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On Singular Solutions of a Second Order Differential Equation

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Abstract

In the paper, sufficient conditions are given under which all nontrivial solutions of $(g(a(t)y'))' + r(t)f(y) = 0$ are proper where $a > 0, r > 0, f(x)x > 0, g(x)x > 0$ for $x \neq 0$ and g is increasing on R. A sufficient condition for the existence of a singular solution of the second kind is given.

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1 Introduction

Consider the differential equation

$$(g(a(t)y'))' + r(t)f(y) = 0, \tag{1}$$

where $a \in C^0(R_+),\ r \in C^0(R_+), g \in C^0(R), f \in C^0(R), R_+ = [0,\infty), R = (-\infty,\infty)$, g is increasing on R and

$$a > 0, r > 0 \quad \text{on} \quad R_+,\ f(x)x > 0 \quad \text{and} \quad g(x)x > 0 \quad \text{for} \quad x \neq 0. \tag{2}$$

Sometimes the following condition will be assumed.

$$\lim_{z\to\infty} g(z) = -\lim_{z\to-\infty} g(z) = \infty. \tag{3}$$

Definition. A function y defined on $J \subset R_+$ is called a solution of (1) if $y \in C^1(J), g(a(t)y') \in C^1(J)$ and (1) holds on J.

It is clear that (1) is equivalent to the system $y_1 = y, \quad y_2 = g(a(t)y')$,

$$y_1' = \frac{g^{-1}(y_2)}{a(t)}, \quad y_2' = -rf(y_1), \tag{4}$$

where g^{-1} is the inverse function to g. Hence, as the right-hand sides of (4) are continuous, the Cauchy problem for (1) has a solution.

Definition. Let y be a continuous function defined on $[0,\tau) \subset R_+$. Then y is called oscillatory if there exists a sequence $\{t_k\}_{k=1}^{\infty}, t_k \in [0,\tau), k = 1,2,\ldots$ of zeros of y such that $\lim_{k\to\infty} t_k = \tau$ and y is nontrivial in any left neighbourhood of τ.

Definition. A solution y of (1) is called proper if it is defined on R_+ and $\sup_{\tau\le t<\infty} |y(t)| > 0$ for every $\tau \in (0,\infty)$. It is called singular of the first kind if it is defined on R_+, there exists $\tau \in (0,\infty)$ such that $y \equiv 0$ on $[\tau,\infty)$ and $\sup_{T\le t<\tau} |y(t)| > 0$ for every $T \in [0,\tau)$. It is called singular of the second kind if it is defined on $[0,\tau), \tau < \infty$, and cannot be defined at $t = \tau$. A singular solution y is called oscillatory if it is an oscillatory function on $[0,\tau)$.

In the sequel we will investigate only solutions that are defined either on R_+ or on $[0,\tau), \tau < \infty$ and cannot be defined at $= \tau$.

Remark 1. (i) According to (2) every nontrivial solution of (1) is either proper, singular of the first kind, or singular of the second kind.
(ii) A solution is singular of the second kind if and only if

$$\lim_{t\to\tau-}\sup|y'(t)| = \infty.$$

(iii) If y is a singular solution of the first kind then $y(\tau) = y'(\tau) = 0$.

Consider the equation with p-Laplacian

$$(A(t)|y'|^{p-1}y')' + r(t)f(y) = 0, \tag{5}$$

where $p > 0, A \in C^0(R_+)$ and $A > 0$ on R_+. This is a special case of (1) with $g(z) = |z|^{p-1}z$ and $a = A^{\frac{1}{p}}$. It is widely studied now; see e.g. [3], [4], [8] and the references therein.

Recall the following sufficient conditions for the nonexistence of singular solutions of (5).

Theorem A. *(i) If $M > 0, M_1 > 0$ and $|f(x)| \leq M_1|x|^p$ for $|x| \leq M$, then there exists no singular solution of the first kind of* (5)*.*
(ii) If $M > 0, M_1 > 0$ and $|f(x)| \leq M_1|x|^p$ for $|x| \geq M$, then there exists no singular solution of the second kind of (5)*.*
(iii) Let the function $A^{\frac{1}{p}}r$ be locally absolutely continuous on R_+. Then every solution of (5) *is proper.*

Proof. Cases (i) and (ii) are simple applications of results in [8, Theorems 1.1 and 1.2] (also see [1]). Case (iii) is proved in [3, Theorem 2] . ☐

Theorem A (iii) shows that if A and r are smooth enough, singular solutions do not exist. But the following theorem shows that singular solutions may exist.

Theorem B ([3] Theorem 4). *Let $0 < \lambda < p$ $(0 < p < \lambda)$. Then there exists a positive continuous function r defined on R_+ such that the equation*

$$(|y'|^{p-1}y')' + r(t)|y|^\lambda \operatorname{sgn} y = 0 \tag{6}$$

has a singular solution of the first (of the second) kind.

Note that the proof of Theorem B uses ideas from [5] and [6] for the case $p = 1$.

The goal of this paper is to generalize results of Theorems A and B to Eq. (1).

2 Main results

We begin our investigations with simple properties of singular solutions.

Lemma 1. *Let y be a singular solution of* (1) *and τ be the number in its definition. Then y is oscillatory if and only if y' is an oscillatory function on $[0,\tau)$.*

Proof. It follows directly from system (4) since, due to (2), y' is an oscillatory function on $[0,\tau)$ if and only if $y_2 = g(a(t)y')$ is oscillatory on the same interval. ☐

Theorem 1. *(i) Every singular solution of the first kind of* (1) *is oscillatory. (ii) If* (3) *holds, then every singular solution of the second kind of* (1) *is oscillatory.*

Proof. (i) Let y be a singular solution of the first kind of (1) and $\tau < \infty$ be the number from its definition. Suppose, contrarily, that $y > 0$ in a left neighborhood of τ (the case $y < 0$ can be studied similarly). Then (1) and (2) yield $g(ay')$ is decreasing and hence, ay' is decreasing on I. From this and from Remark 1 (iii), we have $y'(\tau) = 0$ and hence $y' > 0$ on I; this contradicts the fact that $y > 0$ on I and $y(\tau) = 0$.
(ii) Let y be a singular solution of the second kind of (1) defined on $[0,\tau), \tau < \infty$. Suppose, contrarily, that $y > 0$ in a left neighbourhood $I = [\tau_1, \tau)$ of τ (the case $y < 0$ can be studied similarly). Then (1) and (2) yield ay' is decreasing on I and according to Remark 1 (ii) and Lemma 1 $\lim_{t\to\tau_-} y'(t) = -\infty$. Hence y is positive and decreasing in a left neighbourhood of τ and $rf(y)$ is bounded on I. From this, we have

$$-\infty = g(a(\tau)y'(\tau)) - g(a(\tau_1)y'(\tau_1)) = -\int_{\tau_1}^{\tau} r(t)f(y(t))dt > -\infty.$$

This contradiction proves the statement. ☐

The following example shows that singular solutions of the second kind may be nonoscillatory if (3) does not hold.

Example 1. The differential equation

$$\left(\left(1 - \frac{1}{(|y'|+1)^2}\right)\operatorname{sgn} y'\right)' + r(t)y = 0$$

with $r(t) = \frac{8}{(2\sqrt{1-t}+1)^4}$ for $t \in [0,1]$ and $r(t) = 8$ for $t > 1$ has a nonoscillatory singular solution of the second kind of the form $y = \frac{1}{2} + \sqrt{1-t}$.

The first result for the nonexistence of singular solutions follows from more common results of Mirzov [8] that are specified for (1).

Theorem 2. *Let* $d_1(z) = \max(|g^{-1}(z)|, |g^{-1}(-z)|)$ *and*

$$d_2(z) = \max\left(\max_{0\le s\le |z|} f(s), -\min_{0\le s\le |z|} f(-s)\right) \quad \text{for } \ z \in R.$$

(i) If for every $t^* \in R_+$ *the problem*

$$z' = \frac{1}{a(t)} d_1(d_2(z)\int_{t*}^{t} r(s)ds), \ y(t^*) = 0 \tag{7}$$

has the trivial solution on $[t^*, \infty)$ *only, then* (1) *has no singular solution of the first kind.*
(ii) If for every $c_1 \ge 0$ *and* $c_2 \ge 0$ *the Cauchy problem*

$$z' = \frac{1}{a(t)} d_1\left(c_1 + d_2(z)\int_0^t r(s)ds\right), z(0) = c_2 \tag{8}$$

has the upper solution defined on R_+*, then* (1) *has no singular solution of the second kind.*

Proof. This follows from [8, Theorems 1.1 and 1.2 and Remark 1.1] setting $\varphi_1(t,z) = \frac{1}{a(t)} d_1(z)$ and $\varphi_2(t,z) = r(t)d_2(z)$. □

Corollary 1. *Let* $g(z) = -g(-z), f(z) = -f(-z)$*, and let* f *be nondecreasing on* R_+*.*
(i) If there exists a continuous function $R(t)$ *and a right neighbourhood* I *of* $z = 0$ *such that*

$$f(z)\int_0^t r(s)\,ds \le g(R(t)z)$$

for $t \in R_+$ *and for* $z \in I$*, then* (1) *has no singular solution of the first kind.*
(ii) For any $c > 0$ *let there exist a continuous function* $R_1(c,t)$ *and a neighbourhood* $I_1(c)$ *of* ∞ *such that* $c + f(z)\int_0^t r(s)\,ds \le g(R_1(c,t)z)$, $t \in R_+$, $z \in I_1(c)$*. Then there exists no singular solution of the second kind of* (1)*.*

Proof. In our case, $d_1(z) = g^{-1}(z)$ and $d_2(z) = f(z), z \in R_+$. Moreover,

$$d_1(z) = d_1(-z) \quad \text{and} \quad d_2(z) = d_2(-z). \tag{9}$$

(i) It is clear that (7) can be studied only for $|z| \in I$. Then

$$0 \le d_1\big(d_2(z)\int_{t^*}^t r(s)ds\big) = g^{-1}\big(f(z)\int_{t^*}^t r(s)ds\big) \le g^{-1}\big(f(z)\int_0^t r(s)ds\big) \\ \le R(t)z,$$

$t \in R_+$ and $z \in I$. From this and from (9), Eq. (7) is sublinear in I, the trivial solution $z \equiv 0$ is unique, and the statement follows from Th. 2 (i).
(ii) We have $0 \le d_1\big(c_1 + d_2(z)\int_0^t r(s)ds\big) = g^{-1}\big(c_1 + f(z)\int_0^t r(s)ds\big) \le R_1(c_1,t)z,\ t \in R_+, z \in I_1(c_1)$. From this and from (9), Eq. (8) is sublinear for large values of z, (8) has the upper solution defined on R_+, and the statement follows from Theorem 2 (ii). $\square$

Corollary 2. *Let $p > 0, M > 0$ and $M_1 > 0$.*
(i) Let

$$|g(z)| \ge M|z|^p \quad \textit{and} \quad |f(z)| \le M_1|z|^p \tag{10}$$

hold in a neighbourhood I of $z = 0$. Then (1) *has no singular solution of the first kind.*
(ii) Let $z_0 \in R_+$ be such that (10) *holds for $|z| \ge z_0$. Then* (1) *has no singular solution of the second kind.*

Proof. Let d_1 and d_2 be defined as in Theorem 2.
(i) Since (10) yields $d_1(z) \le \frac{|z|^{\frac{1}{p}}}{M}$ and $d_2(z) \le M_1|z|^p$ for $z \in I$, we have

$$0 \le d_1\big(d_2(z)\int_{t^*}^t r(s)ds\big) \le \frac{1}{M}\left(M_1|z|^p\int_{t^*}^t r(s)ds\right)^{\frac{1}{p}} \\ = M^{-1}\big(M_1\int_{t^*}^t r(s)ds\big)^{\frac{1}{p}}|z|,\ z \in I, t^* \in R_+.$$

The remainder of the proof is similar to that of Cor. 1 (i).
(ii) Similarly, $d_1(z) \le \frac{|z|^{\frac{1}{p}}}{M}$ and $d_2(z) \le M_1|z|^p$ for $|z| \ge z_0$, and so

$$0 \le d_1\big(c_1 + d_2(z)\int_0^t r(s)ds\big) \le \frac{1}{M}\left(c_1 + M_1|z|^p\int_0^t r(s)ds\right)^{\frac{1}{p}}, \\ t \in R_+,\ |z| \ge z_0,\ c_1 \ge 0.$$

From this, equation (8) is sublinear for large $|z|$, the problem (8) has the upper solution defined on R_+, and the statement follows from Theorem 2 (ii). $\square$

Remark 2. Theorem A (i), (ii) is special case of Corollary 2 with $g(z) = |z|^{p-1}z,\ a = A^{\frac{1}{p}}$, and $M = 1$.

The following theorem generalizes Theorem A (iii); sufficient conditions for the nonexistence of singular solutions are posed on the functions a and r only.

Theorem 3. *Let the function ar be locally absolute continuous on R_+, y be a nontrivial solution of* (1) *defined on $[0, b), b \le \infty, ar = r_0 - r_1$ on R_+, and*

$$\rho(t) = \int_0^{g(a(t)y'(t))} g^{-1}(\sigma)d\sigma + a(t)r(t)\int_0^{y(t)} f(\sigma)d\sigma \ge 0, \tag{11}$$

where r_0 and r_1 are nonnegative, nondecreasing and continuous functions. Then, for $0 \le s < t < b$,

$$\rho(s)\exp\left\{-\int_s^t \frac{r_1'(\sigma)d\sigma}{a(\sigma)r(\sigma)}\right\} \le \rho(t) \le \rho(s)\exp\left\{\int_s^t \frac{r_0'(\sigma)d\sigma}{a(\sigma)r(\sigma)}\right\}. \tag{12}$$

Moreover, y is not singular of the first kind, and if (3) *holds, then y is proper.*

Proof. Since ar is of locally bounded variation, the continuous nondecreasing functions r_0 and r_1 exist such that $ar = r_0 - r_1$, and they can be chosen to be nonnegative on R_+. Moreover, $r_0' \in L_{\mathrm{loc}}(R_+)$ and $r_1' \in L_{\mathrm{loc}}(R_+)$. Then ρ is absolute continuous on $[s, t]$ and

$$\rho'(\tau) = (a(\tau)r(\tau))'\int_0^{y(\tau)} f(\sigma)d\sigma, \tau \in [s, t] \quad \text{a.e.}$$

Let $\varepsilon > 0$ be arbitrary. Then (2) implies $\rho(\tau) \ge 0$ on $[s, t]$, both terms in (11) are nonnegative, and

$$\frac{\rho'(\tau)}{\rho(\tau)+\varepsilon} = \frac{a(\tau)r(\tau)}{\rho(\tau)+\varepsilon}\int_0^{y(\tau)} f(\sigma)d\sigma \frac{r_0'(\tau) - r_1'(\tau)}{a(\tau)r(\tau)};$$

hence,

$$-\frac{r_1'(\tau)}{a(\tau)r(\tau)} \le \frac{\rho'(\tau)}{\rho(\tau)+\varepsilon} \le \frac{r_0'(\tau)}{a(\tau)r(\tau)} \quad \text{a.e. on } [s, t].$$

An integration and (11) yield

$$\exp\left\{-\int_s^t \frac{r_1'(\sigma)d\sigma}{a(\sigma)r(\sigma)}\right\} \le \frac{\rho(t)+\varepsilon}{\rho(s)+\varepsilon} \le \exp\left\{\int_s^t \frac{r_0'(\sigma)ds}{a(\sigma)r(\sigma)}\right\}.$$

Since $\varepsilon > 0$ is arbitrary, (12) holds.

Let y be singular of the first kind. Then according to its definition and Remark 1 (iii), there exists $\tau \in (0, \infty)$ such that $y(\tau) = 0, y'(\tau) = 0$, and

$$\sup_{T \le t < \tau} |y(t)| > 0 \quad \text{for every} \quad T \in [0, \tau). \tag{13}$$

Hence, (11) and (12) yield $\rho(\tau) = 0$ and $\rho(t) = 0$ on $[0, \tau]$. From this and from (2), we have $y = 0$ on $[0, \tau]$. This contradiction to (13) proves that y is not singular of the first kind.

Let (3) be valid and y be a singular solution of the second kind. Then according to Remark 1 (ii), there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \in [0, b), \lim_{k\to\infty} t_k = b$, and $\lim_{k\to\infty} |y'(t_k)| = \infty$. Hence, (3) yields $\lim_{k\to\infty} g(a(t_k)y'(t_k)) = \infty$. From this and from (12) we have for $s = 0$ and $t = t_k, k = 1, 2, \ldots$, that

$$\infty = \lim_{k\to\infty} \rho(t_k) \le \rho(0) \exp\left\{ \int_0^{\tau} \frac{r_0'(\sigma) d\sigma}{a(\sigma) r(\sigma)} \right\}.$$

The contradiction proves that y is not singular of the second kind and, according to Remark 1 (i), it is proper. $\square$

Theorem 4. *Let the assumptions of Theorem 3 be valid and let*

$$\rho_1(t) = \frac{1}{a(t)r(t)} \int_0^{g(a(t)y'(t))} g^{-1}(\sigma) d\sigma + \int_0^{y(t)} f(\sigma) d\sigma. \tag{14}$$

Then for $0 \le s < t < b$ we have

$$\rho_1(s) \exp\left\{ -\int_s^t \frac{r_0'(\sigma) d\sigma}{a(\sigma) r(\sigma)} \right\} \le \rho_1(t) \le \rho_1(s) \exp\left\{ \int_s^t \frac{r_1'(\sigma) d\sigma}{a(\sigma) r(\sigma)} \right\}. \tag{15}$$

Proof. The proof is similar to that of Theorem 3 since

$$\rho_1'(\tau) = -\frac{(a(\tau)r(\tau))'}{(a(\tau)r(\tau))^2} \int_0^{g(a(\tau)y'(\tau))} g^{-1}(\sigma) d\sigma = \frac{r_1'(\tau) - r_0'(\tau)}{a(\tau) r(\tau)} \frac{\int_0^{g(a(\tau)y'(\tau))} g^{-1}(\sigma) d\sigma}{a(\tau) r(\tau)}$$

a.e. on $[s, t]$. $\square$

Remark 3. Inequalities (12) and (15) are proved in [7] for Equation (5) with $p = 1$ and $a \equiv 1$, in [3] for $g(z) = |z|^{p-1} z$ with $p > 0$, and in [8] for Equation (6).

Corollary 3. *Let ar be locally absolute continuous on R_+. Let ρ and ρ_1 be given by* (11) *and* (14)*, respectively.*
(i) If ar is nondecreasing on R_+, then for an arbitrary solution y of (1)*, ρ is nondecreasing and ρ_1 is nonicreasing on R_+.*
(ii) If ar is nonincreasing on R_+, then for an arbitrary solution y of (1)*, ρ is nonincreasing and ρ_1 is nondecreasing on R_+.*

Proof. It follows from (12) and (15) as $r_0 \equiv r$ and $r_1 \equiv 0$ in case (i), and $r_0 \equiv r(0), r_1 = r(0) - r$ in case (ii). ☐

In [1] there is an example of Eq. (1) with $a \equiv 1, g(z) \equiv z, f(z) = |z|^\lambda \operatorname{sgn} z$ and $0 < \lambda < 1$ for which there exists a proper solution y with infinitely many accumulation points of zeros. The following corollary gives a sufficient condition under which every solution of (1) has no accumulation point of zeros in its interval of definition.

Corollary 4. *If ar is locally absolute continuous on R_+, then every nontrivial solution y of* (1) *has no accumulation point of its zeros and has no double zero in its interval of definition.*

Proof. Let τ be an accumulation point of zeros or a double zero of a solution y of (1) lying in its definition interval. Hence, $y(\tau) = 0$ and $y'(\tau) = 0$. Then, $\bar{y}(t) = y(t)$ on $[0, \tau]$ and $\bar{y}(t) = 0$ for $t > \tau$ is a singular solution of the first kind of (1) that contradits Theorem 3. ☐

Corollary 5. *Let ar be locally absolute continuous and nondecreasing (nonincreasing) on R_+. Let y be a solution of* (1) *defined on $[0, b), b \le \infty$, and $\{t_k\}_{k=1}^N, N \le \infty$, be a (finite or infinite) increasing sequence of zeros of y' lying in $[0, b)$. Then the sequence of local extrema $\{|y(t_k)|\}_{k=1}^N$ is nonincreasing (nondecreasing).*

Proof. Let ar be nondecreasing on R_+. As all assumptions of Corollary 3 are fulfilled, ρ_1 is nonincreasing and the statement follows from $\rho_1(t_k) = \int_0^{y(t_k)} f(\sigma) d\sigma$ and (2). If ar is nonincreasing, the proof is similar. ☐

The following corollary generalizes Theorem B and it shows that singular solutions may exist if ar is not locally absolutely continuous on R_+.

Corollary 6. *Let $A \equiv 1, 0 < \lambda < p$ $(0 < p < \lambda)$ and $\lim_{z \to 0} \frac{f(z)}{|z|^\lambda \operatorname{sgn} z} = M \in (0, \infty)$. Then there exists a positive continuous function r such that Equation* (5) *has a singular solution of the first (second) kind.*

Proof. Let $0 < \lambda < p$. Then Theorem B yields the existence of a positive continuous function $\bar{r}$ defined on R_+ such that (6) (with $r = \bar{r}$) has a singular solution y of the first kind. Put

$$r(t) = \bar{r}(t)\frac{|y(t)|^\lambda \operatorname{sgn} y(t)}{f(y(t))} \quad \text{if} \quad y(t) \neq 0$$

and $r(t) = \frac{\bar{r}(t)}{M}$ if $y(t) = 0$. From this and from (2), the function r is positive and continuous on R_+, and

$$\left(|y'(t)|^{p-1}y'(t)\right)' = -\bar{r}(t)|y(t)|^\lambda \operatorname{sgn} y(t) = -r(t)f(y(t));$$

hence y is also a solution of (5).
If $0 < p < \lambda$, then the proof is similar. $\square$

Example 1 shows that the statement of Theorem 3 does not hold if (3) is not valid; singular solutions of the second kind may exist. The following theorem gives sufficient conditions for the existence of such solutions.

Theorem 5. *Let* $M \in (0, \infty), a \equiv 1$ *on* R_+, *and* $g \in C^1(R)$.
(i) If $\beta \in \{-1, 1\}, \lambda > 2$, *and*

$$0 < g'(z) \leq |z|^{-\lambda} \quad \textit{for} \quad \beta z \geq M, \tag{16}$$

then (1) *possesses a singular solution of the second kind.*
(ii) If

$$g'(z) \geq |z|^{-2} \quad \textit{for} \quad |z| \geq M, \tag{17}$$

then (1) *has no nonoscillatory singular solution of the second kind.*

Proof. (i) Let $\beta = 1$; if $\beta = -1$, the proof is similar. Consider the differential equation

$$y'' = -r(t)f(y)G(y'), \tag{18}$$

where $G \in C^0(R), G(z)z > 0$ for $z \neq 0$, and

$$G(z) = (g'(z))^{-1} \quad \text{for} \quad z \geq M. \tag{19}$$

Put $M_1 = [(\lambda - 1)\min_{0\leq s\leq 1} r(s) \min_{-3\leq s\leq -\frac{1}{2}} |f(s)|]^{-\frac{1}{\lambda-1}}$. Let τ be such that

$$0 < \tau \leq 1, \tau \leq 2M_1^{-\frac{\lambda-1}{\lambda-2}}, \tau \leq \left[\max_{0\leq s\leq 1} r(s) \max_{-3\leq s\leq -\frac{1}{2}} |f(s)|\right]^{-1} \int_M^{2M} \frac{ds}{G(s)}$$

and

$$\tau \le g(M)\left[\max_{0\le s\le 1} r(s) \max_{-4\le s\le -3}|f(s)|\right]^{-1}. \tag{20}$$

Then (16) and (19) yield $G(z) \ge z^\lambda$ for $z \ge M$ and according to Theorem 1 in [2] (with $n = 2, M = M, \beta = 1, c_0 = -1, \alpha = -1, T = \frac{\tau}{2}, N = 3$; see the proof of Theorem 1 and (13) – (17) as well), there exists a solution y of (18) defined in $[\frac{\tau}{2}, \tau)$ such that

$$\lim_{t\to\tau_-} y(t) = -1,\ \lim_{t\to\tau_-} y'(t) = \infty,$$

and

$$-3 \le y(t) \le -\frac{1}{2},\ M \le y'(t) \le M_1(\tau - t)^{-\frac{1}{\lambda-1}},\ t \in [\frac{\tau}{2}, \tau). \tag{21}$$

Hence, (16), (19) and (21) yield y is the solution of Eq. (1) on $[\frac{\tau}{2}, \tau)$. We will prove that y can be defined on $[0, \tau)$ and, thus, y is singular of the second kind. Let, to the contrary, y be defined on $(\bar\tau, \tau) \subset [0, \tau)$ so that it cannot be defined at $\bar\tau$. Then

$$\limsup_{t\to\bar\tau_+} |y'(t)| = \infty. \tag{22}$$

First, we prove that

$$y'(t) > 0 \quad \text{on} \quad (\bar\tau, \tau). \tag{23}$$

Suppose, that $\tau_1 \in (\bar\tau, \tau)$ exists such that $y'(\tau_1) = 0$ and $y'(t) > 0$ on (τ_1, τ); according to (21), $\tau_1 < \frac{\tau}{2}$. Hence, y is increasing on (τ_1, τ) and negative. From this, (1), and (2), the functions $g(y')$ and y' are increasing on (τ_1, τ). Further, we estimate y on $[\tau_1, \frac{\tau}{2}]$ using (21) and the definition of τ. We have

$$\begin{aligned} -3 \ge y(t) = {} & y(\frac{\tau}{2}) + \int_{\frac{\tau}{2}}^{t} y'(s)ds \ge y\left(\frac{\tau}{2}\right) - y'\left(\frac{\tau}{2}\right)\left(\frac{\tau}{2} - t\right) \\ \ge -3 - M_1(\frac{\tau}{2})^{-\frac{1}{\lambda-1}}\frac{\tau}{2} \ge -3 - M_1(\frac{\tau}{2})^{1-\frac{1}{\lambda-1}} \ge -4, t \in [\tau_1, \frac{\tau}{2}]. \end{aligned} \tag{24}$$

An integration of (1) on $[\tau_1, \frac{\tau}{2}]$, (2), (21), (24), and $\tau \le 1$, yield

$$\begin{aligned} g(M) \le g\left(y'(\frac{\tau}{2})\right) - g(y'(\tau_1)) = - \int_{\tau_1}^{\frac{\tau}{2}} r(s) f\left(y(s)\right) ds \\ \le \max_{0\le s\le 1} r(s) \max_{-4\le s\le -3} |f(s)| \frac{\tau}{2}. \end{aligned}$$

This contradiction to (20) proves that (23) is valid. From this and from (21), $y < 0$ on $(\bar{\tau}, \tau)$, and (1) yields $g(y')$ and y' are incerasing on this interval. Thus, according to (23), y' is bounded in a right neighbourhood of $\bar{\tau}$ which contradicts (22), and so y is defined on so $[0, \tau)$.
(ii) Suppose, that y is a nonoscillatory singular solution of (1) of the second kind defined on $[0, \tau)$. Then Lemma 1 and Remark 1 (ii) yield $\lim_{t\to\tau_-} |y'(t)| = \infty$ and $\lim_{t\to\tau_-} y(t) = C \in [-\infty, \infty]$. Suppose that

$$\lim_{t\to\tau_-} y'(t) = \infty \tag{25}$$

(the opposite case can be studied similarly).

Let $C \in (-\infty, \infty)$. Due to (1) and (17), y is a solution of Eq. (18) and (19) on $[T, \tau) \in [0, \tau)$ where T is such that $y'(t) \geq M$ on $[T, \tau)$. But this contradicts a result in [2, Theorem 2].

Let $C = \infty$. Then $\lim_{t\to\tau_-} y(t) = \infty$. But according to (1) and (2), the functions $g(y')$ and y' are decreasing in a left neighbourhood of τ, which contradicts (25). Clearly, the case $C = -\infty$ is impossible due to (25). ☐

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References

[1] Bartušek M., Asymptotic properties of oscillatory solutions of differential equations on n-th order, *Folia F.S.N. Univ. Brun. Masar. Math. 3* (1992).

[2] Bartušek M., On existence of singular solutions of n-th order differential equations, *Arch. Math. (Brno)* 36, 395–404 (2000).

[3] Bartušek M., Singular solutions for the differential equation with p-Laplacian, *Arch. Math. (Brno)* 41, 123–128 (2005).

[4] Cecchi M. & Došlá Z. & Marini M., On nonoscillatory solutions of differential equations with p-Laplacian, *Advances in Math. Scien. Appl.* 11, 419–436 (2001).

[5] Coffman C. V. & Ullrich D.F., On the continuation of solutions of a certain non-linear differential equation, *Monath. Math.* B 71, 385–392 (1967).

[6] Heidel J. W., Uniqueness, continuation and nonoscillation for a second order differential equation, *Pacif. J. Math.* 32, 715–721 (1970).

[7] Kiguradze I. & Chanturia T., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Kluwer, Dordrecht (1993).

[8] Mirzov D. D., *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations.* Adygea, Maikop (1993) (in Russian); *Folia F.S.N. Univ. Masar. Brunen. Math. 14* (2004).

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