1}} + B - \frac{A}{T^{\gamma-1}}. \tag{4.1}$$

For T sufficiently large, the last term is small. By hypotheses, there exists a T so large that

$$R(T) > \frac{\gamma^2}{4(\gamma-1)} + B = \frac{\gamma}{2}. \tag{4.2}$$

Using the definition of R (2.5), we see that $y'(T) < 0$, and the theorem is proved. ∎

To illustrate how Theorem 1 can be applied, consider the classical Kneser example

$$y''(t) + \frac{k}{t^2} y(t) = 0. \tag{4.3}$$

Then

$$\frac{Q_\gamma(T)}{T^{\gamma-1}} = \frac{k}{\gamma-1}. \tag{4.4}$$

Condition (C3) becomes

$$k \geq \frac{\gamma^2}{4}. \tag{4.5}$$

Since we can choose γ as close to 1 as we like, we see that (4.3) is oscillatory if $k > 1/4$.

Let us now consider the case $\gamma < 1$, and suppose that (C1) fails. Let

$$\bar{Q}_\gamma(t) = \int_t^\infty t^\gamma q(t)\, dt. \tag{4.6}$$

Theorem 2 *Assume (C0). If for some $\gamma < 1$,*

$$\limsup_{T\to\infty} T^{1-\gamma}\bar{Q}_\gamma(T) > 1 + \frac{\gamma^2}{4(1-\gamma)}, \tag{C4}$$

then, (1.1) is oscillatory.

Proof. Given any a, assume that $y(t)$ is a solution of (1.1) with $y(a) = 0$ and $y'(a) > 0$. It suffices to show that y' vanishes at some point beyond a. Let $T_1 > a$ be one of the points which realizes condition (C4), i.e.

$$T_1^{1-\gamma}\bar{Q}_\gamma(T_1) > 1 + \frac{\gamma^2}{4(1-\gamma)}. \tag{4.7}$$

By (C0), the solution $y(t)$ is concave in $[a, T_1]$, so

$$y'(T_1) \le \frac{y(T_1)}{T_1 - a}, \tag{4.8}$$

from which we get

$$\frac{T_1 y'(T_1)}{y(T_1)} \le \frac{T_1}{T_1 - a}. \tag{4.9}$$

Since a is fixed, we can choose T_1 sufficiently large that the right hand side of (4.9) is so very close to 1 that (4.7) becomes

$$T_1^{1-\gamma}\bar{Q}_\gamma(T_1) > \frac{T_1 y'(T_1)}{y(T_1)} + \frac{\gamma^2}{4(1-\gamma)}, \tag{4.10}$$

which can be rewritten as

$$\bar{Q}_\gamma(T_1) > C. \tag{4.11}$$

Let $T > T_1$. From (2.8), we get

$$T^{\gamma-1}R(T) > BT^{\gamma-1} + \bar{Q}_\gamma(T) + \int_{T_1}^{T} t^{\gamma-2}R^2(t)\,dt. \tag{4.12}$$

Since $\bar{Q}_\gamma(T) \to 0$, we actually have, for T large enough

$$T^{\gamma-1}R(T) > BT^{\gamma-1} + \int_{T_1}^{T} t^{\gamma-2}R^2(t)\,dt > BT^{\gamma-1}. \tag{4.13}$$

Hence, for T large enough

$$R(T) = \frac{\gamma}{2} - \frac{Ty'(T)}{y(T)} > B = \frac{2\gamma - \gamma^2}{4(1-\gamma)}. \tag{4.14}$$

It follows that

$$\frac{Ty'(T)}{y(T)} < \frac{\gamma}{2} - \frac{2\gamma - \gamma^2}{4(1-\gamma)} = -\frac{\gamma^2}{4(1-\gamma)} < 0. \tag{4.15}$$

This completes the proof. ∎

When we apply condition (C4) to Kneser's example (4.3), we get oscillation when

$$k > 1 - \gamma + \frac{\gamma^2}{4}, \quad \text{for some } \gamma < 1. \tag{4.16}$$

If we choose γ close to 1, we again get the classical criterion of $k > 1/4$.

5 Case When $q(t)$ May Change Signs

In this section, we consider general $q(t)$ that may change signs. Our first observation is that (C1) by itself is sufficient for oscillation. The author would like to thank the referee for pointing out that this result is indeed known. It can be found, for example, in Hartman [1], and is proved by means of a change of variable and the usual Fite-Wintner result. An alternative nice proof can be found in the book by Kelley and Peterson [4], based on a variational approach using an appropriate auxiliary function.

Theorem 3 *Condition (C1) implies oscillation.*

Proof. We can choose T_2 large enough that

$$Q_\gamma(T) + BT^{\gamma-1} - A > 1, \quad \text{for } T > T_2. \tag{5.1}$$

From (2.8), we see that

$$T^{\gamma-1} > 1 + \int_{T_2}^{T} t^{\gamma-2} R^2(t)\, dt. \tag{5.2}$$

By solving this integral inequality, we see that $R(T)$ blows up to infinity at a finite point T. So the differential equation (1.1) must be oscillatory. ∎

We next extend Theorems 1 and 2. For $\gamma > 1$, define

$$Q^*_\gamma(t) = \inf_{u>t} Q_\gamma(u). \tag{5.3}$$

Likewise, for $\gamma < 1$, define

$$\bar{Q}^*_\gamma(t) = \inf_{u>t} \bar{Q}_\gamma(u). \tag{5.4}$$

If (C0) holds, $Q^*_\gamma(t) = Q_\gamma(t)$ or $\bar{Q}^*_\gamma(t) = \bar{Q}_\gamma(t)$. Hence, Theorems 1 and 2 are special cases of the following results.

Theorem 4 *If for some $\gamma > 1$,*

$$\limsup_{T\to\infty} \frac{Q_\gamma^*(T)}{T^{\gamma-1}} > \frac{\gamma^2}{4(\gamma-1)}, \tag{C5}$$

then, (1.1) is oscillatory.

Proof. As in the proof of Theorem 1, given T_1, we need to show that (2.8) does not have a solution on $[T_1, \infty)$. Let T_2 be so large that

$$Q_\gamma^*(T_2) \geq \left(\frac{\gamma^2}{4(\gamma-1)} + \delta\right) T_2^{\gamma-1} + A, \tag{5.5}$$

for some $\delta > 0$. Then for all $T > T_2$,

$$Q_\gamma(T) \geq \left(\frac{\gamma^2}{4(\gamma-1)} + \delta\right) T_2^{\gamma-1} + A, \tag{5.6}$$

and (2.8) gives

$$T^{\gamma-1}R(T) \geq \left(\frac{\gamma^2}{4(\gamma-1)} + \delta\right) T_2^{\gamma-1} + BT^{\gamma-1} + \int_{T_2}^{T} t^{\gamma-2}R^2(t)\,dt, \quad T > T_2. \tag{5.7}$$

By the theory of integral inequalities, one can compare (5.7) with the solution of the integral equation

$$T^{\gamma-1}S(T) = \left(\frac{\gamma^2}{4(\gamma-1)} + \delta\right) T_2^{\gamma-1} + BT^{\gamma-1} + \int_{T_2}^{T} t^{\gamma-2}S^2(t)\,dt. \tag{5.8}$$

Differentiating (5.8), we get the differential equation

$$tS'(t) = \left(S(t) - \frac{\gamma}{2}\right)\left(S(t) - \frac{\gamma-2}{2}\right), \quad t > T_2. \tag{5.9}$$

Substituting $T = T_2$ into (5.8) gives the initial condition

$$S(T_2) = \frac{\gamma}{2} + \delta. \tag{5.10}$$

It is now easy to see that $S(t)$ must be positive and, in fact, it blows up to infinity at some finite point greater than T_2. This complete the proof. ∎

If we examine the proof more closely, we can find that we need only to require, instead of (5.6), that

$$Q_\gamma^*(T) + \int_{T_1}^{T_2} R^2(t)\,dt \geq \left(\frac{\gamma^2}{4(\gamma-1)} + \delta\right) T_2^{\gamma-1} + A \tag{5.11}$$

to make the proof work. Any method that gives a lower bound on the integral on the left hand side of (5.11) will lead to an improvement of the theorem. For instance, in the special case $\gamma = 2$, if we have $Q_2(t) \geq 0$, then $\int_{T_1}^{T} (Q_2(t)/t)^2 \, dt$ can serve as such a lower bound, and hence (C5) can be improved to

$$\limsup_{T\to\infty} \frac{1}{T^{2-1}} \left(Q_2^*(T) + \int_0^T \left(\frac{Q_2(t)}{t}\right)^2 dt \right) > 1. \tag{5.12}$$

The same remark applies to condition (C6) below.

Theorem 5 *If for some $\gamma < 1$,*

$$\limsup_{T\to\infty} T^{1-\gamma} \bar{Q}_\gamma^*(T) > 1 + \frac{\gamma^2}{4(1-\gamma)}, \tag{C6}$$

then, (1.1) is oscillatory.

Proof. The proof merely combines the techniques used in the proofs of Theorem 2 and Theorem 4 and is omitted. ∎

So far we have ignored the case $\gamma = 1$. A simple application of the theory of integral inequalities gives the following result, which is actually known in previous work of many authors.

Theorem 6 *The condition*

$$\lim_{t\to\infty} Q(t) - \frac{\ln t}{4} = \infty \tag{C7}$$

implies oscillation of (1.1)

The next set of criteria involves the more stringent requirement of a lower bound on the *liminf*, instead of *limsup*, but we gain in a different aspect by requiring smaller constants than those in (C3) and (C4).

Theorem 7 *If for some $\gamma > 1$,*

$$\liminf_{T\to\infty} \frac{Q_\gamma(T)}{T^{\gamma-1}} > \frac{1}{4(\gamma-1)}, \tag{C7'}$$

then, (1.1) is oscillatory.

Proof. It is not difficult to prove this result directly using the Riccati equation (2.8) and the theory of integral inequalities. Instead we show that this theorem follows from Theorem 6.

For large enough T, (C7) implies that for some $\delta > 0$

$$\int_0^T t^\gamma q(t)\,dt > \left(\frac{1}{4(\gamma-1)} + \delta\right) T^{\gamma-1}. \tag{5.13}$$

Integration by parts gives the inequality

$$T^{\gamma-1} Q_1(t) \geq \left(\frac{1}{4(\gamma-1)} + \delta\right) T^{\gamma-1} + \int_0^T (\gamma-1) t^{\gamma-2} Q_1(t)\,dt. \tag{5.14}$$

Applying Lemma 1, we obtain

$$Q_1(t) \geq Y(t), \tag{5.15}$$

where $Y(t)$ is the solution of

$$T^{\gamma-1} Y(t) = \left(\frac{1}{4(\gamma-1)} + \delta\right) T^{\gamma-1} + \int_0^T (\gamma-1) t^{\gamma-2} Y(t)\,dt. \tag{5.16}$$

By differentiating this equation, we can easily determine that

$$Y(t) = \left(\frac{1}{4} + (\gamma-1)\delta\right) \ln t + C, \tag{5.17}$$

for some integration constant C. It follows that (C6) is satisfied. ∎

We remark that the same method used in the proof of Theorem 7 can be used to prove the more general fact that if (C7') is satisfied for any γ_0, then it is satisfies for any $\gamma < \gamma_0$.

We present Theorem 7 even though it is subsumed under Theorem 6, because of its symmetry with the case $\gamma < 1$. Also in the next section we shall see that in some situations, Theorem 7 can turn out to be useful.

Let us now turn to the case $\gamma < 1$. We assume that $\lim_{T\to\infty} \int_a^T t^\gamma q(t)\,dt$ exists and as before we define $\bar{Q}_\gamma(t) = \int_t^\infty t^\gamma q(t)\,dt$. Suppose (1.1) is nonoscillatory, then (2.8) has a solution that exists on $[a, \infty)$ for some a. By letting $T \to \infty$ in (2.8) and replacing T_1 by T, we obtain the Riccati equation

$$\bar{R}(T) = T^{1-\gamma} \bar{Q}_\gamma(T) + \frac{2\gamma - \gamma^2}{4(1-\gamma)} + T^{1-\gamma} \int_T^\infty \frac{\bar{R}^2(2)}{t^{2-\gamma}}\,dt, \tag{5.18}$$

where $\bar{R}(T) = -R(T)$. Note that $\bar{R}(T) > 0$, for all T.

Theorem 8 *Suppose* $\lim_{T\to\infty}\int_a^T t^\gamma q(t)\,dt$ *exists, and*

$$\liminf T^{1-\gamma}\bar{Q}_\gamma(T) > \frac{1}{4(1-\gamma)}. \tag{C8}$$

Then (1.1) is oscillatory.

Proof. If (1.1) is nonoscillatory, then (5.18) has a solution on $[a,\infty)$ for some a. By hypotheses, $\bar{R}(T)$ satisfies the integral inequality

$$\bar{R}(T) \geq \frac{1-\gamma}{4} + T^{1-\gamma}\int_T^\infty \frac{\bar{R}^2(2)}{t^{2-\gamma}}\,dt. \tag{5.19}$$

We can apply Lemma 2 to obtain a contradiction and this completes the proof. ∎

Let us consider some examples. The perturbed Kneser's equation

$$y''(t) + \frac{k + ht^\alpha\cos(t)}{t^2}y(t) = 0, \quad t > 0 \tag{5.20}$$

is covered by (C5) or (C6) when $k > 1/4$, $\alpha < 1$, and any h. Note that for $\alpha > 0$ or for $\alpha = 1$ and $h > k$, the coefficient $q(t)$ changes sign infinitely often.

Let $k(t)$ be a function such that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T k(t)\,dt > 1. \tag{5.21}$$

Then by (C3), with $\gamma = 2$, the differential equation

$$y''(t) + \frac{k(t) + ht^\alpha\cos(t)}{t^2}y(t) = 0, t > 0 \tag{5.22}$$

is oscillatory for any $\alpha < 1$ and h.

From these two examples we see that neither one of (C3) and (C5) (or (C4) and (C6)) implies the other.

The following nonoscillation criterion complements (C7) and (C8). The proofs are essentially the same as those of Theorem 7 and Theorem 8, with the directions of the Riccati integral inequalities reversed. Instead of showing that the solution of the Riccati inequality in question is bounded from below, as in the proofs of Theorem 7 and Theorem 7, we show that the solution is bounded from above by a bounded function and as a result the solution exists.

Theorem 9 *If for some $\gamma > 1$,*

$$\limsup \to \infty \frac{Q_\gamma(T)}{T^{\gamma-1}} < \frac{1}{4(\gamma-1)}, \tag{N1}$$

or for some $\gamma < 1$,

$$\limsup T^{1-\gamma} \bar{Q}_\gamma(T) < \frac{1}{4(1-\gamma)}, \tag{N2}$$

then, (1.1) is nonoscillatory.

6 Further Results Using $Q_2(t)$

One shortfall of the results in Section 5 is that they do not cover cases in which $Q_\gamma(t)$ takes on small values for large t, or perhaps even changes signs frequently.

Some simple examples are

1. $q(t) = (1 + t\cos t)/t^2$, for which , $Q_2(t) = t + t\sin t + \cos t$.
2. $q(t) = \cos t$, for which $Q_2(t) = (t^2 - 2)\sin t + 2t\cos t$.
3. $q(t) = \cos t + 2\sin t/t$, for which $Q_2(t) = t^2 \sin t$.

In this section, we present several results that are applicable to such cases. Although a very general theory can be developed for all $\gamma > 0$, we shall confine ourselves to $\gamma = 2$ in which the conditions are particularly simple to state. We shall omit the subscript and write $Q(t) = \int_0^t t^2 q(t)\,dt$.

A classical technique used to deal with wildly oscillatory coefficients is the use of iterated averages. This approach is useful for some coefficients (such as example 1 above) and has the advantage of being easy to apply. To adopt the technique to our current situation, we define

$$\mathcal{M}Q(t) = \int_0^t \frac{Q(s)}{s}\,ds. \tag{6.1}$$

Theorem 10 *Let (C) be any known oscillation criterion for (1.1) involving $Q(t)$. If (in place of $Q(t)$) one can show that there exists some $\alpha < 1$ such that for any $\beta > 0$, $\alpha\mathcal{M}Q(t) - \beta\ln t$ satisfies the same criterion, then (1.1) is oscillatory.*

Proof. In the case $\gamma = 2$ the Riccati equation (2.8) takes the simple form

$$TR(T) = Q(T) + A + \int_{T_1}^{T} R^2(t)\,dt. \tag{6.2}$$

Divide by T and integrate to obtain

$$\int_{T_1}^{T} R(t)\, dt = \mathcal{M}Q(T) + A \ln t + \int_{T_1}^{T} \frac{1}{t} \int_{T_1}^{t} R^2(s)\, ds\, dt. \tag{6.3}$$

Define $R_1(t) = \dfrac{\alpha}{t} \displaystyle\int_{T_1}^{t} R(s)\, ds$, and use Hölder's inequality, we get

$$TR_1(T) \geq \alpha \mathcal{M}Q(T) + \alpha A \ln t + \int_{T_1}^{T} \frac{(t-T_1)R_1^2(t)}{\alpha t}\, dt. \tag{6.4}$$

Let T_2 be so large that

$$\frac{T_2 - T_1}{\alpha T_2} > 1. \tag{6.5}$$

Then it follows that from (6.4) that

$$TR_1(T) \geq \alpha \mathcal{M}Q(T) + \alpha A \ln t + \int_{T_2}^{T} R_1^2(t) \alpha t\, dt, \quad t > T_2. \tag{6.6}$$

Note that (6.6) is similar to (6.2), only with $Q(T) + A$ replaced by $\alpha \mathcal{M}Q(t) + \alpha A \ln t$. By hypotheses, the latter function satisfies some oscillation criterion, so every solution of the Riccati equation obtained by replacing $\geq$ in (6.6) by $=$ must blow up at some finite point. By the theory of integral inequalities, the solution $R_1(t)$ of (6.6) must also blow up at some finite point. Hence $R(t)$ cannot exist on $[0, \infty)$. ∎

The need to verify the hypotheses of Theorem 10 for some α and for all β may make Theorem 10 seem hardly useful at all. However, in some situations, such as when (C5) and (C7) are used, we only need to verify that $\mathcal{M}Q(t)$ satisfies the conditions. Finally, we can easily extend the result to iterated averages. For $n = 2, 3, \ldots$ Define

$$\mathcal{A}^n Q(t) = \int_0^t \frac{\mathcal{M}^{n-1}Q(s)}{s}\, ds. \tag{6.7}$$

Corollary 1 *If $\mathcal{M}^n Q(t)$ satisfies (C5) or (C7), for some $n \geq 1$, then (1.1) is oscillatory.*

It is easy to see that this Corollary covers Example 1 given at the beginning of this section. The result, however, is useless for Examples 2 and 3, since all the iterated averages of $Q(t)$ change signs infinitely often. Let us show how the equivalent Riccati equation (6.2) can still be used to deduce oscillation for Example 3. A similar proof can be given for Example 2.

The first step is to obtain an estimate of $\int R^2(t)\, dt$. Since the constant A in (6.2) is very small relative to the other terms when T is large, we just ignore A to simplify the discussion. It is not difficult to modify the proof to take care of general A. From (6.2), we have

$$R(T) \geq T \cos T. \tag{6.8}$$

Hence,

$$R^2(T) \geq T^2 \cos^2 T, \quad T \in [s_n, t_n], \tag{6.9}$$

for n large enough, where $s_n = 2n\pi$, and $t_n = (2n+1)\pi$. It follows that

$$\int_{T_1}^{T} R^2(t)\, dt \geq \kappa T^3, \tag{6.10}$$

for some $\kappa > 0$ and T large enough.

Next, we study the differential inequality (6.2) in $[s_n, t_n]$. Using (6.10) and the fact that $Q(T) \geq 0$ in $[s_n, t_n]$, (6.2) gives

$$TR(T) \geq \kappa s_n^3 + \int_{s_n}^{T} R^2(t)\, dt. \tag{6.11}$$

We compare this to the solution of the integral equation

$$TS(T) = \kappa s_n^3 + \int_{s_n}^{T} S^2(t)\, dt. \tag{6.12}$$

Differentiating (6.12), and separating the variables, we get

$$\frac{S'}{S^2 - S} = \frac{1}{T}. \tag{6.13}$$

If $S(t)$ has a solution in the interval $[s_n, t_n]$, then integrating over this interval gives

$$\int_{S(s_n)}^{S(t_n)} \frac{dS}{S^2 - S} = \ln \frac{t_n}{s_n} = \ln\left(1 + \frac{\pi}{2s_n}\right). \tag{6.14}$$

We claim that the two sides of this equality has different orders of magnitude, and hence it cannot hold (for large n). As a consequence, (6.11) cannot have a solution and so (1.1) is oscillatory. First, the right hand side of (6.14) is of the order $1/s_n$. Then since S is very large in the interval $[s_n, t_n]$, we have $S^2 - S > S/2$, and we can estimate the left hand side of (6.14) by

$$\int_{S(s_n)}^{S(t_n)} \frac{dS}{S^2 - S} < \int_{S(s_n)}^{S(t_n)} \frac{dS}{2S^2} < \frac{1}{2S(s_n)} = \frac{1}{2\kappa s_n^2.} \tag{6.15}$$

This is of order less than $1/s_n^2$, proving the claim.

We would like to extend this idea to more general coefficients. Let us examine the proof again. There are two crucial conditions that lead to the nonexistence of $S(t)$ in $[s_n, t_n]$.

First, $S(t)$ has a large initial value at $T = s_n$ (as $n \to \infty$). Second, there is enough room in the interval $[s_n, t_n]$ for $S(t)$ to grow (the length of $[s_n, t_n]$ remains bounded from below as $n \to \infty$ so that $\ln(t_n/s_n)$ is of the order $1/s_n$).

The first condition can be derived from the fact that

$$\int_0^T \frac{Q_+^2(t)}{t^2}\, dt \geq \kappa T^\alpha, \quad T \text{ sufficiently large,} \tag{C9}$$

for some $\kappa > 0$ and $\alpha > 2$, where $Q_+(t) = \max\{Q(t), 0\}$.

The second condition requires the existence of an infinite number of intervals, each of length greater than some positive constant, and in each interval $Q(t) \geq 0$. This can be relaxed to

> There are constants $\delta > \epsilon > 0$, and an infinite number of intervals $[s_n, s_n + \delta]$ such that the measure of the set $\{t \in [s_n, s_n + \delta] : Q(t) \geq 0\} \geq \epsilon$. (C10)

This permits $Q(t)$ to be nonnegative in many small subintervals of $[s_n, t_n]$ as long as the total length of the subintervals exceeds a lower bound.

Theorem 11 *Conditions (C9) and (C10) imply oscillation of (1.1)*

Proof. If $Q(t) \geq 0$ in the entire interval $[s_n, s_n + \delta]$, then the proof is essentially the same as the proof for the oscillation of Example 2. We shall give the proof for the case in which $Q(t) \geq 0$ in two disjoint subintervals. The proof for the general case will then be obvious.

The proof is based on the telescoping principle introduced in [6]. Let the two subintervals be $[a, b]$ and $[c, d] \subset [s_n, t_n]$, $a < b < c < d$. As above, the proof is reduced to showing the nonexistence of a solution to the integral equation

$$TS(T) = \kappa s_n^\alpha + \int_a^T S^2(t)\, dt, \quad t \in [a, b] \cup [c, d]. \tag{6.16}$$

There is a gap in between the two subintervals. Now we push the interval $[a, b]$ towards $[c, d]$ to close the gap, to obtain a larger interval $[c-b+a, d]$, of length $> \epsilon$. On this interval we consider the integral equation

$$TU(T) = \kappa s_n^\alpha + \int_{c-b+a}^T U^2(t)\, dt, \quad t \in [c - b + a, d]. \tag{6.17}$$

We already know that when n is large, (6.17) does not have a solution (any $U(T)$ that satisfies the initial condition at the left endpoint must blow up within the interval). The significance of the constant δ in (C10) is to guarantee that the initial value of $U(\bar{T}) =$

$\kappa s_n^\alpha > (\kappa s_n^\alpha/(s_n+\delta)^\alpha)\bar{T}^\alpha$ at $T = c - b + a$ is still large. We claim that $S(T) \geq U(T)$ in $[c,d]$ at wherever the solutions exist. It then follows that $S(T)$ cannot exist in $[s_n, t_n]$ and the proof of the theorem is complete.

For $T > c$, we can rewrite (6.16) and (6.17) as

$$TS(T) = \kappa s_n^\alpha + \int_a^c S^2(t)\,dt + \int_c^T S^2(t)\,dt, \tag{6.18}$$

$$TU(T) = \kappa s_n^\alpha + \int_{c-b+a}^c U^2(t)\,dt + \int_c^T U^2(t)\,dt, \tag{6.19}$$

The assertion follows from the theory of integral inequalities if we can show that

$$\int_a^c S^2(t)\,dt \geq \int_{c-b+a}^c U^2(t)\,dt. \tag{6.20}$$

In fact, this follows from

$$\int_a^b S^2(t)\,dt \geq \int_{c-b+a}^c U^2(t)\,dt, \tag{6.21}$$

which in turn follows from

$$S(t) \geq U(t-b+a), \quad t \in [a,b]. \tag{6.22}$$

The last assertion can be easily proved by comparing (6.16) in $[a,b]$ with (6.17) in $[c-b+a, c]$ (after a translation). ∎

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(Received May 26, 2006)