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On the iterated order and the fixed points of entire solutions of some complex linear differential equations

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Abstract. In this paper, we investigate the iterated order of entire solutions of homogeneous and non-homogeneous linear differential equations with entire coefficients.

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1 Introduction and statement of results:

For the definition of the iterated order of an entire function, we use the same definition as in [9], [2, p. 317], [10, p. 129]. For all $r \in \mathbf{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbf{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbf{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 Let f be an entire function. Then the iterated $p-$order $\sigma_p(f)$ of f is defined by

$$\sigma_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r, f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r, f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.1}$$

where $T(r,\ f)$ is the Nevanlinna characteristic function of f and $M(r,f) = \max_{|z|=r} |f(z)|$ (see [7], [13]). For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 1.2 The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ polynomial,} \\ \min\{j \in \mathbf{N} : \boldsymbol{\sigma}_j(f) < \infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbf{N} \text{ with } \boldsymbol{\sigma}_j(f) < \infty \text{ exists,} \\ \infty, & \text{for } f \text{ with } \sigma_j(f) = \infty \text{ for all } j \in \mathbf{N}. \end{cases} \tag{1.2}$$

Definition 1.3 Let f be an entire function. Then the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r}, \tag{1.3}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$. Thus $\overline{\lambda}_p(f - z)$ is an indication of oscillation of the fixed points of $f(z)$.

For $k \geq 2$, we consider the linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{1.4}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = F(z), \tag{1.5}$$

where $A_0(z), ..., A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equations (1.4) and (1.5) are entire functions.

Extensive work in recent years has been concerned with the growth of solutions of complex linear differential equations. Many results have been obtained. Examples of such results are the following two theorems:

Theorem A [4]. *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions such that there exists one transcendental A_s $(0 \leq s \leq k-1)$ satisfying $\sigma(A_j) \leq \sigma(A_s)$ for all $j \neq s$. Then equation* (1.4) *has at least one solution f that satisfies $\sigma_2(f) = \sigma(A_s)$.*

Theorem B [4]. *Let* $A_0(z), ..., A_{k-1}(z)$ *satisfy the hypotheses of Theorem A and* $F(z) \not\equiv 0$ *be an entire function with* $\sigma(F) < +\infty$. *Assume that* f_0 *is a solution of* (1.5), *and* $g_1, ..., g_k$ *are a solution base of the corresponding homogeneous equation* (1.4) *of* (1.5). *Then there exists a* g_j $(1 \le j \le k)$, *say* g_1, *such that all the solutions in the solution subspace* $\{cg_1 + f_0,\ c \in \mathbf{C}\}$ *satisfy* $\sigma_2(f) = \overline{\lambda_2}(f) = \sigma(A_s)$, *with at most one exception.*

The purpose of this paper is to extend the above two results by considering the iterated order. We will prove the following theorems:

Theorem 1.1 *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions such that there exists one transcendental* A_s $(0 \le s \le k-1)$ *satisfying* $\sigma_p(A_j) \le \sigma_p(A_s) < +\infty$ *for all* $j \ne s$. *Then equation* (1.4) *has at least one solution* f *that satisfies* $i(f) = p+1$ *and* $\sigma_{p+1}(f) = \sigma_p(A_s)$.

Theorem 1.2 *Let* $A_0(z), ..., A_{k-1}(z)$ *satisfy the hypotheses of Theorem 1.1 and* $F(z) \not\equiv 0$ *be an entire function with* $i(F) = q$. *Assume that* f_0 *is a solution of* (1.5), *and* $g_1, ..., g_k$ *are a solution base of the corresponding homogeneous equation* (1.4) *of* (1.5). *If either* $i(F) = q < p+1$ *or* $q = p+1$ *and* $\sigma_{p+1}(F) < \sigma_p(A_s) < +\infty$, *then there exist a* g_j $(1 \le j \le k)$, *say* g_1, *such that all the solutions in the solution subspace* $\{cg_1 + f_0,\ c \in \mathbf{C}\}$ *satisfy* $i(f) = p+1$ and $\sigma_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \sigma_p(A_s)$, *with at most one exception.*

Set $g(z) = f(z) - z$. Then clearly $\overline{\lambda}_{p+1}(f - z) = \overline{\lambda}_{p+1}(g)$ and $\sigma_{p+1}(g) = \sigma_{p+1}(f)$. By Theorem 1.1 and Theorem 1.2, we can get the following corollaries.

Corollary 1 *Under the hypotheses of Theorem 1.1, if* $A_1 + zA_0 \not\equiv 0$, *then equation* (1.4) *has at least one solution* f *that satisfies* $i(f) = p+1$ *and* $\overline{\lambda}_{p+1}(f - z) = \sigma_{p+1}(f) = \sigma_p(A_s)$.

Corollary 2 *Under the hypotheses of Theorem 1.2, if* $F - A_1 - zA_0 \not\equiv 0$, *then every solution* f *of* (1.5) *with* $i(f) = p+1$ *and* $\sigma_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \sigma_p(A_s)$ *satisfies* $\overline{\lambda}_{p+1}(f - z) = \sigma_p(A_s)$.

2 Preliminary Lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 2.1 $([3],[11])$. *Let* $f(z)=\sum\limits_{n=0}^{\infty}a_n z^n$ *be an entire with* $\sigma_{p+1}(f)=\sigma$, *let* $\mu(r)$ *be the maximum term, i.e.,* $\mu(r)=\max\{|a_n|\, r^n;\ n=0,1,...\}$ *and let* $\nu_f(r)$ *be the central index of* f, *i.e.,* $\nu_f(r)=\max\{m,\ \mu(r)=|a_m|\, r^m\}$. *Then*

$$\overline{\lim_{r\to\infty}}\frac{\log_{p+1}\nu_f(r)}{\log r}=\sigma. \tag{2.1}$$

Lemma 2.2 (*Wiman-Valiron,* $[8]$, $[12]$). *Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z|=r$ *at which* $|f(z)|=M(r,f)$. *Then the estimation*

$$\frac{f^{(k)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^k(1+o(1))\quad (k\ \text{ is an integer}), \tag{2.2}$$

holds for all $|z|$ *outside a set* E_2 *of* r *of finite logarithmic measure* $lm(E_2)=\int_1^{+\infty}\frac{\chi_{E_2}(t)}{t}dt$, *where* χ_{E_2} *is the characteristic function of* E_2.

Lemma 2.3 (See Remark 1.3 of $[9]$). *If* f *is a meromorphic function with* $i(f)=p\geq 1$, *then* $\sigma_p(f)=\sigma_p\left(f'\right)$.

Lemma 2.4 $([5])$. *Let* $f_1,...,f_k$ *be linearly independent meromorphic solutions of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_1(z)f'+A_0(z)f=0, \tag{2.3}$$

with meromorphic coefficients $A_0(z),...A_{k-1}(z)$. *Then*

$$m(r,A_j)=O\left\{\log\left(\max_{1\leq n\leq k}T(r,f_n)\right)\right\}\quad (j=0,...,k-1). \tag{2.4}$$

Lemma 2.5 $([9])$. *Let* f *be a meromorphic function for which* $i(f)=p\geq 1$ *and* $\sigma_p(f)=\sigma$, *and let* $k\geq 1$ *be an integer. Then for any* $\varepsilon>0$,

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\exp_{p-2}\left\{r^{\sigma+\varepsilon}\right\}\right), \tag{2.5}$$

outside of a possible exceptional set E_3 of finite linear measure.

To avoid some problems caused by the exceptional set we recall the following Lemma.

Lemma 2.6 $([1,\ \text{p. } 68],\ [9])$. *Let $g : [0, +\infty) \to \mathbf{R}$ and $h : [0, +\infty) \to \mathbf{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

3 Proof of Theorem 1.1

Suppose that f is a solution of (1.4). We can rewrite (1.4) as

$$\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + ... + A_{s+1}(z)\frac{f^{(s+1)}}{f} + A_s(z)\frac{f^{(s)}}{f}$$

$$+A_{s-1}(z)\frac{f^{(s-1)}}{f} + ... + A_1(z)\frac{f'}{f} + A_0(z) = 0. \qquad (3.1)$$

By Lemma 2.2, there exists a set $E_2 \subset (1, +\infty)$ with logarithmic measure $lm(E_2) < +\infty$ and we can choose z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$, such that (2.2) holds. For given small $\varepsilon > 0$ and sufficiently large r, we have

$$|A_j(z)| \leq \exp_p\left\{r^{\sigma_p(A_s)+\varepsilon}\right\} \quad (j = 0, 1, ..., k-1). \qquad (3.2)$$

Substituting (2.2) into (3.1), we obtain by using (3.2)

$$\left(\frac{\nu_f(r)}{|z|}\right)^k |1 + o(1)| \leq k\left(\frac{\nu_f(r)}{|z|}\right)^{k-1} |1 + o(1)| \exp_p\left\{r^{\sigma_p(A_s)+\varepsilon}\right\}, \qquad (3.3)$$

$(r \notin [0, 1] \cup E_2)$. By Lemma 2.1, Lemma 2.6 and (3.3), we obtain that $i(f) \leq p + 1$ and

$$\sigma_{p+1}(f) = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} \nu_f(r)}{\log r} \leq \sigma_p(A_s) + \varepsilon. \qquad (3.4)$$

Since $\varepsilon > 0$ is arbitrary, then $\sigma_{p+1}(f) \leq \sigma_p(A_s)$.

Assume that $\{f_1, ..., f_k\}$ is a solution base of (1.4). Then by Lemma 2.4

$$m(r, A_s) \leq M \log \left(\max_{1\leq n\leq k} T(r, f_n) \right). \tag{3.5}$$

We assert that there exists a set $E \subset (0, +\infty)$ of infinite linear measure such that

$$\lim_{\substack{r\to\infty \\ r\in E}} \frac{\log_p m(r, A_s)}{\log r} = \sigma_p(A_s). \tag{3.6}$$

In fact, there exists a sequence $\{r_n\}$ $(r_n \to \infty)$ such that

$$\lim_{r_n\to\infty} \frac{\log_p m(r_n, A_s)}{\log r_n} = \sigma_p(A_s). \tag{3.7}$$

We take $E = \bigcup_{n=1}^{\infty} [r_n, 2r_n]$. Then on E, (3.6) holds obviously. Now by setting $E_n = \{r : r \in E \text{ and } m(r, A_s) \leq M \log T(r, f_n) \quad (n = 1, ..., k)\}$, we have $\bigcup_{n=1}^{k} E_n = E$. It is easy to see that there exists at least one E_n, say E_1, which has an infinite linear measure and on which

$$\lim_{\substack{r\to\infty \\ r\in E_1}} \frac{\log_p m(r, A_s)}{\log r} = \sigma_p(A_s), \tag{3.8}$$

and

$$m(r, A_s) \leq M \log T(r, f_1) \quad (r \in E_1). \tag{3.9}$$

From (3.8) and (3.9) we have $i(f_1) \geq p+1$ and $\sigma_{p+1}(f_1) \geq \sigma_p(A_s)$. This and the fact that $i(f_1) \leq p+1$ and $\sigma_{p+1}(f_1) \leq \sigma_p(A_s)$ yield $i(f_1) = p+1$ and $\sigma_{p+1}(f_1) = \sigma_p(A_s)$. The proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

Assume that f is a solution of (1.5) and $g_1, ..., g_k$ are k entire solutions of the corresponding homogeneous equation (1.4). Then by the proof of Theorem 1.1, we know that $i(g_j) \leq p+1$, $\sigma_{p+1}(g_j) \leq \sigma_p(A_s)$ $(j = 1, 2, 3..., k)$ and there exists a g_j, say g_1, satisfying $i(g_1) = p+1$, $\sigma_{p+1}(g_1) = \sigma_p(A_s)$. Thus by variation of parameters, f can be expressed in the form

$$f(z) = B_1(z) g_1(z) + ... + B_k(z) g_k(z), \tag{4.1}$$

where $B_1(z), ..., B_k(z)$ are determined by

$$\begin{aligned} B_1'(z) g_1(z) + ... + B_k'(z) g_k(z) &= 0 \\ B_1'(z) g_1'(z) + ... + B_k'(z) g_k'(z) &= 0 \end{aligned}$$

....

$$B_1'(z) g_1^{(k-1)}(z) + ... + B_k'(z) g_k^{(k-1)}(z) = F. \tag{4.2}$$

Noting that the Wronskian $W(g_1, g_2, ..., g_k)$ is a differential polynomial in $g_1, g_2, ..., g_k$ with constant coefficients, it follows that

$$\sigma_{p+1}(W) \leq \max\{\sigma_{p+1}(g_j) : j = 1, ..., k\} \leq \sigma_p(A_s).$$

Set

$$W_j = \begin{vmatrix} g_1, ..., \overset{(j)}{0}, ..., g_k \\ \cdots \\ \cdots \\ g_1^{(k-1)}, ..., F, ..., g_k^{(k-1)} \end{vmatrix} = F.G_j \quad (j = 1, ..., k), \tag{4.3}$$

where $G_j(g_1, g_2, ..., g_k)$ are differential polynomials in $g_1, g_2, ..., g_k$ and of their derivatives with constant coefficients. So

$$\sigma_{p+1}(G_j) \leq \max\{\sigma_{p+1}(g_j) : j = 1, ..., k\} \leq \sigma_p(A_s) \quad (j = 1, ..., k),$$

$$B_j' = \frac{W_j}{W} = \frac{F.G_j}{W} \quad (j = 1, ..., k). \tag{4.4}$$

Since $i(F) = q < p+1$ or $i(F) = p+1$, $\sigma_{p+1}(F) < \sigma_p(A_s)$, then by Lemma 2.3, we obtain

$$\sigma_{p+1}(B_j) = \sigma_{p+1}\left(B_j'\right) \leq \max(\sigma_{p+1}(F),\ \sigma_p(A_s)) = \sigma_p(A_s) \quad (j = 1, ..., k). \tag{4.5}$$

Then from (4.1) and (4.5), we get $i(f) \leq p+1$ and

$$\sigma_{p+1}(f) \leq \max\{\sigma_{p+1}(g_j),\ \sigma_{p+1}(B_j) : j=1,...,k\} \leq \sigma_p(A_s). \tag{4.6}$$

Now we set

$$H = \{f_c = cg_1 + f_0,\ c \in \mathbf{C}\}, \tag{4.7}$$

where f_0 is a solution of (1.5). Obviously, every f_c in H is a solution of (1.5). Now we prove that for any two solutions f_a and f_b $(a \neq b)$ in H, there is at least one solution, say f_a, among f_a and f_b satisfying $i(f_a) = p+1$ and $\sigma_{p+1}(f_a) = \overline{\lambda}_{p+1}(f_a) = \sigma_p(A_s)$. Since $f_a = (a-b)\,g_1 + f_b$, then

$$T(r,g_1) \leq T(r,f_a) + T(r,f_b) + O(1). \tag{4.8}$$

Assume that the set E_1 satisfies the condition as required in proof of Theorem 1.1. Then there exists at least one of f_a and f_b, say f_a, such that there is a subset E_4 of E_1 with infinite linear measure and

$$T(r,f_b) \leq T(r,f_a),\ \text{for } r \in E_4. \tag{4.9}$$

We get from (4.8) and (4.9)

$$T(r,g_1) \leq 2T(r,f_a) + O(1), \quad \text{for } r \in E_4. \tag{4.10}$$

Thus, $i(f_a) \geq p+1$ and $\sigma_{p+1}(f_a) \geq \sigma_{p+1}(g_1) = \sigma_p(A_s)$ and hence $i(f_a) = p+1$, $\sigma_{p+1}(f_a) = \sigma_p(A_s) = \sigma$.

Now we prove that $\sigma_{p+1}(f_a) = \overline{\lambda}_{p+1}(f_a) = \sigma$. By (1.5), it is easy to see that if f_a has a zero at z_0 of order $\alpha\,(> k)$, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f_a}\right) \leq k\,\overline{n}\left(r, \frac{1}{f_a}\right) + n\left(r, \frac{1}{F}\right) \tag{4.11}$$

and

$$N\left(r, \frac{1}{f_a}\right) \leq k\,\overline{N}\left(r, \frac{1}{f_a}\right) + N\left(r, \frac{1}{F}\right). \tag{4.12}$$

Now (1.5) can be rewritten as

$$\frac{1}{f_a} = \frac{1}{F}\left(\frac{f_a^{(k)}}{f_a} + A_{k-1}\frac{f_a^{(k-1)}}{f_a} + ... + A_1\frac{f_a'}{f_a} + A_0\right). \tag{4.13}$$

By (4.13), we have

$$m\left(r,\frac{1}{f_a}\right)\leq\sum_{j=1}^{k}m\left(r,\frac{f_a^{(j)}}{f_a}\right)+\sum_{j=1}^{k}m\left(r,A_{k-j}\right)+m\left(r,\frac{1}{F}\right)+O\left(1\right). \quad (4.14)$$

Applying the Lemma 2.5, we have

$$m\left(r,\frac{f_a^{(j)}}{f_a}\right)=O\left(\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}\right)\quad\left(j=1,...,k-1\right),\quad\left(\sigma_{p+1}\left(f_a\right)=\sigma\right), \quad (4.15)$$

holds for all r outside a set $E_3\subset(0,+\infty)$ with a linear measure $m\left(E_3\right)=\delta<+\infty$. By (4.12), (4.14) and (4.15), we get

$$T\left(r,f_a\right)=T\left(r,\frac{1}{f_a}\right)+O\left(1\right)$$

$$\leq k\overline{N}\left(r,\frac{1}{f_a}\right)+\sum_{j=1}^{k}T\left(r,A_{k-j}\right)+T\left(r,F\right)+O\left(\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}\right)\quad\left(|z|=r\notin E_3\right). \quad (4.16)$$

For sufficiently large r, we have

$$T\left(r,A_0\right)+...+T\left(r,A_{k-1}\right)\leq k\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}. \quad (4.17)$$

If $i\left(F\right)=q<p+1$, then $q-1\leq p-1$ and

$$T\left(r,F\right)\leq\exp_{q-1}\left\{r^{\sigma_q(F)+\varepsilon}\right\}\leq\exp_{p-1}\left\{r^{\sigma_q(F)+\varepsilon}\right\}\qquad\left(\sigma_q\left(F\right)<\infty\right). \quad (4.18)$$

Thus, by $(4.16)-(4.18)$, we have

$$T\left(r,f_a\right)\leq k\ \overline{N}\left(r,\frac{1}{f_a}\right)+k\ \exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}$$

$$+\exp_{p-1}\left\{r^{\sigma_q(F)+\varepsilon}\right\}+O\left(\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}\right)\qquad\left(|z|=r\notin E_3\right). \quad (4.19)$$

Hence for any f_a with $\sigma_{p+1}\left(f_a\right)=\sigma$, by (4.19) and Lemma 2.6, we have $\sigma_{p+1}\left(f_a\right)\leq\overline{\lambda}_{p+1}\left(f_a\right)$. Therefore, $\overline{\lambda}_{p+1}\left(f_a\right)=\sigma_{p+1}\left(f_a\right)=\sigma$.
If $i\left(F\right)=p+1$ and $\sigma_{p+1}\left(F\right)<\sigma_p\left(A_s\right)=\sigma$, then

$$T\left(r,F\right)\leq\exp_p\left\{r^{\sigma_{p+1}(F)+\varepsilon}\right\}\leq\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}. \quad (4.20)$$

Thus, by (4.16) – (4.17) and (4.20), we have

$$T(r, f_a) \leq k\, \overline{N}\left(r, \frac{1}{f_a}\right) + k\, \exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}$$

$$+\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\} + O\left(\exp_{p-1}\left\{r^{\sigma+\varepsilon}\right\}\right) \qquad (|z| = r \notin E_3)\,. \qquad (4.21)$$

By using similar reasoning as above, we obtain from (4.21) and Lemma 2.6 that $\overline{\lambda}_{p+1}(f_a) = \sigma_{p+1}(f_a) = \sigma$. The proof of Theorem 1.2 is complete.

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