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The Shooting Method and Multiple Solutions of Two/Multi-Point BVPs of Second-Order ODE

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Abstract

Within the last decade, there has been growing interest in the study of multiple solutions of two- and multi-point boundary value problems of nonlinear ordinary differential equations as fixed points of a cone mapping. Undeniably many good results have emerged. The purpose of this paper is to point out that, in the special case of second-order equations, the shooting method can be an effective tool, sometimes yielding better results than those obtainable via fixed point techniques.

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1 Introduction

The goal of this paper is not to attempt to derive the most general results possible, but to underscore the point that the classical shooting method can be effectively used for the subject matter at hand. Because of this, we confine ourselves to some particularly simple cases, in order to keep our arguments and computations both elementary and straightforward. All results in this paper will be developed with regard to the simple nonlinear ordinary differential equations of second-order

$$y''(t) + f(y(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

where $f : [0,\infty) \to [0,\infty)$ is a continuous nonnegative function. The simplest two-point boundary value problem is the Dirichlet problem, which requires

$$y(0) = y(1) = 0. \tag{1.2}$$

The primary interest lies in obtaining conditions on f that guarantee the existence of one or more positive solution. This problem has been studied, for instance, in [1] and [4].

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To formulate a three-point boundary value problem, we take a point $\eta \in (0,1)$, and a positive number $\mu > 0$ and require

$$y(0) = 0, \quad \mu y(\eta) = y(1). \tag{1.3}$$

Due to the concavity of the solution $y(t)$, a necessary condition for the existence of a solution is $\mu\eta < 1$. Many authors have given conditions on f that yield the existence of positive solutions; see for example [7], [8], and [3]. As explained above, we are not going to consider the most general problem in this paper. Let us pick a particular choice $\eta = 1/2$, while keeping μ arbitrary in $[0,2)$. Our three-point problem, therefore, imposes the conditions

$$y(0) = 0, \quad \mu y(1/2) = y(1). \tag{1.4}$$

We could have chosen $\mu = 1$ with an arbitrary $\eta < 1$ (just as in [3]). However, one can show, by taking into account the symmetry of the solutions, that this special case is equivalent to the Dirichlet problem on the interval $[0, 1+\tau]$. Hence, any results obtained for this case is automatically subsumed under results for the Dirichlet problem.

A voluminous literature has appeared in the last decade on various generalizations of these problems. We briefly mention some of these generalizations below but we will not discuss them any further in the rest of the paper. Results have been obtained for equations of the form

$$y''(t) + h(t)f(t,u) = 0, \tag{1.5}$$

where h is a positive continuous function that is required not to vanish in any subinterval of $(0,1)$, and $f(t,u)$ is continuous, but need not be always positive. Equations with a p-Laplacian term have also been studied, e.g. in [2] :

$$(|y'|^{p-2}y'(t))' + h(t)f(t,u) = 0. \tag{1.6}$$

There is a parallel theory in the study of difference equations, equations on time scales, and measure chains. And of course, there are higher order equations, and equations in higher dimensions.

General m-point problems are formulated by replacing the second equation in (1.3) with

$$\sum_{i=1}^{m-2} \mu_i y(\eta_i) = y(1), \tag{1.7}$$

where $\eta_i \in (0,1)$, $\mu_i > 0$. For simplicity the assumption $0 < \sum \mu_i < 1$ (which is stronger than the necessary condition $0 < \sum \mu_i \eta_i < 1$) is often imposed.

A majority of the results in this subject area are obtained using some form of fixed point theorems of cone mappings, the grandfather of which is the famous Krasnoselskii Theorem. Authors who have contributed to generalizing the cone theorem include Leggett, Williams, Avery, Henderson, and many many others. This technique is described below in a nutshell. Since we are not going to make use of cone theorems in this paper, we will not formulate any of them precisely.

Let P be a cone in a Banach space (one can take the positive quadrant of the plane R^2 as an example) and $T : P \to P$ be a completely continuous map from P into itself. What follows is a simplified version of the Krasnoselskii Theorem, which has two parts. The first part (the expansive form) states that if we can find an annular subset of P, i.e. $\{x \in P : a \leq ||x|| \leq b\}$, such that T pushes the inner boundary $\{x \in P : ||x|| = a\}$ towards the origin but pushes the outer boundary $\{x \in P||x|| = b\}$ away from the origin, then T has a fixed point inside the annulus. The second part (the compressive form) is complementary to the first. This time T pushes the inner boundary away from the origin and pushes the outer boundary towards the origin. A fixed point is then still assured.

To make it possible to use a cone theorem, one rewrites the boundary value problem (1.1)–(1.2), or (1.1)–(1.3), in the form of an integral equation via the use of a Green's function. The Banach space is the space of continuous functions over $[0, 1]$ with the sup norm (or its higher order extensions) and the positive cone is the set of continuous positive functions or some suitable subset of it (such as the space of positive continuous concave functions). If we can find suitable constants a and b such that the two boundaries of the annular region have the behavior as required in the cone theorem, then the annular region has a fixed point which is equivalent to a positive solution of the boundary value problem.

A natural approach to try to get multiple solutions is to stack several annular regions together and apply the alternate forms of Krasnoselskii's Theorem to these regions. For instance, if one can find three suitable numbers $a < b < c$ such that T pushes the innermost and outermost boundaries towards the origin, while it pushes the middle boundary in the other direction, then each of the two annular regions, $\{a \leq ||x|| \leq b\}$ and $\{b \leq ||x|| \leq c\}$, must have a fixed point. In reality, it may not be easy or even possible to find three such numbers. That is where the generalizations of Krasnoselskii's Theorem come in. Conceptually, such generalizations aim to allow one to find two (or more) annulus-like regions, such that T maps the boundaries (or some superset of the boundaries) in more or less the manner described above and still guarantee a fixed point for each region. For instance, in the Leggett-Williams Theorem, roughly speaking, the middle boundary is not determined by the norm equality $||x|| = b$, but by some suitable concave functional.

In the next two sections, we consider the two-point Dirichlet problem (1.1)–(1.2) and the three-point problem (1.1)–(1.4), respectively, in detail and show how the classical shooting method can lead to better results obtained so far by the cone method. I have chosen the results from a few authors for comparison, only for the reason that their results can be most easily stated and understood, and that their papers are more easily accessible.

In spite of the remarks made in this paper, cone method remains an indispensable tool in cases when the shooting method is not available (for example for higher order equations and partial differential equations) or when it is hard to use.

2 Symmetric Solutions of the Dirichlet Problem

The following result is well-known (Henderson and Thompson [4]). If there exist three numbers $0 < a < b < c/2$, and the nonlinear function f satisfies

1. $f(w) < 8a, \quad \text{for } 0 \le w \le a,$
2. $f(w) \ge 16b, \quad \text{for } b \le w \le 2b,$
3. $f(w) \le 8c, \quad \text{for } 0 \le w \le c,$

then the Dirichlet problem (1.1)–(1.2) has two symmetric positive solutions. An earlier result by Avery [1] requires condition 2 to hold on $[b, 4b]$. We are going to further reduce this to $[b, 3b/2]$. Conditions 1 and 3 will also be improved.

Since the solutions we sought are symmetric, they satisfy $y'(1/2) = 0$. After translation, we see that the Dirichlet problem is equivalent to the Neumann-Dirichlet boundary value problem consisting of equation (1.1) coupled with the boundary conditions

$$y'(0) = 0, \quad y(1/2) = 0. \tag{2.8}$$

The shooting method converts the problem into finding suitable initial heights $h > 0$, such that the solution of the initial value problem (1.1) with

$$y(0) = h, \quad y'(0) = 0 \tag{2.9}$$

vanishes for the first time at $t = 1/2$. Let us denote by $y(t; h)$ the solution to (1.1)–(2.9), if it uniquely exists. Then solving the boundary value problem is equivalent to finding h such that $y(1/2; h) = 0$, but $y(t; h) > 0$ for $t \in [0, 1/2)$.

The first set of obstacles we encounter is due to the fact that assuming just the continuity of the nonlinear function f does not ensure the uniqueness of the solution to the initial value problem, nor does it ensure the existence of a solution on the entire interval $[0, 1/2]$.

The first difficulty can be overcome by approximating $f(w)$ by a C^1 function $f_\epsilon(w)$ (for instance, in the sense of the sup norm). The stronger continuity guarantees that $y(t, h)$ is uniquely defined and it depends continuously on both t and h. We can then apply the shooting method to get a solution of the boundary value problem, with f replaced by f_ϵ in (1.1). We then let $\epsilon \to 0$ and a compactness argument can then be applied as usual to yield a solution to the original problem. If we are seeking multiple solutions, we just have to make sure that the multiple solutions obtained for the approximating boundary value problem have norms belonging to non-overlapping ranges of values, so that after passing to the limit, the solutions of the original problem so obtained also have distinct norms.

The second difficulty is due to the fact that $f(w)$ is not defined for $w < 0$. Even if we can find a solution $y(t;h)$ for (2.9) in a neighborhood of $t = 0$ and we can extend it to the right as much as we can, once $y(t;h)$ hits the t-axis, we cannot extend it any further. So how can we talk about $y(1/2;h)$ in this case. We can overcome this difficulty in one of two ways. We can either extend f by defining $f(w) = f(0)$, for $w < 0$ and study $y(1/2;h)$ for all h. Or we can track the point where the solution curve cuts either the interval $[0, 1/2]$ on the t-axis or the part of the vertical line $t = 1/2$ above the t-axis, whichever occurs first. We shall adopt the second approach. This intersection point is a continuous function of the initial height h (this follows from the fact that a solution cannot touch the t-axis tangentially). If the intersection point happens to be exactly at $(0, 1/2)$ of the t-y-plane, then we have a solution of the boundary value problem.

The principle of the shooting method for the Dirichlet problem is stated as follows. It is a simple corollary of the intermediate value theorem of continuous functions.

Lemma 1 *If there exist two initial heights $a > 0$ and $b > 0$ such that*

- *(S1) the solution $y(t;a)$ intersects the line $t = 1/2$ before or when it intersects the t-axis,*
- *(S2) the solution $y(t;b)$ of (1.1)–(2.9) intersects the t-axis before or when it reaches $t = 1/2$,*

then the boundary value problem (1.1)–(2.8), and hence also the Dirichlet boundary value problem (1.1)–(1.2), has a solution with sup norm $||y||$ between a and b.

Note that we do not need to specify whether $a > b$ or $a < b$ and the principle works in either case.

It is an elementary fact that equation (1.1) is explicitly solvable. After multiplying both sides of (1.1) by $u'(t)$ and integrating, we get

$$u'(t) = \sqrt{2\int_u^h f(w)\,dw}. \tag{2.10}$$

If the solution $y(t;h)$ vanishes at a finite point t_0, then

$$t_0 = \int_0^h \frac{du}{\sqrt{2\int_u^h f(w)\,dw}}. \tag{2.11}$$

The following Lemma is a simple consequence of this fact.

Lemma 2 *Suppose there exists $a > 0$ such that*

$$\int_0^a \frac{du}{\sqrt{2\int_u^a f(w)\,dw}} \geq \frac{1}{2}. \tag{2.12}$$

Then the solution $y(t; a)$ satisfies (S1).

It is easy to verify that condition 1 at the beginning of the section implies (2.12). An example of f that satisfies (2.12) but not condition 1 is

$$f(w) < 2a, \text{ for } w \in [a/2, a] \tag{2.13}$$

but $f(w)$ can be arbitrary in $[0, a/2)$. In particular, $f(w)$ can assume values that are much larger than $8a$ in $[0, a/2]$.

As another example, take any f such that $f(w) \leq \pi^2 w$. Then (2.12) is satisfied. Obviously not all such f (for instance, $f(w) = 9w$) satisfy condition 1.

Likewise whenever condition 3 is used, we can often replace it with condition (2.12), with a replaced by c.

The next Lemma gives an improvement on condition 2.

Lemma 3 *Suppose there exists $b > 0$ such that*

$$f(w) \geq 16b, \quad \textit{for } b \leq w \leq 3b/2, \tag{2.14}$$

Then the solution $y(t; b)$ satisfies (S2).

Proof. Let us define $g(w) = 16b$, for $b \leq w \leq 3b/2$ and $g(w) = 0$ elsewhere. Then

$$f(w) \geq g(w) \text{ for all } w. \tag{2.15}$$

Consider the initial value problem

$$z''(t) + g(z(t)) = 0, \quad t > 0, \tag{2.16}$$

$$z(0) = b, \quad z'(0) = 0. \tag{2.17}$$

Even though the function $g(w)$ has a jump discontinuity, the initial value problem is well-posed. It is also easy to show from the comparison hypothesis (2.15) that if both $y(t; b)$ and $z(t)$ are nonnegative in a right neighborhood $[0, \tau]$ of 0, then

$$y(t) \leq z(t). \tag{2.18}$$

As a consequence, if we can show that $z(t)$ satisfies (S2), then $y(t;b)$ satisfies (S2). It is easy to solve for $z(t)$ explicitly to get

$$z(t) = \begin{cases} 3b/2 - 8bt^2, & t \in [0, 1/4] \\ 2b - 4bt, & t \in [1/4, 1/2] \end{cases} \tag{2.19}$$

and to verify that $z(1/2) = 0$. Hence $y(t;b)$ satisfies (S2) and the Lemma is proved. ∎

By combining Lemmas 1, 2, and 3, we obtain the main theorem of this section, which is an improvement of the result in [4].

Theorem 1 *If there exist a constant $a > 0$ such that condition (2.12) is satisfied and another constant $b > 0$ such that condition (2.14) is satisfied, then $a \neq 3b/2$ and the Dirichlet problem (1.1)–(1.2) has a positive solution with sup norm between a and $3b/2$.*

Note that we do not specify whether $a < 3b/2$ or $a > 3b/2$. Even if it happens that $a < 3b/2$, it is not required that $a < b$.

Although Theorem 1 only yields one solution, it is obvious how it can be applied to yield multiple solutions. All we need to do is to find two interleaving sequences of numbers such as

$$a_1 < 3b_1/2 < a_2 < 3b_2/2 < ..., \tag{2.20}$$

for which f satisfies (2.12) at each a_i, and (2.14) at each b_i. The a, b, and c in [4] corresponds to a_1, b_1, and a_2.

The proof of Lemma 2 seems to suggest that condition (2.12) is best possible. As for condition (2.14), it can be shown that the length of the interval $[b, 3b/2]$ cannot be further reduced if we fix the left endpoint b and the constant 16. It is possible to derive a further improvement of (2.14) in a format similar to that of (2.12), but (2.14) is good enough for having allowed us to make our point, so we did not spend the effort to look for the most general result.

3 Three-Point Problem

The following result by Ma [7] (the equation considered in [7] is slightly more general than (1.1)) and its extensions are obtained using the cone approach.

If either

1. $\lim_{w\to 0} \dfrac{f(w)}{w} = 0$ and $\lim_{w\to\infty} \dfrac{f(w)}{w} = \infty$ (the superlinear case), or

2. $\lim_{w\to\infty} \frac{f(w)}{w} = 0$ and $\lim_{w\to 0} \frac{f(w)}{w} = \infty$ (the sublinear case),

then the three-point boundary value problem (1.1)–(1.3) has a positive solution.

Raffoul [8] studies the same problem in an eigenvalue setting. After translating back to the framework of the original problem, the results are equivalent to asserting that the four limit values in conditions 1 and 2 can be replaced by some suitable non-zero and finite values and the existence of a solution is still affirmed. The exact values of these constants are not quoted here. It suffices to say that, in the setting of the simpler boundary conditions (1.4), they are weaker than those derived below using the shooting method.

To tackle the three-point problem, we shoot our solution from the origin but vary the initial slope. Let $y(t;s)$ be the solution of (1.1) satisfying the initial conditions

$$y(0;s) = 0, \quad y'(0;s) = s > 0. \tag{3.21}$$

Just as in the case of the Neumann-Dirichlet problem discussed in Section 2, the solution may or may not vanish (i.e. hit the t-axis) before reaching $t = 1$. The case when $y(t;s)$ vanishes at some point $t \leq 1$ is analogous to the case (S2) in Section 2.

Suppose $y(t;s) \geq 0$ in $[0,1]$. We define the function

$$k(s) = \frac{y(1;s)}{y(1/2;s)} \tag{3.22}$$

which is obviously a continuous function of $s \in [0,\infty)$. A solution of the three-point problem corresponds to a suitable initial slope s that gives $k(s) = \mu$.

The principle of the shooting method for the three-point problem can be stated as follows.

Lemma 4 *If there exist two initial slopes $\alpha > 0$ and $\beta > 0$ such that*

(T1) the solution $y(t;\alpha)$ remains positive in $(0,1)$ and $k(\alpha) \geq \mu$,

(T2) the solution $y(t;\beta)$ of (1.1)–(3.21) either vanishes before it reaches $t = 1$, or satisfies $k(\beta) \leq \mu$,

then the three-point boundary value problem (1.1)–(1.3) has a solution with initial slope $y'(0)$ between α and β.

Proof. If $y(t;\beta)$ does not vanish before $t = 1$, and $k(\beta) \leq \mu$, then the existence of a solution with $k(s) = \mu$ is a consequence of the intermediate value theorem.

If $y(t;\beta)$ vanishes before $t=1$, we first conclude that there must be a γ between α and β such that $y(t;\gamma)$ vanishes at $t=1$. Then for this solution, $k(\gamma)=0$. Again the intermediate value theorem gives a suitable initial slope s between γ and α such that $k(s)=\mu$. ∎

Let us first give a heuristic argument on how the shooting method can lead to Ma's result. Suppose that condition 1 at the beginning of the section holds. Let us shoot a solution $y(t;\alpha)$ with a very small initial slope α. Then $y(t;\alpha)$ remains small throughout $[0,1]$. The first equality in condition 1 means that we can make $f(w)/w$ as small as we can by choosing α sufficiently small. As a result, $y(t;\alpha)$ can be approximated by the solution of

$$Z''(t)=0, \quad Z'(0)=\alpha. \tag{3.23}$$

which is $z(t)=\alpha t$. Hence, $k(\alpha)$ is approximately $2>\mu$ and so $y(t,\alpha)$ satisfies (T1). On the other hand, if we shoot a solution $y(t;\beta)$ with a very large initial slope β, then $y(t;\beta)$ will increase to a large value before coming back down. Suppose that $y(t;\beta)$ does not vanish before $t=1$, then $y(t;\beta)$ can be made as large as we please over the interval $[\epsilon,1-\epsilon]$, where ϵ can be chosen as small as we please. Hence, $f(w)/w$ can be made as large as we please over $[\epsilon,1-\epsilon]$ and this contradicts the assumption that $y(t;\beta)$ does not vanish before $t=1$. Thus with β large enough, $y(t;\beta)$ satisfies (T2). Lemma 4 then gives us a solution of the three-point problem.

The following comparison result is useful to make the above arguments rigorous.

Lemma 5 *Let $q(t)$ and $Q(t)$ be two piecewise continuous functions defined on $[0,1]$ and suppose it is known that, for a given solution $y(t;\alpha)$ of (1.1) which does not vanish in $[0,1]$*

$$q(t)\le \frac{f(y(t;\alpha))}{y(t;\alpha)}\le Q(t), \quad t\in[0,1]. \tag{3.24}$$

Let $z(t)$ and $Z(t)$ be the solution of the two comparison initial value problems, respectively,

$$z''(t)+q(t)z(t)=0, \quad z(0)=0, z'(0)=\alpha, \tag{3.25}$$

$$Z''(t)+Q(t)Z(t)=0, \quad Z(0)=0, Z'(0)=\alpha, \tag{3.26}$$

and suppose that $Z(t)$ does not vanish in $[0,1]$. Then

$$\frac{z(1)}{z(\eta)}\ge\frac{y(1)}{y(\eta)}\ge\frac{Z(1)}{Z(\eta)}. \tag{3.27}$$

In particular, if we take $\eta=1/2$, we have

$$\frac{z(1)}{z(1/2)}\ge k(\alpha)\ge\frac{Z(1)}{Z(1/2)}. \tag{3.28}$$

Proof. The classical Sturm Comparison Theorem gives us the inequalities

$$\frac{z'(t)}{z(t)} \geq \frac{y'(t;\alpha)}{y(t;\alpha)} \geq \frac{Z'(t)}{Z(t)}, \quad \text{for all } t \in [0,1]. \tag{3.29}$$

Integrating these inequalities from $t = \eta$ to $t = 1$ yields (3.27). ∎

The following theorem extends the results of Ma and Raffoul.

Theorem 2 *If either*

1. $\limsup_{w\to 0} \frac{f(w)}{w} < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2 < \liminf_{w\to\infty} \frac{f(w)}{w}$ *, or*
2. $\limsup_{w\to \infty} \frac{f(w)}{w} < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2 < \liminf_{w\to 0} \frac{f(w)}{w}$ *,*

then the three point boundary value problem (1.1)–(1.3) has a positive solution.

Proof. We only give the proof for 1, that for 2 being similar. Suppose that the first inequality in 1 holds. We consider solutions $y(t;\alpha)$ with α small. For convenience, we use $A = 2\cos^{-1}(\mu/2)$ to denote the constant in condition 1.

Our first claim is that, as long as α is chosen small enough, $y(t;\alpha)$ cannot vanish in $[0,1]$. Suppose this is false, let τ be the first zero and $y(t;\alpha) > 0$ in $(0,\tau)$. By convexity, $y(t;\alpha) \leq \alpha\tau \leq \alpha$ in $(0,\tau)$. The first inequality in 1 implies that if we let α be small enough, then $f(y)/y < A^2$ for all $t \in (0,\tau)$. Then by comparing (1.1) (using the classical Sturm Comparison Theorem) with the linear equation

$$Z''(t) + A^2 Z(t) = 0 \tag{3.30}$$

over $[0,\tau]$, we see that $y(t;\alpha)$ oscillates slower than $Z(t) = \sin(At)$ which does not have a zero in $[0,1]$ and we get a contradiction.

Now that we know that $y(t;\alpha)$ does not vanish in $(0,1)$, convexity again gives us $y(t;\alpha) \leq \alpha$ in $[0,1]$. By choosing α small enough, we can ensure that $f(y)/y < A^2$ for all $t \in (0,1)$ and we can use Lemma 5 to compare $y(t;\alpha)$ of (1.1) and $Z(t)$ of (3.30) to get

$$k(\alpha) \geq \frac{Z'(1)}{Z(1/2)} = \mu. \tag{3.31}$$

Thus $y(t;\alpha)$ satisfies (T1).

Next we shoot out solutions $y(t;\beta)$ with β sufficiently large. If $y(t;\beta)$ happens to vanish in $[0,1]$, then $y(t;\beta)$ satisfies (T2) and Lemma 4 yields a solution to the three-point problem and we are done.

Now suppose that $y(t;\beta)$ remains positive in $(0,1)$. The second inequality in 1 means that there exists a number $L>0$ such that

$$\frac{f(w)}{w} > (A+\delta)^2, \quad \text{if } w \geq L, \tag{3.32}$$

where $\delta > 0$ is some fixed constant.

One can then choose β so large that

$$y(t;\beta) > L, \quad \text{for } t \in [\epsilon, 1-\epsilon], \tag{3.33}$$

where ϵ is a small number depending on β, with $\epsilon \rightarrow$ as $\beta \rightarrow \infty$. If we define

$$g(t) = \begin{cases} (A+\delta)^2, & t \in [\epsilon, 1-\epsilon] \\ 0, & t \notin [\epsilon, 1-\epsilon] \end{cases}, \tag{3.34}$$

then we can use Lemma 5 to compare $y(t;\beta)$ with the solution of

$$z''(t) + g(t)z(t) = 0, \quad z(0) = 0, z'(0) = \beta, \tag{3.35}$$

to get

$$k(\beta) \leq \frac{z'(1)}{z(1/2)}. \tag{3.36}$$

The necessity to exclude a small neighborhood from each of $t=0$ and $t=1$ is a bit of an annoyance, but the fact that $\epsilon \rightarrow 0$ as $\beta \rightarrow \infty$ means that these small neighborhoods have ignorable effect on the solution. We leave the rigorous details to the readers, and note that

$$z(t) \approx \sin((A+\delta)t) \tag{3.37}$$

so that

$$\frac{z(1)}{z(1/2)} < \mu, \tag{3.38}$$

as long as we make β large enough so that ϵ is small enough. Combining (3.36) and (3.38), we see that $y(t;\beta)$ satisfies (T2) and Lemma 4 now gives a solution of the three-point problem. ∎

Note that condition 1 (same for condition 2) of Theorem 2 is, in fact, comprised of two conditions. The left hand inequality implies the existence of a $y(t;\alpha)$ satisfying (T1) and the right hand inequality implies the existence of a $y(t;\beta)$ satisfying (T2). If we can establish another criterion that leads to a solution satisfying (T2), this criterion can be used to replace the right hand inequality in condition 1 to get a new existence result. The same applies to the left hand inequality in condition 1 and also to condition 2.

For instance, condition (2.14) of Section 2 implies the existence of a solution satisfying (S2) of Lemma 1, and after extending this solution by symmetry and translating it if necessary we obviously get a solution that satisfies (T2). We thus have the following new existence result.

Theorem 3 *If condition (2.14) is satisfied and either*

1. $\limsup_{w\to 0} \frac{f(w)}{w} < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2$ *, or*
2. $\limsup_{w\to \infty} \frac{f(w)}{w} < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2$ *,*

then the three point boundary value problem (1.1)–(1.3) has a positive solution.

We mention another technique, the use of sub- and super-solutions, to prove (T1) or (T2). If we can find a function $G(w)$, such that

$$f(w) \le G(w), \quad \text{for all } w \in [0,a], \tag{3.39}$$

and there exists a solution of the differential equation

$$Z''(t) + G(Z(t)) = 0, \quad t \in (0,1), \tag{3.40}$$

satisfying

$$Z(0) = 0, \quad Z(t) \in [0,a] \text{ for all } t \in [0,1], \quad \frac{Z(1)}{Z(1/2)} \ge \mu, \tag{3.41}$$

then there exists a solution of (1.1)-(3.21) satisfying (T1). This is because $Z(t)$ is a super-solution for (1.1) and so (1.1) has a solution $y(t) \le Z(t)$ having the same boundary condition as $Z(t)$, namely $y(0) = Z(0) = 0$ and $y(1) = Z(1)$. For this solution, $k(\alpha) = y(1)/y(1/2) \ge Z(1)/Z(1/2) \ge \mu$.

On the other hand, suppose we can find another function $g(w)$ such that

$$f(w) \ge g(w), \quad \text{for all } w \in [0,b], \tag{3.42}$$

and a corresponding $z(t)$ satisfying

$$z(0) = 0, \quad z(t) \in [0,a] \text{ for all } t \in [0,1], \quad \frac{z(1)}{z(1/2)} \le \mu. \tag{3.43}$$

Then $z(t)$ serves as a sub-solution of (1.1). However, this condition alone is not sufficient to guarantee the existence of a solution of (1.1) that satisfies the same boundary conditions as $z(t)$ and at the same time dominates $z(t)$ in $(0, 1)$. If we impose an extra condition that can yield the existence of such a solution of (1.1), then the same arguments shows that this solution satisfies (T2). One such extra condition is the existence of another super-solution that dominates $z(t)$.

In actual fact, with some additional effort, one can show that conditions (3.42) and (3.43) are already sufficient to affirm the existence of a solution (which may not dominate $z(t)$) that satisfies (T2), without the extra assumption mentioned above. We leave the verification to the readers.

Using this technique, one can, for example, replace condition 1 of Theorem 3 by the following condition which is similar to condition 1 of Henderson and Thompson's result quoted in Section 2:

There exists a constant $a > 0$ such that

$$f(w) \leq \begin{cases} \dfrac{4(2-\mu)}{\mu}, & \mu > 4/3 \\[2ex] \dfrac{32(2-\mu)^2 a}{(4-\mu)^2}, & \mu \leq 4/3 \end{cases} . \tag{3.44}$$

One can also improve condition (2.14) used in Theorem 3 in a similar way, but the computations are much more complicated, involving several different expressions depending on the value of μ. One can also use a general interval $[b, \delta b]$ instead of $[b, 3b/2]$ and try to determine the optimal δ. We omit the details here.

4 Concluding Remarks

The shooting method has one advantage over the cone method. It only requires showing two particular solutions, one having a property like (S1) or (T1), and the other having (S2) or (T2). On the other hand, the cone method requires that a certain property be satisfied by every point on one of the boundaries of the annulus, and another property be satisfied by every point on the other boundary of the annulus.

Nevertheless, the cone method has the most important advantage over the shooting method, namely that it can be applied to higher order equations, in higher dimensions, and in very general topological settings. In all such situations, the shooting method is either absent or hard to use.

Could there be a technique that is able to bridge the two methods? We will explore this possibility in a forthcoming paper [6], which is based on another forthcoming paper [5] that contains some generalizations of the Krasnoselskii cone Theorem.

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