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Renormalized Solutions of a Nonlinear Parabolic Equation with Double Degeneracy*

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Abstract

In this paper, we consider the initial-boundary value problem of a nonlinear parabolic equation with double degeneracy, and establish the existence and uniqueness theorems of renormalized solutions which are stronger than BV solutions.

Keywords: Renormalized solutions, double degeneracy.
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1 Introduction

This paper is concerned with the following initial-boundary value problem

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)=\frac{\partial}{\partial x}\Big[A\Big(\frac{\partial}{\partial x}B(u)\Big)\Big], \qquad (x,t)\in Q_T\equiv(0,1)\times(0,T), \tag{1.1}$$

$$B(u(0,t))=B(u(1,t))=0, \qquad t\in(0,T), \tag{1.2}$$

$$u(x,0)=u_0(x), \qquad x\in(0,1), \tag{1.3}$$

where $f(s)$ is an appropriately smooth function and

$$A(s)=|s|^{p-2}s,\quad B(s)=\int_0^s b(\sigma)d\sigma,\quad s\in\mathbb{R}$$

with $p\geq 2$ and $b(s)\geq 0$ appropriately smooth.

The equation (1.1) presents two kinds of degeneracy, since it is degenerate not only at points where $b(u)=0$ but also at points where $\dfrac{\partial}{\partial x}B(u)=0$ if $p>2$. Using the method depending on the properties of convex functions, Kalashnikov [10] established the existence of continuous solutions of the Cauchy problem of the equation (1.1) with $f\equiv 0$ under some convexity assumption on $A(s)$ and $B(s)$. Under such assumption, the equation degenerates only at the zero value of the solutions or their spacial derivatives. The more interesting case is that the equation may present strong degeneracy, namely, the set $E=\{s\in\mathbb{R}:b(s)=0\}$ may have interior points. Generally, the equation may have no classical solutions and even continuous solutions for this case and it is necessary to formulate some suitable weak solutions.

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For the semilinear case of the equation (1.1) with $p = 2$, it is Vol'pert and Hudjaev [12] who first introduced BV solutions of the Cauchy problem and proved the existence theorem. Later, Wu and Wang [13] considered the initial-boundary value problem. For the quasilinear case with $p > 2$, the existence and uniqueness of BV solutions of the problem (1.1)–(1.3) under some natural conditions have been studied by Yin [15], where the BV solutions are defined in the following sense

Definition 1.1 *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is said to be a BV solution of the problem* (1.1)–(1.3)*, if*

(i) u satisfies (1.2) *and* (1.3) *in the sense that*

$$B(u^r(0,t)) = B(u^l(1,t)) = 0, \qquad \text{a.e. } t \in (0,T), \tag{1.4}$$

$$\lim_{t\to 0^+} \bar{u}(x,t) = u_0(x), \qquad \text{a.e. } x \in (0,1), \tag{1.5}$$

where $u^r(\cdot,t)$ and $u^l(\cdot,t)$ denote the right and left approximate limits of $u(\cdot,t)$ respectively and $\bar{u}$ denotes the symmetric mean value of u.

(ii) for any $k \in \mathbb{R}$ and any nonnegative functions $\varphi_i \in C^\infty(Q_T)\,(i = 1, 2)$ with $\mathrm{supp}\varphi_i \subset [0,1] \times (0,T)$ and

$$\varphi_1(0,t) = \varphi_2(0,t), \quad \varphi_1(1,t) = \varphi_2(1,t), \quad 0 < t < T,$$

the following integral inequality holds

$$\begin{aligned}\iint_{Q_T} &\mathrm{sgn}(u-k)\Big[(u-k)\frac{\partial\varphi_1}{\partial t} + (f(u)-f(k))\frac{\partial\varphi_1}{\partial x} - A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial\varphi_1}{\partial x}\Big]dxdt \\ &+ \mathrm{sgn}k\iint_{Q_T}\left[(u-k)\frac{\partial\varphi_2}{\partial t} + (f(u)-f(k))\frac{\partial\varphi_2}{\partial x} - A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial\varphi_2}{\partial x}\right]dxdt \ge 0.\end{aligned} \tag{1.6}$$

In this paper we discuss the renormalized solutions of the problem (1.1)–(1.3). Such solutions were first introduced by Di Perna and Lions [8] in 1980's, where the authors studied the existence of solutions of Boltzmann equations. From then on, there have been many results on renormalized solutions of various problems, see [1, 2, 3, 4, 5, 6, 7, 11]. It is shown that such solutions play an important role in prescribing nonsmooth solutions and noncontinuous solutions. The renormalized solutions of the problem (1.1)–(1.3) considered in this paper are defined as follows

Definition 1.2 *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is said to be a renormalized solution of the problem* (1.1)–(1.3)*, if*

(i) u satisfies (1.2) *and* (1.3) *in the sense of* (1.4) *and* (1.5)*;*

(ii) for any $k \in \mathbb{R}$, any $\eta > 0$ and any nonnegative functions $\varphi_i \in C^\infty(Q_T)\,(i = 1, 2)$ with $\mathrm{supp}\varphi_i \subset [0,1] \times (0,T)$ and

$$\varphi_1(0,t) = \varphi_2(0,t), \quad \varphi_1(1,t) = \varphi_2(1,t), \quad 0 < t < T,$$

the following integral inequality holds

$$\iint_{Q_T} \mathcal{H}_\eta(u,k)\frac{\partial\varphi_1}{\partial t}dxdt + \iint_{Q_T} \mathcal{F}_\eta(u,k)\frac{\partial\varphi_1}{\partial x}dxdt$$

$$
-\iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta(u-k)\frac{\partial \varphi_1}{\partial x}dxdt - \iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta^{'}(u-k)\frac{\partial u}{\partial x}\varphi_1 dxdt
$$
$$
+\int_k^0 H_\eta'(\tau-k)(f(\tau)-f(k))d\tau \int_0^T (\varphi_1(1,t)-\varphi_1(0,t))dt
$$
$$
+ H_\eta(k)\iint_{Q_T}\Big[(u-k)\frac{\partial \varphi_2}{\partial t}+(f(u)-f(k))\frac{\partial \varphi_2}{\partial x} - A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial \varphi_2}{\partial x}\Big] dxdt \geq 0, \qquad (1.7)
$$

where

$$
H_\eta(s)=\frac{s}{\sqrt{s^2+\eta}}, \quad H_\eta^{'}(s)=\frac{\eta}{(s^2+\eta)^{3/2}}, \quad s\in\mathbb{R},
$$
$$
\mathcal{H}_\eta(s,k)=\int_k^s H_\eta(\tau-k)d\tau, \quad \mathcal{F}_\eta(s,k)=\int_k^s H_\eta(\tau-k)f^{'}(\tau)d\tau, \quad s,k\in\mathbb{R}.
$$

Such solutions are a natural extension of classical solutions, which will be shown at the beginning of the next section. Comparing the two definitions of weak solutions, there are two additional terms in (1.7), i.e. the forth and fifth ones. Moreover, (1.6) follows by letting $\eta \to 0^+$ in (1.7) since the forth term of (1.7) is nonnegative and the limit of the fifth term is zero. Therefore, renormalized solutions imply more information than BV solutions and thus it is stronger.

Since renormalized solutions are stronger than BV solutions, the uniqueness of renormalized solutions of the problem (1.1)–(1.3) may be deduced directly from the uniqueness of BV solutions (see [15]). Hence

Theorem 1.1 *There exists at most one renormalized solution for the initial-boundary value problem* (1.1)–(1.3).

And we will prove the existence of renormalized solutions of the problem (1.1)–(1.3) in this paper, namely

Theorem 1.2 *Assume* $u_0 \in BV([0,1])$ *with* $u_0(0)=u_0(1)=0$. *Then the initial-boundary value problem* (1.1)–(1.3) *admits one and only one renormalized solution.*

The paper is arranged as follows. The preliminaries are done in §2. We first prove that the classical solution is also a renormalized solution, which shows that the latter is a natural extension of the former. Then we formulate the regularized problem and do some a priori estimates and establish some convergence. Two technical lemmas are introduced at the end of this section. The main result of this paper (Theorem 1.2) is proved in §3 subsequently.

2 Preliminaries

The renormalized solution is a natural extension of the classical solution. In fact, we have

Proposition 2.1 *Let* $u\in C^2(Q_T)\cap C(\overline{Q}_T)$ *be a solution of the equation (1.1) with*

$$
u(0,t)=u(1,t)=0, \quad t\in(0,T). \qquad (2.1)
$$

Then for any $k \in \mathbb{R}$, any $\eta > 0$ and any nonnegative functions $\varphi_i \in C^\infty(Q_T)\,(i = 1, 2)$ with $\mathrm{supp}\varphi_i \subset [0, 1] \times (0, T)$ *and*

$$\varphi_1(0, t) = \varphi_2(0, t), \quad \varphi_1(1, t) = \varphi_2(1, t), \quad 0 < t < T,$$

$$\begin{aligned}
&\iint_{Q_T} \mathcal{H}_\eta(u, k)\frac{\partial \varphi_1}{\partial t}dxdt + \iint_{Q_T} \mathcal{F}_\eta(u, k)\frac{\partial \varphi_1}{\partial x}dxdt \\
&\quad - \iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta(u-k)\frac{\partial \varphi_1}{\partial x}dxdt - \iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta'(u-k)\frac{\partial u}{\partial x}\varphi_1 dxdt \\
&\quad + \int_k^0 H_\eta'(\tau - k)(f(\tau) - f(k))d\tau \int_0^T (\varphi_1(1, t) - \varphi_1(0, t))dt \\
&\quad + H_\eta(k)\iint_{Q_T} \left[(u-k)\frac{\partial \varphi_2}{\partial t} + (f(u) - f(k))\frac{\partial \varphi_2}{\partial x} - A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial \varphi_2}{\partial x}\right] dxdt = 0. \qquad (2.2)
\end{aligned}$$

Therefore, if $u \in C^2(Q_T) \cap C(\overline{Q}_T)$ is a solution of the problem (1.1)–(1.3) *with* (2.1)*, then u is also a renormalized solution of the problem* (1.1)–(1.3) *and the inequality (1.7) can be rewritten as the equality.*

Proof. Let $k \in \mathbb{R}$, $\eta > 0$ and $\varphi_i \in C^\infty(Q_T)\,(i = 1, 2)$ be nonnegative functions with $\mathrm{supp}\varphi_i \subset [0, 1] \times (0, T)$ and

$$\varphi_1(0, t) = \varphi_2(0, t), \quad \varphi_1(1, t) = \varphi_2(1, t), \quad 0 < t < T. \qquad (2.3)$$

On the one hand, multiply (1.1) with $H_\eta(u - k)\varphi_1$ and then integrate over Q_T to get

$$\begin{aligned}
&\iint_{Q_T} \varphi_1\frac{\partial}{\partial t}\mathcal{H}_\eta(u, k)dxdt + \iint_{Q_T} \varphi_1\frac{\partial}{\partial x}\mathcal{F}_\eta(u, k)dxdt \\
=&\iint_{Q_T} \frac{\partial}{\partial x}A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta(u-k)\varphi_1 dxdt. \qquad (2.4)
\end{aligned}$$

From the definition of $\mathcal{F}_\eta(s, k)$, (2.1) and the Newton-Leibniz formula,

$$\begin{aligned}
&\int_0^T \varphi_1\mathcal{F}_\eta(u, k)\Big|_{x=0}^{x=1}dt \\
=&\int_0^T \left(\int_k^{u(1,t)} H_\eta(\tau - k)f'(\tau)d\tau\varphi_1(1, t) - \int_k^{u(0,t)} H_\eta(\tau - k)f'(\tau)d\tau\varphi_1(0, t)\right) dt \\
=&\int_k^0 H_\eta(\tau - k)(f(\tau) - f(k))'d\tau \int_0^T (\varphi_1(1, t) - \varphi_1(0, t))dt \\
=&H_\eta(-k)(f(0) - f(k))\int_0^T (\varphi_1(1, t) - \varphi_1(0, t))dt \\
&\quad - \int_k^0 H_\eta'(\tau - k)(f(\tau) - f(k))d\tau \int_0^T (\varphi_1(1, t) - \varphi_1(0, t))dt. \qquad (2.5)
\end{aligned}$$

Then, by using the formula of integrating by parts in (2.4) and from (2.5), we get

$$\iint_{Q_T} \mathcal{H}_\eta(u, k)\frac{\partial \varphi_1}{\partial t}dxdt + \iint_{Q_T} \mathcal{F}_\eta(u, k)\frac{\partial \varphi_1}{\partial x}dxdt$$

$$
\begin{aligned}
&\quad + H_\eta(k)(f(0)-f(k))\int_0^T \varphi_1(1,t)-\varphi_1(0,t))dt\\
&\quad + \int_k^0 H'_\eta(\tau-k)(f(\tau)-f(k))d\tau \int_0^T (\varphi_1(1,t)-\varphi_1(0,t))dt\\
&= \int_0^T H_\eta(k)\Big[A\Big(\frac{\partial}{\partial x}B(u(1,t))\Big)\varphi_1(1,t) - A\Big(\frac{\partial}{\partial x}B(u(0,t))\Big)\varphi_1(0,t)\Big]dt\\
&\quad + \iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)\Big(H_\eta(u-k)\frac{\partial \varphi_1}{\partial x} + H'_\eta(u-k)\frac{\partial u}{\partial x}\varphi_1\Big)\,dxdt. \qquad (2.6)
\end{aligned}
$$

On the other hand, the equation (1.1) leads to

$$
\frac{\partial}{\partial t}(u-k) + \frac{\partial}{\partial x}(f(u)-f(k)) = \frac{\partial}{\partial x}\Big[A\Big(\frac{\partial}{\partial x}B(u)\Big)\Big], \quad (x,t)\in Q_T.
$$

Multiplying this equation with φ_2 and then integrating over Q_T, we get that by the formula of integrating by parts and (2.1)

$$
\begin{aligned}
&\iint_{Q_T}(u-k)\frac{\partial \varphi_2}{\partial t}dxdt + \iint_{Q_T}(f(u)-f(k))\frac{\partial \varphi_2}{\partial x}dxdt\\
&\quad - (f(0)-f(k))\int_0^T(\varphi_2(1,t)-\varphi_2(0,t))dt\\
&= -\int_0^T \Big[A\Big(\frac{\partial}{\partial x}B(u(1,t))\Big)\varphi_2(1,t) - A\Big(\frac{\partial}{\partial x}B(u(0,t))\Big)\varphi_2(0,t)\Big]dt\\
&\quad + \iint_{Q_T} A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial \varphi_2}{\partial x}dxdt. \qquad (2.7)
\end{aligned}
$$

Multiplying (2.7) with $H_\eta(k)$ and then adding what obtained to (2.6), we get (2.2) owing to (2.3). The proof is complete. □

Remark 1 *The condition* (2.1) *is not egregious. In fact,* (2.1) *and* (1.2) *are identical if* $B(s)$ *is strictly increasing, i.e. the set* $E = \{s\in\mathbb{R} : b(s) = 0\}$ *has no interior point. Otherwise, if the set* E *has interior points, the equation (1.1) is strong degenerate and the equation may have no classical solutions in general, as mentioned in the introduction.*

Since the equation (1.1) presents double degeneracy, we regularize the equation to get the existence of renormalized solutions by doing a prior estimates and passing a limit process. We firstly approximate the given initial data u_0. For any $0<\varepsilon<1$, choose $u_{0,\varepsilon}\in C_0^\infty(0,1)$ satisfying

$$
\begin{aligned}
&\int_0^1 \Big|\frac{\partial u_{0,\varepsilon}}{\partial x}\Big|dx \le C,\\
&\int_0^1 \Big|\frac{\partial}{\partial x}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_{0,\varepsilon})\Big)\Big|dx \le C,\\
&u_{0,\varepsilon}(x) \longrightarrow u_0(x), \quad \text{uniformly in } (0,1) \quad \text{as } \varepsilon\to 0^+,
\end{aligned}
$$

where $C>0$ is independent of ε, and

$$
A_\varepsilon(s) = A(s)+\varepsilon s, \quad B_\varepsilon(s) = B(s)+\varepsilon s, \quad s\in\mathbb{R}.
$$

Consider the regularized problem

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon) = \frac{\partial}{\partial x}\Big[A_\varepsilon\Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon)\Big)\Big], \qquad (x,t) \in Q_T, \tag{2.8}$$

$$u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \qquad t \in (0,T), \tag{2.9}$$

$$u_\varepsilon(x,0) = u_{0,\varepsilon}(x), \qquad x \in (0,1). \tag{2.10}$$

By virtue of the standard theory for uniformly parabolic equations, there exists a unique classical solution $u_\varepsilon \in C^2(\overline{Q}_T)$ of the above problem. To pass the limit process to the problem (1.1)–(1.3), we need do a priori estimates on u_ε. On the one hand, the maximum principle gives

$$\sup_{Q_T} |u_\varepsilon| \le C. \tag{2.11}$$

Here and hereafter, we denote by C positive constants independent of ε and may be different in different formulae. On the other hand, the following BV estimates and $C^{1,1/2}$ estimates have been proved by Yin [15].

Lemma 2.1 *The solutions u_ε satisfy*

$$\sup_{0<t<T} \int_0^1 \Big|\frac{\partial u_\varepsilon}{\partial x}\Big| dx + \sup_{0<t<T} \int_0^1 \Big|\frac{\partial u_\varepsilon}{\partial t}\Big| dx \le C,$$

$$\sup_{0<t<T} \int_0^1 \Big|\frac{\partial}{\partial x} A_\varepsilon\Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon(x,t))\Big)\Big| dx \le C,$$

$$\sup_{Q_T} \Big|\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon)\Big| \le C.$$

Lemma 2.2 *For the function*

$$\omega_\varepsilon(x,t) = B_\varepsilon(u_\varepsilon(x,t)), \quad (x,t) \in Q_T,$$

we have

$$|\omega_\varepsilon(x_1,t) - \omega_\varepsilon(x_2,t)| \le C|x_1 - x_2|, \qquad x_1, x_2 \in (0,1),\ t \in (0,T),$$

$$|\omega_\varepsilon(x,t_1) - \omega_\varepsilon(x,t_2)| \le C|t_1 - t_2|^{1/2}, \qquad x \in (0,1),\ t_1, t_2 \in (0,T).$$

Form the estimate (2.11), Lemma 2.1 and Lemma 2.2, there exist a subsequence of $\varepsilon \in (0,1)$, denoted by itself for convenience, and a function $u \in L^\infty(Q_T) \cap BV(Q_T)$ with $B(u) \in C^{1,1/2}(\overline{Q}_T)$ and (1.4) and (1.5), and a function $\mu \in L^\infty(Q_T)$, such that

$$u_\varepsilon(x,t) \longrightarrow u(x,t), \qquad \text{a.e. in } Q_T, \tag{2.12}$$

$$B_\varepsilon(u_\varepsilon(x,t)) \longrightarrow B(u(x,t)), \qquad \text{uniformly in } Q_T, \tag{2.13}$$

$$\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon) \stackrel{*}{\rightharpoonup} \frac{\partial}{\partial x} B(u), \qquad \text{weakly}^* \text{ in } L^\infty(Q_T), \tag{2.14}$$

$$A_\varepsilon\Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon)\Big) \stackrel{*}{\rightharpoonup} \mu, \qquad \text{weakly}^* \text{ in } L^\infty(Q_T), \tag{2.15}$$

as $\varepsilon \to 0^+$, see more details in Yin [15].

To complete the limit process, we also need the following convergence.

Lemma 2.3 *For the solution u_ε of the regularized problem (2.8)–(2.10) and the above limit function u, we have*

$$A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon(x,t))\Big) \longrightarrow A\Big(\frac{\partial}{\partial x}B(u(x,t))\Big), \qquad \text{a.e. in } Q_T \quad \text{as } \varepsilon\to 0^+. \tag{2.16}$$

Proof. We first prove that

$$\mu = A\Big(\frac{\partial}{\partial x}B(u)\Big), \quad \text{a.e. in } Q_T. \tag{2.17}$$

For convenience, we rewrite

$$A(s)=\int_0^s a(\sigma)d\sigma, \quad s\in\mathbb{R}, \qquad a(\sigma)=\frac{1}{p}|\sigma|^p, \quad \sigma\in\mathbb{R}.$$

Multiplying (2.8) with $(B_\varepsilon(u_\varepsilon)-B(u))$ and then integrating over Q_T, we get that by the formula of integration by parts

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\Big(B_\varepsilon(u_\varepsilon)-B(u)\Big)dxdt+\iint_{Q_T}\frac{\partial}{\partial x}f(u_\varepsilon)\Big(B_\varepsilon(u_\varepsilon)-B(u)\Big)dxdt\\ =&-\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)dxdt,\end{aligned}$$

which yields

$$\lim_{\varepsilon\to0^+}\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)dxdt=0 \tag{2.18}$$

from (2.13). On the other hand, by (2.14) and $B(u)\in C^{1,1/2}(\overline{Q}_T)$,

$$\lim_{\varepsilon\to0^+}\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B(u)\Big)\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)dxdt=0. \tag{2.19}$$

Therefore, combining (2.18) with (2.19) gives

$$\begin{aligned}0=&\lim_{\varepsilon\to0^+}\iint_{Q_T}\Big(A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)-A_\varepsilon\Big(\frac{\partial}{\partial x}B(u)\Big)\Big)\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)dxdt\\ =&\lim_{\varepsilon\to0^+}\iint_{Q_T}a_\varepsilon^*(x,t)\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)^2dxdt,\end{aligned} \tag{2.20}$$

where

$$a_\varepsilon^*(x,t)=\int_0^1\Big[a\Big(\lambda\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-(1-\lambda)\frac{\partial}{\partial x}B(u)\Big)+\varepsilon\Big]d\lambda, \quad (x,t)\in Q_T.$$

Since $a_\varepsilon^*(x,t)$ is positive and uniformly bounded in Q_T,

$$\iint_{Q_T}(a_\varepsilon^*)^2\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)^2dxdt\le C\iint_{Q_T}a_\varepsilon^*\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)-\frac{\partial}{\partial x}B(u)\Big)^2dxdt. \tag{2.21}$$

Then, from the definition of a_ε^*, (2.21) and (2.20),

$$\lim_{\varepsilon\to0^+}\iint_{Q_T}\Big[A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)-A\Big(\frac{\partial}{\partial x}B(u)\Big)\Big]^2dxdt$$

$$= \lim_{\varepsilon\to 0^+} \iint_{Q_T} (a_\varepsilon^*)^2 \Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon) - \frac{\partial}{\partial x} B(u)\Big)^2 dxdt = 0.$$

This, together with (2.15) yields (2.17), namely

$$A_\varepsilon\Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon)\Big) \to A\Big(\frac{\partial}{\partial x} B(u)\Big), \quad \text{in } L^2(Q_T) \quad \text{as } \varepsilon \to 0^+,$$

which deduces (2.16). The proof is complete. $\Box$

In order to reach Theorem 1.2, we need the following two technical lemmas, which may be found in [9], [14].

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\{u_m\}$ be a uniformly bounded sequence in $L^\infty(\Omega)$ and $u \in L^\infty(\Omega)$ with*

$$u_m \overset{*}{\rightharpoonup} u, \quad \text{weakly}^* \text{ in } L^\infty(\Omega) \quad \text{as } m \to \infty.$$

Assume that $A(s), B(s)$ are continuous functions, and $A(s)$ is nondecreasing. If for any $\alpha \in A(\mathbb{R})$, $B(A^{-1}(\alpha))$ contains only a single point, and

$$A(u_m(x)) \to \omega(x), \quad \text{a.e. } x \in \Omega \quad \text{as } m \to \infty,$$

then

$$A(u(x)) = \omega(x), \quad \lim_{m\to\infty} B(u_k(x)) = B(u(x)), \quad \text{a.e. } x \in \Omega.$$

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < q < \infty$. Assume $\{f_m\}$ is a sequence in $L^q(\Omega)$ and $f \in L^q(\Omega)$ with*

$$f_m \rightharpoonup f, \quad \text{weakly in } L^q(\Omega) \quad \text{as } m \to \infty.$$

Then

$$\varliminf_{m\to\infty} \|f_m\|_{L^q(\Omega)} \geq \|f\|_{L^q(\Omega)}.$$

3 Proof of the Main Result

In this section, we will complete the proof of Theorem 1.2 based on the estimates and convergence established in §2.

Proof of Theorem 1.2. For any $k \in \mathbb{R}$, any $\eta > 0$ and any nonnegative functions $\varphi_i \in C^\infty(Q_T)\,(i = 1, 2)$ with $\text{supp}\varphi_i \subset [0, 1] \times (0, T)$ and

$$\varphi_1(0, t) = \varphi_2(0, t), \quad \varphi_1(1, t) = \varphi_2(1, t), \quad 0 < t < T,$$

according to Proposition 2.1,

$$\iint_{Q_T} \mathcal{H}_\eta(u_\varepsilon, k) \frac{\partial \varphi_1}{\partial t} dxdt + \iint_{Q_T} \mathcal{F}_\eta(u_\varepsilon, k) \frac{\partial \varphi_1}{\partial x} dxdt$$
$$- \iint_{Q_T} A_\varepsilon\Big(\frac{\partial}{\partial x} B_\varepsilon(u_\varepsilon)\Big) H_\eta(u_\varepsilon - k) \frac{\partial \varphi_1}{\partial x} dxdt$$

$$
\begin{aligned}
&-\iint_{Q_T} A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)H_\eta'(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x}\varphi_1 dxdt\\
&+\int_k^0 H_\eta'(\tau-k)(f(\tau)-f(k))d\tau\int_0^T(\varphi_1(1,t)-\varphi_1(0,t))dt\\
&+H_\eta(k)\iint_{Q_T}\Big[(u_\varepsilon-k)\frac{\partial\varphi_2}{\partial t}+(f(u_\varepsilon)-f(k))\frac{\partial\varphi_2}{\partial x}-A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\frac{\partial\varphi_2}{\partial x}\Big]dxdt=0. \quad (3.1)
\end{aligned}
$$

By the definitions of $\mathcal{H}_\eta(s,k)$ and $\mathcal{F}_\eta(s,k)$, and by using Lemma 2.4 with (2.12), we get

$$
\lim_{\varepsilon\to0^+}\mathcal{H}_\eta(u_\varepsilon(x,t),k)=\mathcal{H}_\eta(u(x,t),k),\quad \lim_{\varepsilon\to0^+}\mathcal{F}_\eta(u_\varepsilon(x,t),k)=\mathcal{F}_\eta(u(x,t),k),\quad \text{a.e. in } Q_T.
$$

Combining this with (2.16) and (2.12), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to0^+}\Big(\iint_{Q_T}\mathcal{H}_\eta(u_\varepsilon,k)\frac{\partial\varphi_1}{\partial t}dxdt+\iint_{Q_T}\mathcal{F}_\eta(u_\varepsilon,k)\frac{\partial\varphi_1}{\partial x}dxdt\\
&\qquad-\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)H_\eta(u_\varepsilon-k)\frac{\partial\varphi_1}{\partial x}dxdt\Big)\\
&=\iint_{Q_T}\mathcal{H}_\eta(u,k)\frac{\partial\varphi_1}{\partial t}dxdt+\iint_{Q_T}\mathcal{F}_\eta(u,k)\frac{\partial\varphi_1}{\partial x}dxdt\\
&\qquad-\iint_{Q_T}A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta(u-k)\frac{\partial\varphi_1}{\partial x}dxdt \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to0^+}\iint_{Q_T}\Big[(u_\varepsilon-k)\frac{\partial\varphi_2}{\partial t}+(f(u_\varepsilon)-f(k))\frac{\partial\varphi_2}{\partial x}-A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\frac{\partial\varphi_2}{\partial x}\Big]dxdt\\
&=\iint_{Q_T}\Big[(u-k)\frac{\partial\varphi_2}{\partial t}+(f(u)-f(k))\frac{\partial\varphi_2}{\partial x}-A\Big(\frac{\partial}{\partial x}B(u)\Big)\frac{\partial\varphi_2}{\partial x}\Big]dxdt. \qquad (3.3)
\end{aligned}
$$

Note that u satisfies (1.2) and (1.3) in the sense of (1.4) and (1.5). Therefore, owing to (3.2), (3.3) and (3.1), it is shown that u is just a renormalized solution of the problem (1.1)–(1.3) provided

$$
\begin{aligned}
&\varliminf_{\varepsilon\to0^+}\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)H_\eta'(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x}\varphi_1 dxdt\\
&\ge\iint_{Q_T}A\Big(\frac{\partial}{\partial x}B(u)\Big)H_\eta'(u-k)\frac{\partial u}{\partial x}\varphi_1 dxdt, \qquad (3.4)
\end{aligned}
$$

which will be proved below.

Multiplying (2.8) by u_ε and then integrating over Q_T, we derive that by the formula of integration by parts

$$
\frac12\iint_{Q_T}\frac{\partial u_\varepsilon^2}{\partial t}dxdt-\iint_{Q_T}f(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}dxdt=-\iint_{Q_T}A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\frac{\partial u_\varepsilon}{\partial x}dxdt. \qquad (3.5)
$$

From (2.11) and Lemma 2.1,

$$
\Big|\iint_{Q_T}\frac{\partial u_\varepsilon^2}{\partial t}dxdt\Big|=\Big|\int_0^1\Big(u_\varepsilon^2(x,T)-u_{0,\varepsilon}^2(x)\Big)dx\Big|\le C,
$$

$$\left|\iint_{Q_T} f(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}dxdt\right| \le \sup_{Q_T}|f(u_\varepsilon)| \iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial x}\right|dxdt \le C.$$

These two estimates and (3.5) yield

$$\iint_{Q_T} A_\varepsilon\Big(\frac{\partial}{\partial x}B_\varepsilon(u_\varepsilon)\Big)\frac{\partial u_\varepsilon}{\partial x}dxdt \le C.$$

From the definitions of A_ε and B_ε, the above inequality leads to

$$\iint_{Q_T}\left|\frac{\partial K_\varepsilon(u_\varepsilon)}{\partial x}\right|^p dxdt \le C,$$

where

$$K_\varepsilon(s) = \int_0^s (b(\sigma)+\varepsilon)^{(p-1)/p}d\sigma, \quad s \in \mathbb{R}.$$

Thus

$$\iint_{Q_T}\left|\frac{\partial K(u)}{\partial x}\right|^p dxdt \le C,$$

namely $\dfrac{\partial K(u)}{\partial x} \in L^p(Q_T)$ with

$$K(s) = \int_0^s b^{(p-1)/p}(\sigma)d\sigma, \quad s \in \mathbb{R}.$$

Moreover, (2.12) and (2.14) imply

$$\frac{\partial K_\varepsilon(u_\varepsilon)}{\partial x} \rightharpoonup \frac{\partial K(u)}{\partial x}, \quad \text{weakly in } L^p(Q_T) \quad \text{as } \varepsilon \to 0^+. \tag{3.6}$$

Therefore, for fixed $\eta > 0$, $\dfrac{\partial K(u)}{\partial x}\big(H_\eta'(u-k)\varphi_1\big)^{1/p} \in L^p(Q_T)$, and (3.6) implies

$$\frac{\partial K_\varepsilon(u_\varepsilon)}{\partial x}\big(H_\eta'(u_\varepsilon-k)\varphi_1\big)^{1/p} \rightharpoonup \frac{\partial K(u)}{\partial x}\big(H_\eta'(u-k)\varphi_1\big)^{1/p}, \quad \text{weakly in } L^p(Q_T) \quad \text{as } \varepsilon \to 0^+.$$

By Lemma 2.5,

$$\left\|\frac{\partial K(u)}{\partial x}\big(H_\eta'(u-k)\varphi_1\big)^{1/p}\right\|_{L^p(Q)} \le \varliminf_{\varepsilon\to0^+}\left\|\frac{\partial K_\varepsilon(u_\varepsilon)}{\partial x}\big(H_\eta'(u_\varepsilon-k)\varphi_1\big)^{1/p}\right\|_{L^p(Q)}.$$

This is just (3.4). The proof of Theorem 1.2 is complete. $\square$

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